National Curriculum (Vocational) Mathematics Level 2

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SUBJECT OUTCOME I

NUMBERS: USE COMPUTATIONAL TOOLS AND STRATEGIES TO MAKE ESTIMATIONS AND APPROXIMATIONS

Subject outcome 1.1

Use computational tools and strategies to make estimations and approximations.

Learning outcomes

- Use a scientific calculator correctly to solve expressions involving addition, subtraction, multiplication, division, squares, cubes, square roots and cube roots.
- Estimate and approximate physical quantities to solve problems in practical situations. Quantities include length, time, mass and temperature.

Unit 1

By the end of this unit you will be able to:

- Use a scientific calculator competently and efficiently.
- Execute algorithms appropriately in calculations.
- Take and record measurements to the degree of accuracy of the instrument.

Unit 1: Make estimations and approximations

GILL SCOTT

Unit 1

By the end of this unit you will be able to:

- Use a scientific calculator competently and efficiently.
- Execute algorithms appropriately in calculations.
- Take and record measurements to the degree of accuracy of the instrument.

What you should know

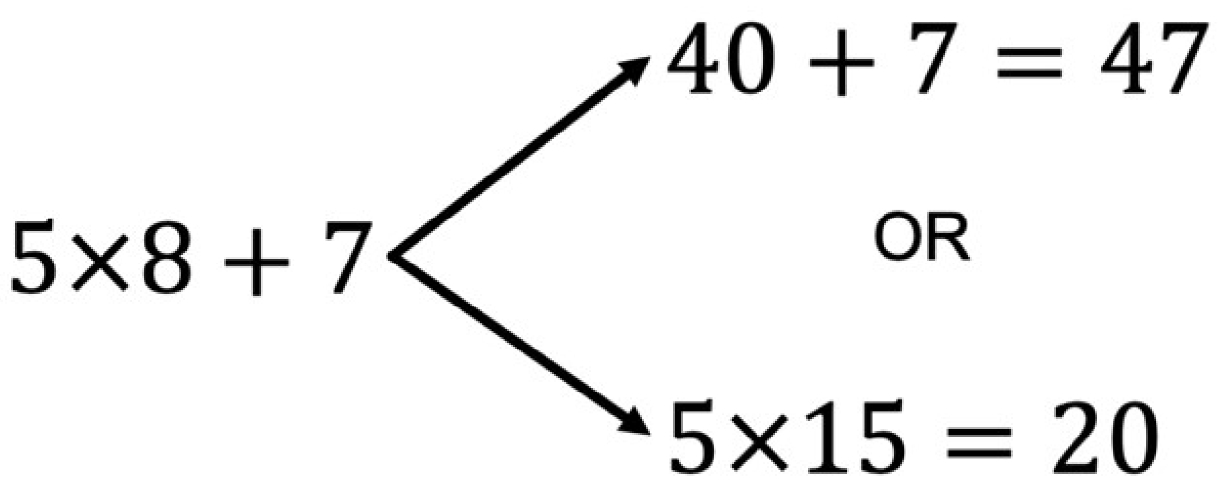
Before you start this unit, make sure you can:

- Use basic arithmetic operations.
- Work with the decimal system.

Introduction

A scientific calculator is a useful tool for people working with numbers and doing arithmetical and mathematical calculations. As with all tools, to be effective, users need to know what tasks (or 'operations') must be done, what steps to follow and how to use the tools correctly.

Numbers are often combined in a sequence of arithmetical operations (such as addition, subtraction, multiplication, division, raising to powers and finding square, cube and other roots). The order in which calculations are made often affects the answers. For example, is:



In other words, do we multiply five by eight first or do we add eight and seven first?

Although many calculators automatically do the calculations in the correct order, it is necessary to know the rule defining the proper order of priority in which arithmetic operations should be made.

A useful rule to remember is one called BODMAS. BODMAS is a mnemonic (pronounced nemonik). A mnemonic is a system such as a pattern of letters, ideas, or associations which helps you to remember something. BODMAS is a mnemonic for Brackets, Of, Division, Multiplication, Addition, Subtraction.

Take note!

Order of operations – BODMAS

Brackets – work out whatever is inside of any brackets first. You may need to apply the BODMAS order inside brackets as well.

Of – 'Of' in an expression means $\times$; or the **O**rders **O**f numbers – i.e. Powers, and Square Roots, Cube Roots, etc.

DM – **D**ivision and **M**ultiplication (start from left to right)

AS – **A**ddition and **S**ubtraction (start from left to right)

Using the BODMAS rule that multiplication must be done before addition, the correct answer to the above calculation is:

$5 \times 8 + 7 = 40 + 7 = 47$

Using a scientific calculator

There are many brands of scientific calculators available, each with its own operator manual explaining how to use it. If you lose the operator manual for your calculator, you can download it from the internet. It is important to have a manual for your calculator to refer to as your studies progress and you learn about more advanced mathematical functions.

Two commonly available brands of calculators are CASIO and SHARP. They are shown in Figure 1 to illustrate how scientific calculators can be used in general. YouTube has many calculator tutorials, if ever you are stuck. Here are two you might want to watch now.

How To Use Scientific Calculators (Duration: 3.11)

Using a Scientific Calculator (Duration: 17.12)

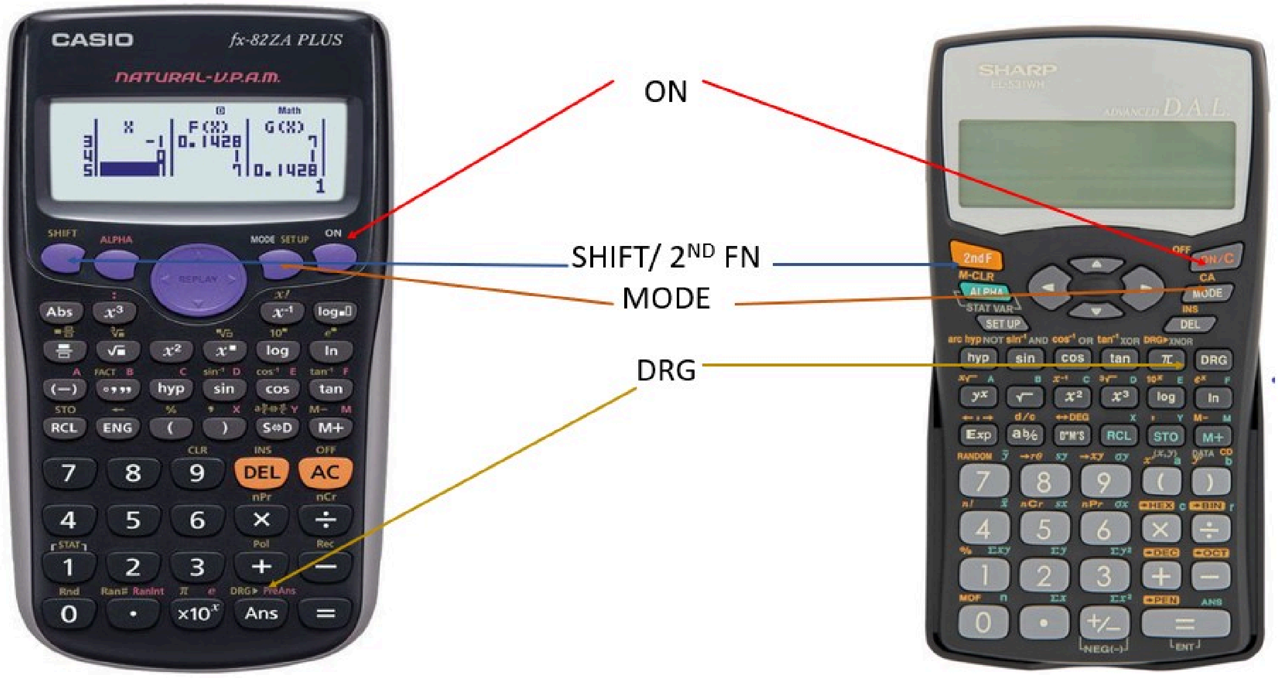
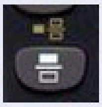


Figure 1: An example of a CASIO scientific calculator and a SHARP scientific calculator

In Figure 1 you can see that each key on the calculator has a number or operator on its face, as well as functions above it, to the left and sometimes the right. Pressing keys in a particular order enters numbers and functions onto the display screen showing the calculation being made, and giving the answer. The functions of some of the keys that you will need are explained below (some are for CASIO and some for SHARP – check on your calculator to see which is relevant).

ON/C **AC** **OFF** **DEL**	Turns the calculator on, or clears the data. Clears all of the calculation you are keying in. Turns the calculator off, although most will turn off automatically if left for a while. Deletes a single character from the string you are entering .
← ↑ → ↓	These position arrows move the cursor so you can insert or delete a character or function in a specific position in a calculation that you are entering.
MODE	Different MODEs are available, including statistical (STAT). The calculator is in 'statistical' mode if the display screen shows the word 'STAT'. You will need 'normal' mode for normal arithmetic and computation (COMP) calculations.
 	Pressing **SHIFT** (CASIO) or **2nd F** (2nd Function) (SHARP) before pressing a function key uses the function printed above the function key. For example: • Pressing **SHIFT** before  will turn the **CASIO** calculator off • Pressing **2nd F** before  will turn the **SHARP** off.

•	Decimal point key
π	Enters the value for π (3.14159...).
DRG	This key indicates values for angles, with units in DEGREEs, RADIANs or GRADIANs, with this key showing the units in which the value was entered. Make sure that your calculator is set for DEG before you start.
(and **)**	Insert brackets where you think necessary for calculations to be done in the correct order.

Exp or **SCI** or **ENG**	Numbers with too many digits to display on the screen in normal decimal format are displayed in **standard notation** (also called exponential form – **Exp**, **scientific notation - SCI**, or sometimes **engineering notation** - **ENG**). For example, $2\,800\,000 = 2.8 \times 10^{6}$ - a *positive* exponent indicates a very *large* number, and $0.00000056 = 5.6 \times 10^{-7}$ - a *negative* exponent indicates a very *small* number
 And  	Calculations can be done with fractions or mixed numbers, and conversions can be made between these and their decimal forms. Press the fraction key after entering each number part of the fraction, and then continue with the calculation as usual. The **CASIO** uses  to convert between fractional and decimal forms of an entered number.

 (CASIO) and  (SHARP)	Calculates the inverse of a number. Note: with the **SHARP**, the **2nd F** key would need to be pressed first
x^2 and x^3 $x^{\square}$ (CASIO) and (SHARP) y^x	Enter a number and press x^2 or x^3 to get its square or cube. $x^{\square}$ and y^x raise numbers to any power: - enter the base number, press the key, enter the required power, press the $=$
 (CASIO) and  with **SHIFT** (CASIO) or  with **2nd F** (SHARP)	Enter a number and press the relevant square of cube root button to get the required root. When using keys where you must input the root that you want, - first enter the order of the root required, press the key (using **SHIFT** or **2nd F** as necessary), enter the number whose root you want, and then press the $=$

Note

If you would like a short demonstration of calculator basics, watch the video called "Basic Calculator Use".

Basic Calculator Use (Duration: 16.23)

Example 1.1

Calculate the value of the following expression:

$11 - 12 \div 4 + 3 \times (6 - 2)$

Solution

If you enter this string of operations into your calculator as written and then press the key, you should get an answer of 20.

Here is the correct order of operations that your calculator will perform.

$$
\begin{aligned}
&11 - 12 \div 4 + 3 \times (6 - 2) \quad \text{The value of the bracket is calculated first}\\
&= 11 - 12 \div 4 + 3 \times 4 \quad \text{Division is calculated next } (12 \div 4)\\
&= 11 - 3 + 3 \times 4 \quad \text{Multiplication is calculated next } (3 \times 4)\\
&= 11 - 3 + 12 \quad \text{Subtraction appears first on the left: calculate } (11 - 3)\\
&= 8 + 12 \quad \text{Addition appears on the right of the subtraction: calculate } (8 + 12)\\
&= 20
\end{aligned}
$$

Example 1.2

Calculate the value of the following expression:

$2.58^2 - 0.89^3$

Solution

$$
\begin{aligned}
2.58^2 - 0.89^3 &= 6.6564 - 0.89^3 && \text{Enter 2.58 then press the } x^2 \text{ key then press the } - \text{ key}\\
&= 6.6564 - 0.704969 && \text{Enter 0.89 then press the } x^{\square} \text{ key CASIO}\\
& && \text{(or for SHARP press the } y^x \text{ key) followed by 3}\\
&= 5.951431 && \text{Press the } = \text{ key to get the answer}
\end{aligned}
$$

Once you are sure of the steps, you need not write them all down, but just carefully press the keys as necessary.

Exercise 1.1

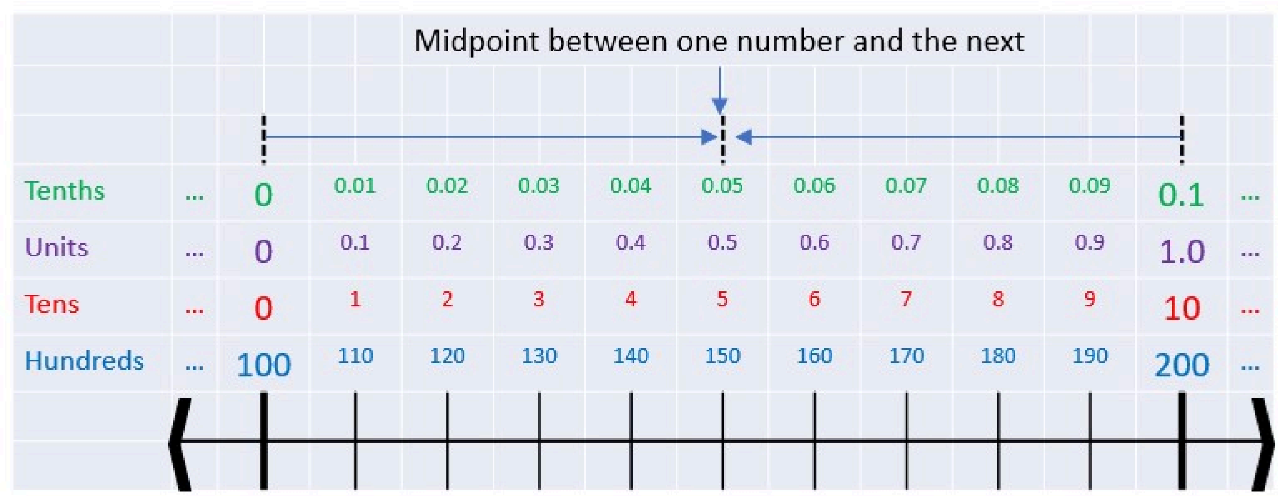
Use your calculator to solve the following expressions:

1. $3 + 803.308 \times 10^3 - \sqrt{64}$
2. $603.8 \times 21.7 - 52^2$
3. $21.1^3 - \sqrt[3]{9393.931}$
4. $52.785 \div 2.3 + 471.03 \times 2.825 \times 0.44$

The full solutions are at the end of the unit.

Approximations

The decimal system is based on powers of 10, and each interval between one number and the next can be divided into ten sub-intervals, with 5 being the mid-point between the two numbers. Numbers less than 5 are closer to the lower number, and numbers greater than 5 are closer to the larger number. By agreement, 5 is treated as though it is closer to the larger number.



This idea is used in approximating numbers, when an answer is required to a specific number of decimal places, or to a given number of 'significant figures'.

- If the figure to be discarded is 5 or more, the previous figure is increased by 1.
- If the figure to be discarded is less than 5, the previous figure is not changed.

If an answer must be correct to a given number of decimal places, that number of figures must be given after the decimal point.

Example 1.3

1. Round 0.07823 off to 3 decimal places.
2. Round 0.07823 off to 2 decimal places.
3. Round 0.07823 off to 1 decimal place.
4. Round 23.492 off to 2 decimal places.
5. Round 23.492 off to 1 decimal place.

Solutions

1. $0.07823 = 0.078 \text{ correct to } 3 \text{ decimal places}$
   We start counting digits from the decimal point to the right. In this case, we keep the first three digits after the decimal point and discard from the fourth digit. Because the first digit to be discarded is 2 (less than 5) we do not need to change the value of the third digit.
2. $0.07823 = 0.08 \text{ correct to } 2 \text{ decimal places}$
   We start counting digits from the decimal point to the right. In this case, we keep the first two digits following the decimal point, and discard from the third digit. Because the first digit to be discarded is 8 (which is greater than or equal to 5) we have to round the second digit *up* i.e. we need to change the 7 to an 8.
3. $0.07823 = 0.1 \text{ correct to } 1 \text{ decimal place}$
   We start counting digits from the decimal point to the right. In this case, we keep the first digit only and discard from the second digit. Because the first digit to be discarded is 7 (greater than or equal to 5) we have to round the first digit *up* i.e. we need to change the 0 into a 1.
4. and 5.
   $$\begin{aligned}23.492 &= 23.49 \text{ correct to } 2 \text{ decimal places}\\ &= 23.5 \text{ correct to } 1 \text{ decimal place}\end{aligned}$$

Example 1.4

1. Solve $27.4144 \div 6.4$.
2. Solve $27.4144 \div 6.4$ correct to 3 decimal places.
3. Solve $27.4144 \div 6.4$ correct to 2 decimal places.
4. Solve $27.4144 \div 6.4$ correct to 1 decimal place.

Solutions

1. $27.4144 \div 6.4 = 4.2835$ The answer is exact, to 4 decimal places.
2. $$\begin{aligned}27.4144 \div 6.4 &= 4.2835\\ &= 4.284 \text{ correct to } 3 \text{ decimal places}\end{aligned}$$
   We keep the first three digits after the decimal point, and discard the fourth digit. Because the digit being discarded (5 in this case) is greater than or equal to 5, we round the third digit *up* i.e. we

change the 3 into a 4.

3.

$$\begin{aligned} 27.4144 \div 6.4 &= 4.2835 \\ &= 4.28 \text{ correct to 2 decimal places} \end{aligned}$$

We keep the first two digits after the decimal point, and discard the third digit. Because the digit being discarded (3 in this case) is less than 5, we keep the second digit the same i.e. 8.

4.

$$\begin{aligned} 27.4144 \div 6.4 &= 4.2835 \\ &= 4.3 \text{ correct to 1 decimal place} \end{aligned}$$

We keep the first digit after the decimal point, and discard from the second digit. Because the first digit being discarded (8 in this case) is greater than or equal to 5, we round the first digit *up* i.e we change the 2 into a 3.

Note

All non-zero digits are **significant**. Zeros that are between other significant figures are significant and must be kept.

$$\begin{array}{rl} 4204 & 4 \text{ significant figures} \\ 32.002 & 5 \text{ significant figures} \\ 50.90602 & 7 \text{ significant figures} \end{array}$$

Zeros that come before all non-zero significant figures are not significant. The leading zeros serve to place the decimal point correctly.

$$\begin{array}{rl} 0.23 & 2 \text{ significant figures} \\ 0.00023 & 2 \text{ significant figures} \\ 0.00203 & 3 \text{ significant figures} \end{array}$$

Zeros that follow non-zero digits are not significant unless the decimal point is indicated:

$$\begin{array}{l} 150 \text{ is correct to the tens digit: it has 2 significant figures. But} \\ 150.0 \text{ is correct to the tenths digit. it has 0 units and 0 tenths: it has 4 significant figures} \end{array}$$

$$\begin{array}{rl} 43\ 000 & 2 \text{ significant figures} \\ 43.000 & 5 \text{ significant figures} \\ 43\ 000\ 564\ 001 & 11 \text{ significant figures} \end{array}$$

Example 1.5

1. Write $27\ 283$ correct to 4 significant figures.
2. Write $27\ 283$ correct to 3 significant figures.
3. Write $27\ 283$ correct to 2 significant figures.
4. Write 0.0887 correct to 2 significant figures.

5. Write 0.0887 correct to 1 significant figure.
6. Write 36.7504 correct to 5 significant figures.
7. Write 36.7504 correct to 3 significant figures.

Solutions

1. $27\ 283 = 27\ 280$ correct to 4 significant figures
   Here we need to replace the units digit with a zero to keep the decimal point in its proper position. If we did not do this then we would have changed the whole value of the number by essentially dividing it by 10.
2. $27\ 283 = 27\ 300$ correct to 3 significant figures
   Here we use two zeros to keep the decimal point in its proper position.
3. $27\ 283 = 27\ 000$ correct to 2 significant figures
4. and 5.
   $$\begin{aligned} 0.0887 &= 0.089 \text{ correct to } 2 \text{ significant figures} \\ &= 0.09 \text{ correct to } 1 \text{ significant figure} \end{aligned}$$
6. and 7.
   $$\begin{aligned} 36.7504 &= 36.750 \text{ correct to } 5 \text{ significant figures} \\ &= 36.8 \text{ correct to } 3 \text{ significant figures} \end{aligned}$$

Exercise 1.2

1. Give the following numbers correct to the number of significant figures stated:
   a. 32.86583
      i. to 6 significant figures
      ii. to 4 significant figures
      iii. to 2 significant figures
   b. 0.0047582
      i. to 4 significant figures
      ii. to 3 significant figures
      iii. to 2 significant figures
   c. 4.97848
      i. to 5 significant figures
      ii. to 3 significant figures
      iii. to 1 significant figure
   d. 21.987
      i. to 2 significant figures
      ii. to 4 significant figures
      iii. to 6 significant figures

e. 62.602
   i. to 4 significant figures
   ii. to 6 significant figures
f. $32\ 638\ 319$
   i. to 5 significant figures
   ii. to 2 significant figures
   iii. to 10 significant figures
g. 4.14976
   i. to 5 significant figures
   ii. to 4 significant figures
   iii. to 3 significant figures

2. Give the following numbers correct to the number of decimal places stated:
   a. 5.14987
      i. to 4 decimal places
      ii. to 3 decimal places
      iii. to 2 decimal places
   b. 35.285
      i. to 2 decimal places
      ii. to 1 decimal place
   c. 0.004977
      i. to 5 decimal places
      ii. to 4 decimal places
      iii. to 3 decimal places
   d. 8.4076
      i. to 3 decimal places
      ii. to 2 decimal places
      iii. to 1 decimal place
3. Find the value of:
   a. $18.76 \div 14.3$ correct to 2 decimal places
   b. $0.0396 \div 2.51$ correct to 2 significant figures
   c. $7.21 \div 0.038$ correct to 3 significant figures
   d. 5.13×7.34 correct to 2 decimal places
   e. $27 \times 13 \div 17$ correct to 3 significant figures

The full solutions are at the end of the unit.

Estimation

Estimating the answer to a calculation is useful to check that you have not made a mistake in entering it into your calculator. Estimation is particularly useful to check that you have not misplaced the decimal point.

Example 1.6

First estimate the calculation, then use your calculator to get an answer. Check that your answer is fairly close to the estimate.

1. 0.34×0.62
2. $213.8 \div 39.4$
3. $\dfrac{8.198 \times 19.56 \times 30.82 \times 0.198}{6.52 \times 3.58 \times 0.823}$

Solutions

1. 0.34×0.62
   For a rough estimate, we could use 0.3 and 0.6.
   $\text{Estimated product} = 0.3\times 0.6 = 0.18$
   $\text{Correct product} = 0.34\times 0.62 = 0.2108$
   The estimate shows that the answer of 0.2108 is correct and that it is not 2.108 or 0.02108.
2. $213.8 \div 39.4$
   For a rough estimate we could use 200 and 40.
   $\text{Estimated answer} = 200 \div 40 = 5$
   $\text{Correct answer} = 213.8 \div 39.4$
   $= 5.426 \text{ (correct to 3 decimal places)}$
   Note that the estimate and the correct answer are of the same order.
3. $\dfrac{8.198 \times 19.56 \times 30.82 \times 0.198}{6.52 \times 3.58 \times 0.823}$
   Estimated as $= \dfrac{8 \times 20 \times 30 \times 0.2}{6 \times 4 \times 1} = 40$
   Correct answer $= 50.938 \text{ (correct to 3 decimal places)}$
   The estimate shows that the answer should be 50.938, and not 509.38 or 5.0938.

Exercise 1.3

Use estimates to check your answers for the following:

1. 32.7×0.259
2. $0.682 \times 0.097 \times 2.38$
3. $78.41 \div 23.78$
4. $0.059 \div 0.00268$
5. $\dfrac{0.728 \times 0.00625}{0.0281}$
6. $\dfrac{27.5 \times 30.52}{11.3 \times 2.73}$

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to use the basic functions of a scientific calculator.
- How to apply the order of operations when doing calculations.
- How to approximate and round off in the decimal system, to give answers correct to a given number of significant figures, and a given number of decimal places.
- How to use estimation to check calculations.

Unit 1: Assessment

Suggested time to complete: 10 *minutes*

1. Use your calculator to give the answers to the following, correct to 3 decimal places.
   a. $27 + 162.51^2 - 24^3$
   b. $\sqrt{249.35} - 36.55 \times 0.52 + 12.8^2$
   c. $13.44 \times 0.864 + 12.55 \div 3.2 - \sqrt[3]{36}$
   d. $(\sqrt{125} + \sqrt[3]{128}) \times 12.4$
2. Use your calculator to give the answers to the following, correct to 3 significant figures.
   a. $384.6^2 \times \sqrt{0.925} + (12.35 \div 7.57)^3$
   b. $\sqrt[3]{0.4388} \div 12.765 - 265.22 \times 0.08549^2$

The full solutions are at the end of the unit.

Unit 1: Solutions

Exercise 1.1

1. $803\ 303$
2. $10\ 398.46$
3. $9\ 372.831$
4. 22.95

Back to Exercise 1.1

Exercise 1.2

1.
   a.
      i. 32.8658
      ii. 32.87

iii. 33

b.

i. 0.004758

ii. 0.00476

iii. 0.0048

c.

i. 4.9785

ii. 4.98

iii. 5.0

d.

i. 22

ii. 21.99

iii. 21.9870

e.

i. 62.60

ii. 62.6020

f.

i. $32\ 638\ 000$

ii. $33\ 000\ 000$

iii. $32\ 638\ 310.00$

g.

i. 4.1498

ii. 4.150

iii. 4.15

2.

a.

i. 5.1499

ii. 5.150

iii. 5.15

b.

i. 35.29

ii. 35.3

c.

i. 0.00498

ii. 0.0050

iii. 0.005

d.

i. 8.408

ii. 8.41

iii. 8.4

3.

a. 1.31

b. 0.016
c. 190
d. 37.65
e. 20.6

Back to Exercise 1.2

Exercise 1.3

1.
32.7×0.259
Estimated as $30 \times 0.3 = 9$
Correct answer $= 8.469$ correct to 3 decimal places

2.
$0.682 \times 0.097 \times 2.38$
Estimated as $1 \times 0.1 \times 2 = 0.2$
Correct answer $=0.157$ correct to 3 decimal places

3.
$78.41 \div 23.78$
Estimated as $80 \div 20 = 4$
Correct answer $= 3.297$ correct to 3 decimal places

4.
$0.059 \div 0.00268$
Estimated as $0.06 \div 0.003 = 20$
Correct answer $= 22.015$ correct to 3 decimal places

5.
$\dfrac{0.728 \times 0.00625}{0.0281}$
Estimated as $\dfrac{1 \times 0.006}{0.03} = 0.2$
Correct answer $= 0.162$ correct to 3 decimal places

6.
$\dfrac{27.5 \times 30.52}{11.3 \times 2.73}$
Estimated as $\dfrac{30 \times 30}{10 \times 3} = 30$
Correct answer is $= 27.207$ correct to 3 decimal places

Back to Exercise 1.3

Unit 1: Assessment

The answers to the steps in the calculations are given to help identify any mistakes that may have been made.

1.

a.
$$\begin{aligned} 27 + 162.51^2 - 24^3 &= 27 + 26409.5001 - 13824 \\ &= 12612.500 \text{ correct to 3 decimal places} \end{aligned}$$

b.

$$\sqrt{249.35} - 36.55 \times 0.52 + 12.8^2$$
$$= 15.7908 - 19.006 + 163.84$$
$$= 160.625 \text{ correct to 3 decimal places}$$

c.

$$13.44 \times 0.864 + 12.55 \div 3.2 - \sqrt[3]{36}$$
$$= 11.6122 + 3.9219 - 3.3019$$
$$= 12.232 \text{ correct to 3 decimal places}$$

d.

$$(\sqrt{125} + \sqrt[3]{128}) \times 12.4$$
$$= (11.1803 + 5.0397) \times 12.4$$
$$= 162200 \times 12.4$$
$$= 201.128 \text{ correct to 3 decimal places}$$

2.

a.

$$384.6^2 \times \sqrt{0.925} + (12.35 \div 7.57)^3$$
$$= 147917.16 \times 0.96177 + (1.63144)^3$$
$$= 142262.287 + 4.3422$$
$$= 142266.6292$$
$$= 142000 \text{ correct to 3 significant figures}$$

b.

$$\sqrt[3]{0.4388} \div 120.765 - 5.2264 \times 0.008549^2$$
$$= 0.7599 \div 120.765 - 5.2264 \times 0.000073085$$
$$= 0.006292 - 0.000381973$$
$$= 0.0059104$$
$$= 0.00591 \text{ correct to 3 significant figures}$$

Back to Unit 1: Assessment

Media Attributions

SUBJECT OUTCOME II

NUMBERS: NUMBERS AND NUMBER RELATIONSHIPS

Subject outcome 1.2

Demonstrate an understanding of numbers and relationships among numbers and number systems and represent numbers in different ways.

Learning outcomes

- Identify rational numbers and convert between terminating and recurring decimals like $\frac{a}{b}; a, b \in \mathbb{Z}, b \neq 0$.
- Round off rational and irrational numbers to an appropriate degree of accuracy.
  Apply the laws of exponents.
- Rationalise fractions with surd denominators.
- Add, subtract, multiply, and divide simple surds.
- Identify and work with arithmetic progressions, sequences and series.

Unit 1: Identify and work with rational and irrational numbers

NATASHIA BEARAM-EDMUNDS

Unit outcomes

By the end of this unit you will be able to:

- Identify rational numbers and convert between terminating and recurring decimals such as $\frac{a}{b}; a, b \in \mathbb{Z},\ b \neq 0$.
- Round off rational and irrational numbers to an appropriate degree of accuracy.

What you should know

Before you start this unit, make sure you can:

- Classify real numbers as natural numbers, whole numbers, or integers.
- Perform calculations using order of operations on fractions.

Test your understanding of real numbers and operations by trying these questions before continuing with this unit.

1. Classify each number below as a natural number ($\mathbb{N}$), whole number ($\mathbb{N}_0$) or integer ($\mathbb{Z}$). Note: List all possible classifications for each number.
   a. $\sqrt{9}$
   b. -2
   c. $\frac{125}{25}$
2. Evaluate:
   a. $24 + 36(\frac{2}{3})$
   b. $(3 \cdot 2)(3 \cdot 2) - 4(6 + 2)$

Solutions

1. a. $\sqrt{9} = 3$ is $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{Z}$.
   b. -2 is an $\mathbb{Z}$
   c. $\frac{125}{25} = 5$ is $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{Z}$
2. a.

$$\begin{aligned} 24 + 36\left(\frac{2}{3}\right) &= 24 + 12(2) \\ &= 48 \end{aligned}$$

b.

$$\begin{aligned}(3\cdot 2)(3\cdot 2)-4(6+2) &= (6)(6)-4(8)\\ &= 36-32\\ &= 4\end{aligned}$$

Introduction

The earliest use of numbers was to count items. Farmers, cattlemen and tradesmen used tokens, stones or markers to signify a single quantity, for example a sheaf of grain, a head of livestock or a fixed length of cloth. Doing so made commerce possible, leading to improved communication and the spread of civilization.

About four thousand years ago, Egyptians introduced fractions to the number system. They first used them to show reciprocals. The reciprocal of a number is 1 divided by the number. Later, they used them to represent the amount when a quantity was divided into equal parts.

In this unit, we will explore the number system further by using rational and irrational numbers.

The number system revised

We will begin with a recap of the number system.

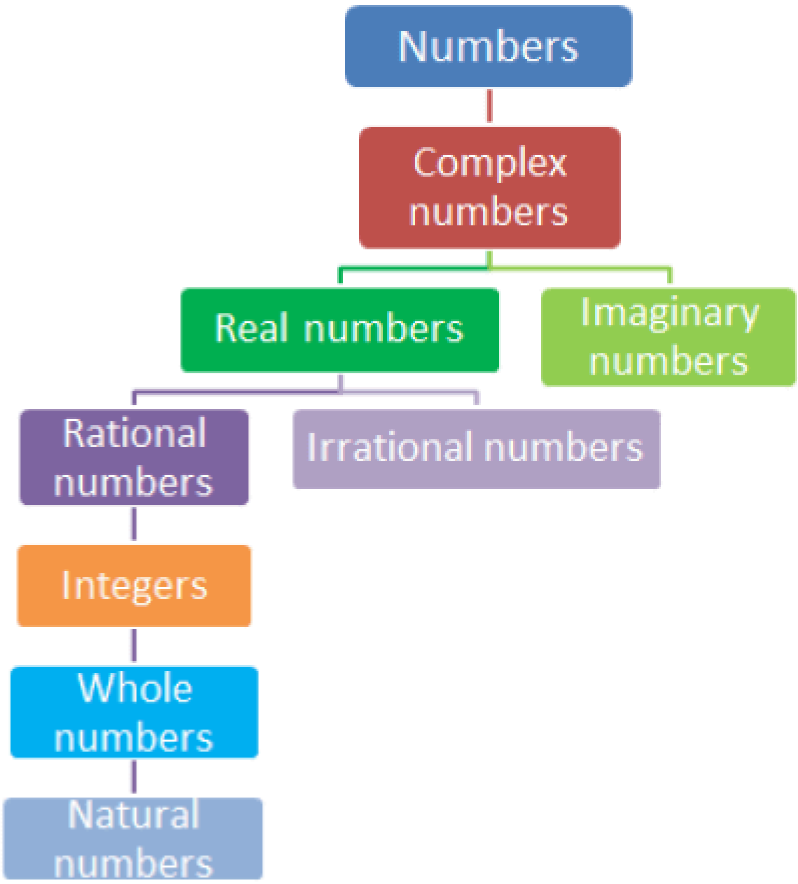


Figure 1: The number system

Complex numbers include all real and imaginary (non-real) numbers. More about complex numbers is covered in Level 3. For now we will focus on real numbers, which are made up of rational and irrational numbers.

Beginning with the natural numbers, we expand each set to form a larger set of numbers.

We use the following definitions:

The set of natural numbers $\mathbb{N}$ includes the numbers used for counting: $\{1,\ 2,\ 3,\ \ldots\}$.

The set of whole numbers $\mathbb{N}_0$ is the set of natural numbers plus zero: $\{0,\ 1,\ 2,\ 3,\ \ldots\}$.

The set of integers $\mathbb{Z}$ adds the negative of the natural numbers to the set of whole numbers $\{\ldots,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ \ldots\ \}$. It is important to remember that integral values are not fractions.

The set of rational numbers $\mathbb{Q}$ includes fractions that can be written as $\frac{a}{b}$ where $a, b \in \mathbb{Z},\ \ b \neq 0$.

The set of irrational numbers $\mathbb{Q}'$ are numbers that cannot be written as a fraction with the numerator and denominator as integers. These fractions give us nonrepeating and nonterminating (they do not end) decimals.

Rational and irrational numbers

A rational number $\mathbb{Q}$ is any number which can be written as $\frac{a}{b}$ where a and b are integers and $b \neq 0$.

We can see from the definition that every natural number, whole number and integer is a rational number with a denominator of 1.

As they are fractions, any rational number can also be expressed in decimal form. Any rational number can be represented as either:

a terminating decimal ($\frac{3}{4} = 0.75$) or

a repeating decimal ($\frac{4}{11} = 0.36363636\ldots = 0.\overline{36}$).

We use a line drawn over the repeating block of numbers to show that those numbers are repeated.

The following numbers are examples of rational numbers:

$\frac{5}{1}, \frac{15}{3}, \frac{-1}{3}, \frac{2}{3}, \frac{12}{24}$

We can see that all the numerators and denominators are integers. You can write any rational number as a decimal number but not all decimal numbers are rational numbers.

Exercise 1.1

1. Write each of the following as a rational number:

a. 6

b. -5

c. 0

2. Use your calculator to determine if the following rational numbers are terminating or recurring decimals.

   a. $\frac{-5}{6}$

   b. $\frac{325}{13}$

   c. $\frac{13}{25}$

The full solutions are at the end of the unit.

Irrational numbers $\mathbb{Q}'$ are numbers that cannot be written as a fraction with the numerator and denominator as integers.

The following numbers are examples of irrational numbers.

$$\sqrt{2}, \sqrt{3}, \pi, \frac{1}{\sqrt{2}}$$

These are not rational numbers, because either the numerator or the denominator is not an integer.

To identify whether a number is rational or irrational, first write the number in decimal form.

If the number terminates then it is rational. If it goes on forever, then look for a repeated pattern of digits. If there is no repeated pattern, then the number is irrational.

Exercise 1.2

1. Determine whether each of the following numbers is rational or irrational. If it is rational, determine whether it is a terminating or repeating decimal.

   a. $\sqrt{9}$

   b. $\sqrt{10}$

   c. $\frac{17}{34}$

   d. 0.3033033303333

The full solutions are at the end of the unit.

Convert terminating decimals to rational numbers

Any terminating decimal can be written as a decimal divided by one. For example, $0.45 = \frac{0.45}{1}$.

To convert $\frac{0.45}{1}$ to a rational number, multiply both the numerator and denominator by 100 (because there are two digits after the decimal point so that is $10 \times 10 = 100$).

$$\frac{0.45}{1} \times \frac{100}{100} = \frac{45}{100}$$

Do you see that the numerator is now a whole number? If you multiply the numerator and denominator by the correct power of ten, you will always end up with a whole number in the numerator.

We can simplify further by dividing both the numerator and denominator by five, to get:

$$\frac{45}{100} = \frac{9}{20}$$

Take note!

These are the steps you can use to convert a terminating decimal to a rational number.

1. Rewrite the terminating decimal as a decimal divided by one.
2. Multiply both the numerator and denominator by 10 for every number after the decimal point in the numerator. For example, if there are three numbers after the decimal point, then multiply by $1\ 000$, if there are four then multiply by, $10\ 000$ and so on.
3. Simplify the fraction if possible.

When there is a whole number before the decimal point, set the whole number aside and bring it back in at the end. As shown in the next example.

Example 1.1

Write 10.585 as a mixed fraction.

Set the whole number 10 aside and remember to bring it back in at the end.

There are three zeroes after the decimal so multiply both the numerator and denominator by $1\ 000$.

$$\begin{aligned}\frac{0.585}{1} &= \frac{0.585 \times 1\ 000}{1\ 000} \\ &= \frac{585}{1\ 000}\end{aligned}$$

Simplify the fraction if possible.

$$\frac{585 \div 5}{1\ 000 \div 5} = \frac{117}{200}$$

Bring the 10 back in and write the answer as a mixed fraction.

$10.585 = 10\frac{117}{200}$

Note: Do not 'add' the 10 to the fraction, you will get an incorrect answer. Simply put the whole number back in front of the fraction.

Exercise 1.3

1. Convert the following decimals to fractions:
   a. 0.2589
   b. 5.24

The full solutions are at the end of the unit.

Activity 1.1: Convert recurring decimal numbers into fractions

Time required: 15 minutes

What you need:

- pen and paper

What to do:

Write $0.\dot{3}$ in the form $\frac{a}{b}$ where a and b are integers.

1. We have $0.\dot{3} = 0.3333...$ a repeating decimal. Let $x = 0.3333...$. We will call this equation one.
2. Multiply both sides of equation one by 10. We will call this equation two.
3. Subtract equation one from equation two.
4. Simplify and solve for x.
5. Write $0.\dot{3}$ in the form $\frac{a}{b}$.

What did you find?

1. Equation one: $x = 0.3333...$
2. The decimal was multiplied by 10 because there was only one digit (3) recurring after the decimal point. In general, if you have one digit recurring, then multiply by 10. If you have two digits recurring, then multiply by 100. If you have three digits recurring, then multiply by $1\ 000$, and so on.
   You should have the following for equation two: $10x = 3.3333...$

3. We want the repeating parts of the decimal to cancel out when subtracting the equations. By subtracting equation one from equation two you will get:

$$\begin{aligned} 10x &= 3.3333... \\ -x &= -0.3333... \\ \therefore 9x &= 3 \end{aligned}$$

4. You can simplify by dividing both sides of the equation by nine.

$$\begin{aligned} \therefore x &= \frac{3}{9} \\ &= \frac{1}{3} \end{aligned}$$

5. Since we started out by defining $x = 0.3333...$ this means that $0.3333... = \frac{1}{3}$.

Exercise 1.4

1. Convert the following decimals to fractions:
   a. $0.\overline{25}$
   b. $0.8\dot{3}$

The full solutions are at the end of the unit.

Rounding off decimals

Rounding off a decimal number to a given number of decimal places is the quickest way to approximate a number.

For example, if you wanted to round off 2.6525272 to three decimal places, you would:

count three places after the decimal place and assess the value of the fourth digit

round up the third digit if the fourth digit is greater than or equal to 5

leave the third digit unchanged if the fourth digit is less than 5

if the third digit is 9 and needs to be rounded up, then the 9 becomes a 0 and the second digit is rounded up.

So, since the fourth digit after the decimal point in 2.6525272 is a 5 , we must round up the digit in the third decimal place to 3 . So the final answer of 2.6525272 rounded to three decimal places is 2.653 .

Exercise 1.5

1. Round off the following numbers to the indicated number of decimal places:
   a. $\frac{131}{9}$ to three decimal places
   b. π to six decimal places
   c. $\sqrt{2}$ to four decimal places.

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- To identify and differentiate between rational and irrational numbers.
- Convert terminating decimals to rational numbers.
- Convert recurring decimals to rational numbers.
- How to round off to a certain number of decimal places to approximate a number.

Unit 1: Assessment

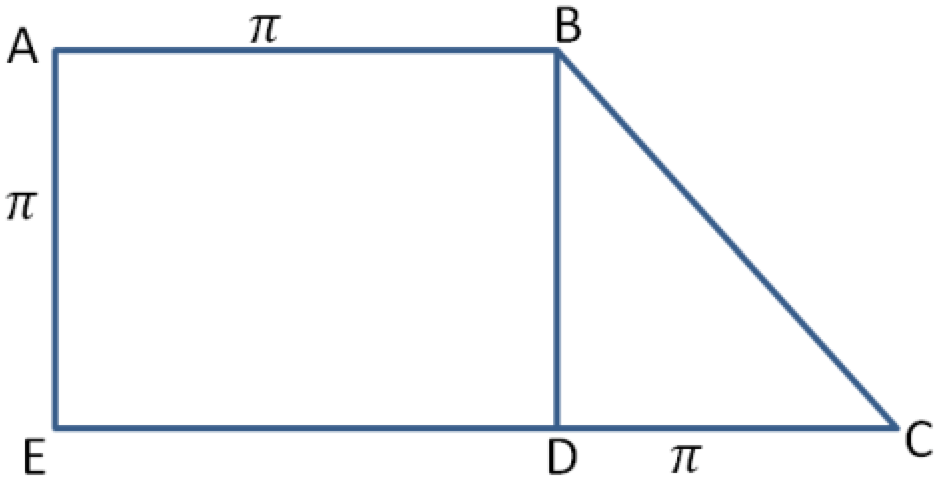
Suggested time to complete: 30 minutes

1. Which of the statements is true?
   a. Every integer is a natural number.
   b. Every natural number is a whole number.
   c. There are no decimals in whole numbers.
2. State whether the following numbers are rational or irrational. If the number is rational, state whether it is a natural number, whole number or an integer.
   a. $\frac{5}{8}$
   b. $0{,}651268962154862\ldots$
   c. $\frac{\sqrt{16}}{4}$
   d. $\sqrt[3]{4}$
3. Write the following as fractions:
   a. 0.68
   b. $2.\dot{3}$
   c. $0.\overline{12}$
   d. $0.6\dot{3}$
4. Write the following in decimal form, using the recurring decimal notation and without using a calculator:
   a. $\frac{7}{33}$

b. $1\frac{3}{11}$

c. $2\frac{1}{9}$

5. Round off the following to 3 decimal places:
   a. $12{,}56637061\ldots$
   b. $0{,}05555555\ldots$
   c. $0{,}2666666\ldots$
6. Study the diagram below



   a. Calculate the area of ABDE to two decimal places.
   b. Calculate the area of BCD to two decimal places.
   c. Using your answers in a) and b), calculate the area of ABCDE.

The full solutions are at the end of the unit.

Unit 1: Solutions

Exercise 1.1

1. Write a fraction with the integer in the numerator and 1 in the denominator.
   a. $6 = \frac{6}{1}$
   b. $-5 = \frac{-5}{1}$
   c. $0 = \frac{0}{1}$
2. Write each fraction as a decimal by dividing the numerator by the denominator.
   a. $\frac{-5}{6} = -0.8\overline{3}$ repeating decimal
   b. $\frac{325}{13} = 25$ (or 25.0) a terminating decimal
   c. $\frac{13}{25} = 0.52$ a terminating decimal

Back to Exercise 1.1

Exercise 1.2

1. a. $\sqrt{9} = 3$ is a terminating rational number

   b. $\sqrt{10} = 3.16227766016837933199...$ The average scientific calculator can only calculate up to ten decimal places. Using a powerful scientific calculator we can get a more precise answer. $\therefore \sqrt{10}$ is irrational.

   c. $\frac{17}{34} = \frac{1}{2} = 0.5$ is a terminating rational number

   d. $0.3033033303333\ldots$ is not a terminating decimal. Also note that there is no repeating pattern because the group of threes increases each time. Therefore it is neither a terminating nor a repeating decimal and, hence, not a rational number. It is an irrational number.

Back to Exercise 1.2

Exercise 1.3

1. a.

$$\begin{aligned}\frac{0.2589}{1} &= \frac{0.2589}{1} \times \frac{10\ 000}{10\ 000} \\ &= \frac{2\ 589}{10\ 000}\end{aligned}$$

   b. Work with only the decimal and bring the whole number back in at the end.

$$\begin{aligned}\frac{0.24}{1} &= \frac{0.24 \times 100}{100} \\ &= \frac{24}{100} \\ &= \frac{6}{25} \\ \therefore 5.24 &= 5\frac{6}{25}\end{aligned}$$

Back to Exercise 1.3

Exercise 1.4

1. a.

$$\begin{aligned}\text{Let } x &= 0.2525 && \text{Equation 1} \\ 100x &= 25.2525 && \text{Equation 2} \\ 99x &= 25 && \text{Subtract equation 1 from equation 2} \\ x &= \frac{25}{99} \\ \therefore 0.\overline{25} &= \frac{25}{99}\end{aligned}$$

   b.

$$\begin{aligned}
\text{Let } x &= 8.83333 \\
10x &= 8.3333 && \text{Equation 1} \\
100x &= 83.333 && \text{Equation 2} \\
90x &= 75 && \text{Subtract equation 1 from equation 2} \\
x &= \frac{75}{90} \\
&= \frac{5}{6} \\
\therefore 0.8\dot{3} &= \frac{5}{6}
\end{aligned}$$

Back to Exercise 1.4

Exercise 1.5

1. a. $\frac{131}{9} = 14.556$ to three decimal places.
   b. $\pi = 3.141593$ to six decimal places.
   c. $\sqrt{2} = 1.4142$ to four decimal places.

Back to Exercise 1.5

Unit 1: Assessment

1. Statements b) and c) are true.
2. a. $\frac{5}{8}$ is a rational number ($\mathbb{Q}$) but not also an integer ($\mathbb{Z}$).
   b. $0{,}651268962154862\ldots$ is a non-repeating non-terminating decimal. Therefore it is irrational number ($\mathbb{Q}'$)
   c. $\frac{\sqrt{16}}{4} = 1$ therefore it is a rational number ($\mathbb{Q}$). But it is also an integer ($\mathbb{Z}$), a whole number ($\mathbb{N}_0$) and a natural number ($\mathbb{N}$).
   d. $\sqrt[3]{4}$ is an irrational number ($\mathbb{Q}'$).
3. a.

$$\begin{aligned}
0.68 &= \frac{0.68}{1} \times \frac{100}{100} \\
&= \frac{68}{100} \\
&= \frac{17}{25}
\end{aligned}$$

   b.

$$\begin{aligned}
\text{Let } x &= 2.3333 \\
10x &= 23.333 \\
\therefore 9x &= 23.333 - 2.333 \\
9x &= 21 \\
\therefore x &= \frac{21}{9} = \frac{7}{3} \\
\therefore 2.\dot{3} &= \frac{7}{3}
\end{aligned}$$

   c.

$$0.\overline{12} = 0.121212$$

$$\begin{aligned} \text{Let } x &= 0.1212 \\ \therefore 100x &= 12.1212 \\ \therefore 99x &= (12.1212 - 0.1212) \\ \therefore 99x &= 12 \\ \therefore x &= \frac{12}{99} \\ &= \frac{4}{33} \\ \therefore 0.\overline{12} &= \frac{4}{33} \end{aligned}$$

d.

$$0.6\dot{3} = 0.6333$$
$$\begin{aligned} \text{Let } x &= 0.6333 \\ 10x &= 6.333 \\ 100x &= 63.333 \\ \therefore 90x &= (63.333 - 6.333) \\ &= 57 \\ \therefore x &= \frac{57}{90} \\ &= \frac{19}{30} \\ \therefore 0.63 &= \frac{19}{30} \end{aligned}$$

4. a. $\frac{7}{33} = 0.\overline{21}$

b.

$$\begin{aligned} 1\frac{3}{11} &= \frac{14}{11} \\ &= 1.\overline{27}3 \end{aligned}$$

c.

$$\begin{aligned} 2\frac{1}{9} &= \frac{19}{9} \\ &= 2.\dot{1} \end{aligned}$$

5. a. $12.56637061... = 12.566$ correct to 3 decimal places

b. $0.05555555... = 0.056$ correct to 3 decimal places

c. $0.2666666...0.267$ correct to 3 decimal places

6. a. Area of ABDE:
$\pi \times \pi = 9.86960... = 9.87$ correct to 2 decimal places

b. Area of BCD:

$$\begin{aligned} \frac{1}{2}b \cdot h &= \frac{1}{2}(\pi)(\pi) \\ &= 4.934802... = 4.93 \text{ correct to } 2 \text{ decimal places} \end{aligned}$$

c. Area of ABCDE = Area of ABDE + Area of BCD

$$\begin{aligned} \text{Area } &= 9.87 + 4.93 \\ &= 14.8 \end{aligned}$$

Back to Unit 1: Assessment

Media Attributions

Unit 2: Introduction to exponents

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Unit outcomes

By the end of this unit you will be able to:

- Use exponential notation.
- Understand the basic exponential identities $(a^n = a \times a \times a \times a \text{ (n times)}\ ; a^{(-n)} = \frac{1}{a^n}; \frac{1}{a^{-n}} = a^n; a^0 = 1)$.
- Multiply exponents with the same base $(a^m \times a^n = a^{(m+n)})$.
- Divide exponents with the same base $(\frac{a^m}{a^n} = a^{(m-n)})$.

What you should know

Before you start this unit, make sure you can:

Describe the real number system and explain the difference between natural numbers, whole numbers, integers, rational numbers and irrational numbers. Go back to unit 1 in this subject outcome if you need to revise this.

Introduction

We are living in exponential technological times. There are over 100 billion searches on Google every month, a massive increase from 2006 when that number was 2.7 billion. In 1984 the number of connected internet devices was $1\ 000$, in 1992 it was $1\ 000\ 000$, in 2008 it was $1\ 000\ 000\ 000$, around 17 billion in 2016 and just over 26 billion by the end of 2019.

It is estimated that by the end of 2025 there will be approximately 76 billion internet-connected devices worldwide. This form of rapid growth can best be described using exponents. Exponential increases start off slowly but then sharply increase to a tremendous explosion in size.

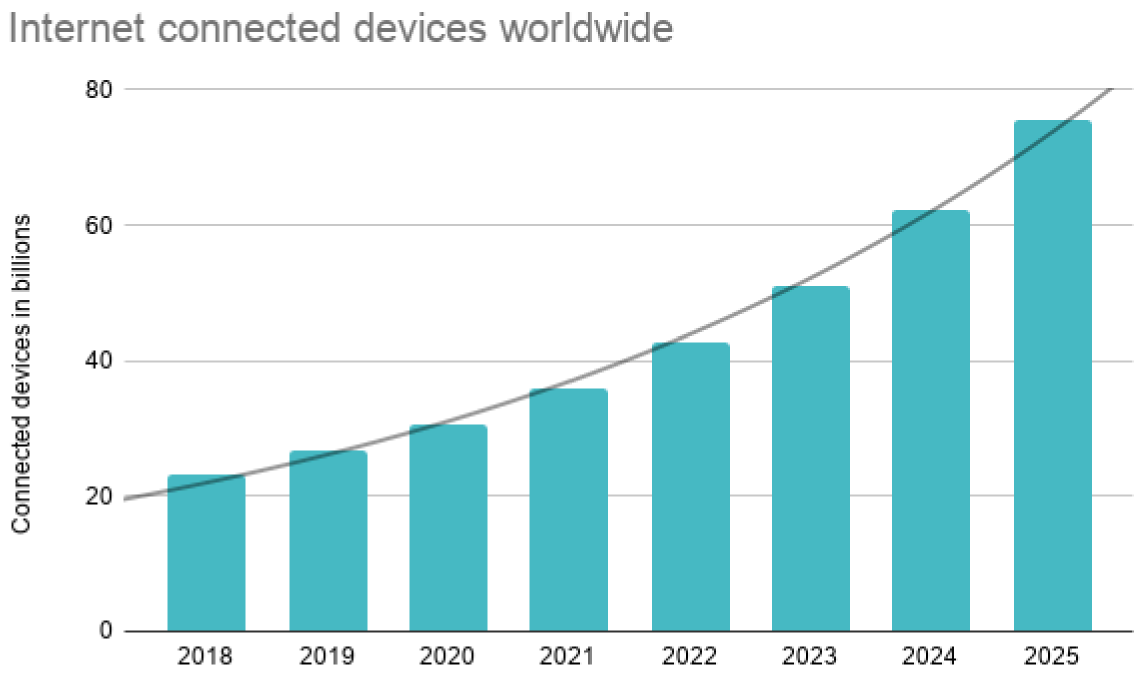


Figure 1: Exponential growth of internet devices

Are new internet devices increasing at the same rate each year? Or is the rate of new devices also growing exponentially? Think about this.

Can you think of another current situation where we are seeing exponential growth?

Exponents are a powerful way to describe rapid increases or decreases in growth – growth where the rate of change itself is increasing or decreasing. Exponential notation is useful to describe very large and very small numbers.

Exponential growth

Exponents are useful for describing growth in many different real-life situations. But what exactly are exponents? The best way to start learning what exponents are is to see them in action. In activity 2.1 we will do just that.

Activity 2.1: Investigate exponential growth using a chessboard and grains of rice

Time required: 5 minutes

What you need:

- a chessboard (if you do not have a chessboard you can make your own 8×8 grid)
- grains of rice

What to do:

Use grains of rice and a chessboard to discover exponential growth.

You will place grains of rice on the board using patterns that follow either repeated multiplication or repeated addition.

Place one grain of rice on the first square, two grains on the second square, four grains on the third square, eight grains on the fourth square and continue adding rice to the board in this doubling (repeated multiplication by two) pattern.

On the other half of the board, place two grains of rice on the first square, four grains on the second square, six grains on the third square, eight grains on the fourth square and continue adding two more grains of rice to the board in this repeated addition of two pattern.

- How many grains of rice do you place on the sixth square when following repeated multiplication?
- How many grains of rice do you place on the twentieth square when following repeated addition?
- Which method (repeated multiplication or repeated addition) will reach at least 100 grains of rice more quickly? Why?
- Which method shows a faster rate of increase?

What did you find?

- Did the doubling grow more quickly than expected?
- Did you see that repeated multiplication results in rapid increases in growth?
- Would you be able to place the correct amount of rice on every square under repeated multiplication and repeated addition?
- If you were able to continue the pattern of doubling all the way up to the last square on the board and add the numbers together there would be over 18 quadrillion grains of rice on the board, which is around $1\ 000$ times more than the current annual global production of rice. The grains of rice would weigh over 460 billion tons, which would be a heap of rice larger than Mount Everest!

Take note!

The video called "Chess and exponents illustrates" the concept of exponential growth, which you have just investigated in the activity.

Chess and exponents (Duration: 2.01)

We often shorten the names of objects or people as it's quicker and easier to remember and say shorter names, for example:

- Maths instead of Mathematics
- CD instead of Compact Disc
- USB instead of Universal Serial Bus.

In Maths it is no different. We find quicker, easier ways to carry out operations by shortening their processes.

Multiplication is a short way to write **repeated addition**.

Would you rather write $2+2+2+2+2$ or 5×2?

How about $a+a+a+a+a$ or $5 \times a$?

Which expressions are quicker to write and easier to work with?

Hopefully you agree that multiplication is a much quicker and easier way to express repeated addition.

Just as multiplication is a short way to write repeated addition, similarly, exponents are a short way to write repeated multiplication.

Look at the following expressions. Which do you think would be easier to write and work with?

$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$ or 2^{10}?

Certainly 2^{10} is quicker to write and leaves less room for error.

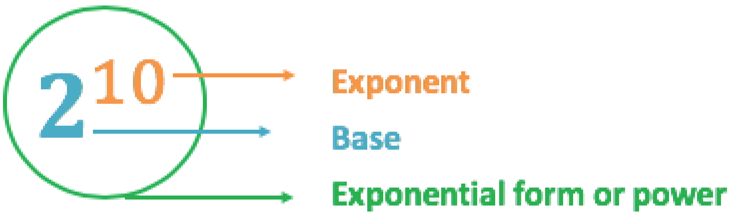
But how did we get from $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$ to 2^{10}?

Count the number of times that 2 has been multiplied by itself. What do you get?

You would have counted that 2 is multiplied 10 times. It is no coincidence that the number 'floating above' the 2, which is called the exponent, is also 10. Let's go over each part of 2^{10}.

Take note!

2^{10} is called **exponential form** or a power. The **base** is the number that is repeatedly multiplied. The **exponent**, which is sometimes also referred to as the 'index' or 'power', tells us how many times the base is multiplied by itself.



We read 2^{10} as 2 to the power of 10. This tells us that the base 2 has been multiplied by itself 10 times.

We can generalise this as:

$a^n = a \times a \times a \ldots \times a \text{ (n times) } (a \in \mathbb{R}\ a \neq 0, n \in \mathbb{N})$

To make sure you understand, work through the next two examples.

Example 2.1

Write $3 \times 3 \times 3 \times 3 \times 3$ in exponential form

Solution

Do you see that the base of 3 is multiplied 5 times?

Since this is repeated multiplication, writing the expression using exponential form will be a shorter way to rewrite it.

$3 \times 3 \times 3 \times 3 \times 3 = 3^5$

Example 2.2

Write 4^3 in expanded form

Solution

Here you have to write the expression without exponents. In other words, you need to show the repeated multiplication (expanded form) of the expression.

The base of 4^3 is 4 and the exponent is 3 so 4 has been multiplied by itself 3 times:

$4^3 = 4 \times 4 \times 4$

Now try this exercise on your own.

Exercise 2.1

1. Rewrite the following expressions in exponential form:
   a. $4 \times 4 \times 4 \times 4 \times 4$
   b. $2 \times 2 \times 2 \times 3 \times 3$
   c. $2 + 2 + 3 + 3 + 3$
2. Write the following in expanded form:
   a. 3^4
   b. $3^2 \times 2^3$
   c. $x^3 y^2$
   d. $(-2)^4$
3. Fill in $>$, $<$ or $=$ to make the statements true:
   a. 3^2___$2 \times 2 \times 2 \times 2$
   b. $3^2.2$___$2^2.3$
   c. $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$___$\left(\frac{1}{2}\right)^3$
4. Write the following numbers in exponential form:
   a. 25
   b. 729
   c. 48

   The full solutions are at the end of the unit.

Multiplying powers with the same base

Every number or variable (letter) that is not already in exponential form can be written with a power of 1.

3 can be written as 3^1

x can be written as x^1

$2xy$ can be written as $2^1x^1y^1$

You can think of the invisible exponent of 1 as a 'ghost' floating above the numbers and variables (figure 2).

Figure 2: 1 as an exponential ghost

The exponent of 1 is always present, even when it is not written. We will use this fact to show that $3 \times 3 \times 3 \times 3 = 3^4$.

We know that each 3 can be written as 3^1 so $3 \times 3 \times 3 \times 3 = 3^1 \times 3^1 \times 3^1 \times 3^1$. You have already seen that $3 \times 3 \times 3 \times 3$ means 3^4. So can you see what we have actually done to the exponents?

$3^1 \times 3^1 \times 3^1 \times 3^1 = 3^4 = 3^{1+1+1+1}$

We have just added the exponents together.

This is a very useful rule especially when we are working with a number of different expressions with the same bases being multiplied. We can apply this rule as long as the bases of the exponents are the same.

To simplify $x^2 \times x^4$ we could expand and rewrite as $x^1 \cdot x^1 \times x^1 \cdot x^1 \cdot x^1 \cdot x^1$ and then add up the exponents to get x^6. But a much quicker way to simplify $x^2 \times x^4$ is to keep the base x and add the exponent $x^2 \times x^4 = x^{2+4} = x^6$.

Now, try to simplify $x^{20} \times x^{12}$.

If you decided to first expand $x^{20} \times x^{12}$ it would take a long time to write out all those bases and it could lead to many errors. So, using a rule would not only make things easier but it would save time too. I'm sure you will agree that the quickest way to simplify $x^{20} \times x^{12}$ is to keep the base x and add the exponents $x^{20} \times x^{12} = x^{20+12} = x^{32}$.

Example 2.3

Write $3^2 \times 3^5$ with a single base of 3.

Solution

To write $3^2 \times 3^5$ with a single base of 3 you could first write each base of 3 in expanded form and then count the total number of threes being multiplied together.

$3^2 \times 3^5 = 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 = 3^7$

But, a quicker way to simplify is to keep a single base of 3 and add the exponents to get $3^2 \times 3^5 = 3^{2+5} = 3^7$.

Take note!

What happens when the bases are different? Say, for example, we had $3^2 \times 2^3$. Can we add the exponents together now? If you answered yes, then which base would get to keep the exponents?

Example 2.4

In $3^2 \times 2^3$ we are working with different bases of 3 and 2, so adding the powers together will not work.

Solution

$3^2 \times 2^3 = 9 \times 8 = 72$

So, whatever rule you use it must get you back to this answer.

There are three different ways you may have tried to simplify the expression.

1. Let's say you kept the base of 3 and added the exponents to get $3^{2+3} \times 2$ what will the answer be? $3^{2+3} \times 2 = 3^5 \times 2 = 486$. This is not equal to 72 so it is **not correct** to simplify $3^2 \times 2^3$ by keeping the base 3.
2. You may have decided to keep the base of 2 and add exponents to get 3×2^5. The answer to 3×2^5 is 96. Once again, this is not equal to 72 so it is **not correct** to simplify $3^2 \times 2^3$ by adding the exponents to base 2.
3. Lastly, you may have done something like this: $(3 \times 2)^{2+3} = (3 \times 2)^5$. If you calculate the answer to $(3 \times 2)^5$ you will get $7\ 776$. This is very far from 72! So it is **not correct** to simplify $3^2 \times 2^3$ by multiplying the bases together and adding the powers.

In fact, $3^2 \times 2^3$ cannot be simplified any further using any exponent rules BUT you can arrive at a rule from this example.

Powers with different bases cannot be combined using any exponent rules.

This leads us to some of the other common mistakes that must be avoided when working with exponents with the same base.

Take note!

$2^2 \times 2^5 \neq 2^{10}$

$3^3 \times 3^4 \neq 9^7$

$2^2 + 2^3 \neq 2^5$

Exercise 2.2

1. Simplify the following where possible:
   a. $3^2 \times 3^4$
   b. $2^3 \times 5^3$
   c. $x^2y^3 \times 2x^3y$
   d. $a^2 \times b^3 \times y^2$
   e. $\left(-x^2y^4\right)\left(-3yx^3\right)$
   f. $\left(-2a^3b^2\right)\left(-bxa\right)\left(a^2x^3b\right)$
   g. $2^{2x} \times 2^{3x}$
   h. $3^{3a} \times 3^{2a+b}$

The full solutions are at the end of the unit.

Dividing powers with the same base

By now, you know that division is the reverse operation of multiplication. So, if we add exponents when multiplying with the same bases, what do you think we do to the exponents when we divide the same bases?

Simplify $\frac{a^5}{a^2}$.

We can rewrite the numerator as $a^5 = a \times a \times a \times a \times a$.

We can rewrite the denominator as $a^2 = a \times a$.

Then $\frac{a^5}{a^2} = \frac{a \times a \times a \times a \times a}{a \times a}$.

Using simple division and cancelling out we get $\frac{a^5}{a^2} = \frac{\times \times a \times a \times a}{\times} = \frac{a \times a \times a}{1}$.

By multiplying powers with the same base we get $\frac{a^5}{a^2} = a^1 \times a^1 \times a^1 = a^3$.

Without expanding the powers, is there a quick way to get from $\frac{a^5}{a^2}$ to a^3?

Hopefully you can see that the quick way to get from $\frac{a^5}{a^2}$ to a^3 is to subtract the exponents with the same base $\frac{a^5}{a^2} = a^{5-2} = a^3$. We keep the base '$a$' and subtract the exponent 2 from 5.

Take note!

When we divide powers with the same base, we keep the base and subtract the exponents. This can be generalised as:

$$\frac{a^m}{a^n} = a^{m-n}$$

Example 2.5

Simplify: $\frac{2^6}{2^4}$ and $\frac{2x^3y^4}{y^2}$

Solution

Let's start with $\frac{2^6}{2^4}$. The base is the same in the numerator and denominator therefore we can apply the rule for dividing exponents with the same base. We will keep the base of 2 in the numerator and subtract the powers:

$$\frac{2^6}{2^4} = 2^{6-2} = 2^4 = 16.$$

NOTE: If the final answer is easy to work out without a calculator, then write it out without exponents.

With the expression $\frac{2x^3y^4}{y^2}$, we see more than one base. You must remember the same rule that we arrived at when multiplying powers; you can only apply the rules for exponents to powers with the same base.

We see that there is an exponent with base y in both the numerator and denominator so those are the powers we can simplify using the rule.

$$\frac{2x^3y^4}{y^2} = 2x^3y^{4-2}$$

Keep the 2 and x^3 as is without changing their exponents.

$$\frac{2x^3y^4}{y^2} = 2x^3y^{4-2} = 2x^3y^2$$

Exercise 2.3

1. Simplify the following where possible:
   a. $\frac{6a^3b}{a}$
   b. $\frac{3^2}{2^4}$
   c. $\frac{xy^2 \times 2x^2y}{3xy}$

The full solutions are at the end of the unit.

There is an important identity that we use often in exponents. The identity states that any base raised to the power of zero is equal to one or $a^0 = 1$. We can easily arrive at this identity using the rule for dividing exponents with the same base, as shown in the example below.

Example 2.6

Use the rule for dividing exponents with the same base to prove that $a^0 = 1$.

Solution

We know that any number or variable, other than 0, divided by itself is equal to 1.

$\frac{2}{2} = 1$; $\frac{3}{3} = 1$; $\frac{a}{a} = 1$

Using the exponent law for dividing:

$$\frac{2}{2} = 2^{1-1} = 2^0 = 1$$
$$\frac{3}{3} = 3^{1-1} = 3^0 = 1$$
$$\frac{a}{a} = a^{1-1} = a^0 = 1$$

$$\frac{2}{2} = 2^{1-1} = 2^0 = 1$$

$$\frac{3}{3} = 3^{1-1} = 3^0 = 1$$

$$\frac{a}{a} = a^{1-1} = a^0 = 1$$

So we can see that an exponent of 0 means that the base has been divided by itself and will be equal to 1.

Therefore, any base raised to the power of zero is equal to one or $a^0 = 1$.

So far, we have worked with powers where the exponent in the numerator is greater than the exponent in the denominator but what happens in a case like this $\frac{3^3}{3^5}$?

Example 2.7

Simplify: $\dfrac{3^3}{3^5}$

Solution

We see the bases are the same and that the exponent in the denominator is greater than the exponent in the numerator: $5 > 3$.

Using basic expansion, we see that $\dfrac{3^3}{3^5} = \dfrac{3 \times 3 \times 3}{3 \times 3 \times 3 \times 3 \times 3} = \dfrac{1}{3^2}$

Using the law for dividing exponents, keep the base of 3 in the numerator and subtract the powers.

$\dfrac{3^3}{3^5} = 3^{3-5}$

Can you see what happens to the sign of the exponent now?

$3^{3-5} = 3^{-2}$

You have already seen by expansion that $\dfrac{3^3}{3^5} = \dfrac{1}{3^2}$ but we now also know that $3^{3-5} = 3^{-2}$. So we can conclude that $3^{-2} = \dfrac{1}{3^2}$.

To write 3^{-2} with a positive exponent you simply find the positive reciprocal. Remember that a reciprocal is a fraction where the numerator and denominator switch places.

So 3^{-2} can be written as $\dfrac{3^{-2}}{1}$ and its positive reciprocal will be $\dfrac{1}{3^{+2}}$. You will notice that the sign of the exponent has changed when its position changed.

$3^{-2} = \dfrac{1}{3^2} = \dfrac{1}{9}$.

This example leads us to an important concept – negative exponents. Negative exponents produce fractions. You will use the following identity when dealing with negative exponents.

$$a^{-m} = \frac{1}{a^{m}}$$

Similarly:

$$\frac{1}{a^{-m}} = a^{m}$$

The table below will help you understand how to change from negative to positive exponents.

Negative exponent	Rewrite as positive exponent	Answer
3^{-2}	$3^{-2} = \frac{1}{3^{+2}}$	$3^{-2} = \frac{1}{3^{+2}} = \frac{1}{9}$
$(2x)^{-2}$	$(2x)^{-2} = \frac{1}{(2x)^2}$	$(2x)^{-2} = \frac{1}{(2x)^2} = \frac{1}{4x^2}$
$\frac{1}{2^{-3}}$	$\frac{1}{2^{-3}} = 2^{+3}$	$\frac{1}{2^{-3}} = 2^{+3} = 8$
$\frac{2}{x^{-3}}$	$\frac{2}{x^{-3}} = 2x^{+3}$	$\frac{2}{x^{-3}} = \frac{2}{1} \times \frac{1}{x^{-3}}$ $= 2 \times x^{+3}$ $= 2x^3$

We can also use patterns to visually understand what negative exponents represent. Have a look at figure 3 and the explanation in the 'Did you know?' box.

Did you know?

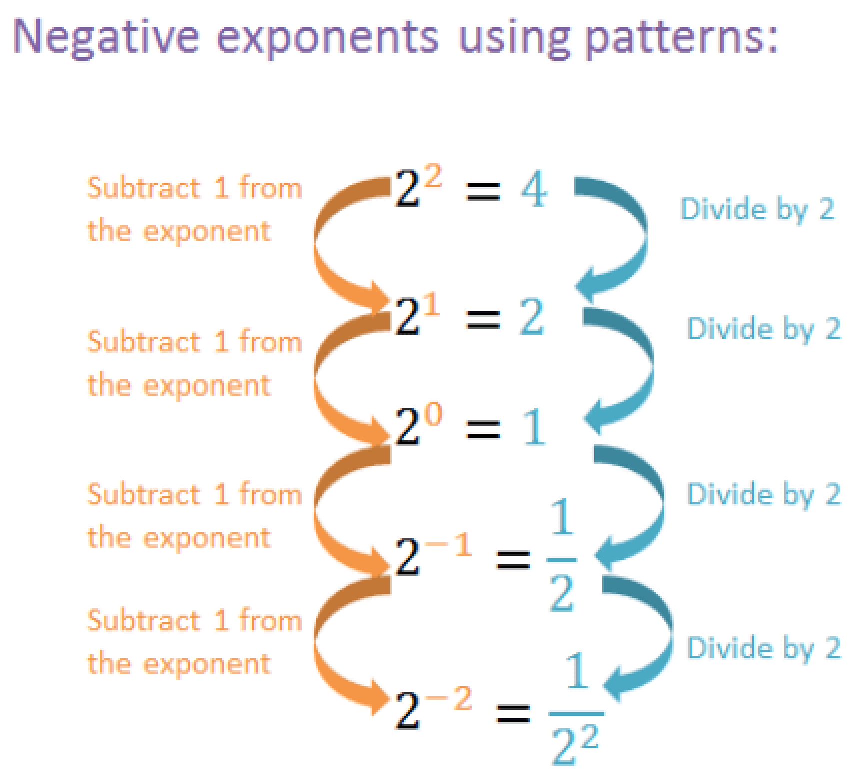


Figure 3: Negative exponents using patterns

If we start at 2^0 and move up the left hand side of the diagram we add 1 to the exponent of 0 and get 2^1. This is the same as $2^1 = 1 \times 2 = 2$. We multiplied by 2 to move up the diagram on the right hand side. Similarly, if we add to the exponent of 2^1 we get 2^2, this is the same as $2^2 = 1 \times 2 \times 2 = 4$. So as we move up the diagram, we add exponents and the answers get larger.

Now, start at 2^2 and move down the left hand side of the diagram by subtracting 1 from the exponent. You will see that each time we decrease the exponent by 1 we divide the previous answer on the right hand side of the diagram by 2. So as we move down the diagram, we subtract exponents and the answers get smaller.

When we get to 2^0 and subtract one from the exponent we will get 2^{-1} so to work out the answer for 2^{-1} we divide the previous answer on the right $2^0 = 1$ by 2 and get that $2^{-1} = \frac{1}{2}$. Similarly, for 2^{-2} we divide the previous answer $2^{-1} = \frac{1}{2}$ by 2 and get $2^{-2} = 1 \div 2 \div 2 = \frac{1}{2^2}$.

Hopefully, you see that positive, zero and negative exponents form a simple pattern as you move up and down the diagram.

Exercise 2.4

1. Simplify without using a calculator and write answers with positive exponents:
   a. $3^{12} \div 3^9$
   b. $\dfrac{15x^{12}y^5}{12x^9y^3}$
   c. $\dfrac{2^{x+3}}{2^{2+x}}$
   d. $\dfrac{3}{2^{-2}}$
   e. $\dfrac{a^2x^{-2}}{a^{-1}x^3}$

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to use exponential form to write a number as a^n where n is any natural number and a is any real number $\neq 0$.
- The law for multiplying exponents with the same base $a^m \times a^n = a^{m+n}$.
- The law for dividing exponents with the same base $\dfrac{a^m}{a^n} = a^{m-n}$.
- The identity $a^0 = 1$.
- The identity for negative exponents: $a^{-m} = \dfrac{1}{a^m}$; $\dfrac{1}{a^{-m}} = a^m$.

Unit 2: Assessment

Suggested time to complete: 10 minutes

1. Simplify as far as possible and write your answers with positive exponents:
   a. $(3^{-2}a^{-6})^0 \times 3^{-2}$
   b. $-8x^2y^3 \div 6x^3y$
   c. $\dfrac{(2x^2y)(-4xy^2)}{-2xy}$
   d. $\dfrac{2^{a-2}3^{a+3}}{2^{-2+a}}$
2. Andy and Alex find different answers to the same question. Look at each method below and state who is correct. Give a reason for your answer.

   Andy's solution

$3 \times 2^2 + 5$
$= 6^2 + 5$ ·
$= 41$

Alex's solution

$3 \times 2^2 + 5$
$= 3 \times 4 + 5$
$= 17$

The full solutions are at the end of the unit.

Note

Think about what you learnt in this unit. Can you:

- Rewrite expressions with repeated addition in exponential form?
- Expand expressions written in exponential form?
- Multiply exponents with the same base?
- Divide exponents with the same base?

Unit 2: Solutions

Exercise 2.1

1. a. $4 \times 4 \times 4 \times 4 \times 4 = 4^5$. The base 4 has been multiplied 5 times so the exponent is 5.
   b. You need to be careful with this expression since there is a combination of numbers being multiplied.
   $2 \times 2 \times 2 = 2^3$ and $3 \times 3 = 3^2$. So $2 \times 2 \times 2 \times 3 \times 3 = 2^3 \times 3^2$.
   c. Did you notice that the numbers are being added in this question so you cannot use exponents? $2 + 2 + 3 + 3 + 3 = 2 \times 2 + 3 \times 3$. Use multiplication to write the expression.
2. a. $3^4 = 3 \times 3 \times 3 \times 3$. The base 3 has been multiplied 4 times.
   b. $3^2 \times 2^3 = 3 \times 3 \times 2 \times 2 \times 2$. There are two different bases so each base must be treated separately.
   c. $x^3y^2 = x \times x \times x \times y \times y$. The variables (letters that represent unknown numbers) must be multiplied separately, the x is multiplied 3 times and y is multiplied twice.
   d. $(-2)^4 = (-2) \times (-2) \times (-2) \times (-2) = 16$. Use brackets to help you multiply negative numbers.
3. a.
   $$\begin{aligned} 3^2 &= 9 \\ 2 \times 2 \times 2 \times 2 &= 16 \\ 9 &< 16 \\ \therefore 3^2 &< 2 \times 2 \times 2 \times 2 \end{aligned}$$
   b.
   $$\begin{aligned} 3^2.2 &= 9 \times 2 = 18 \\ 2^2.3 &= 4 \times 3 = 12 \\ 18 &> 12 \end{aligned}$$

$\therefore 3^2.2 > 2^2.3$

c. $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \left(\frac{1}{2}\right)^3$. The base of $\frac{1}{2}$ is multiplied 3 times hence the expressions are equal.

4. a. $25 = 5^2$

   b. $729 = 3^6$

   c. $48 = 2^4 \times 3$

Back to exercise 2.1

Exercise 2.2

1. a.

$$\begin{aligned} 3^2 \times 3^4 &= 3^{2+4} \\ &= 3^7 \end{aligned}$$

b.

$$\begin{aligned} 2^3 \times 5^3 &= (2 \times 5)^3 \\ &= 10^3 \\ &= 1000 \end{aligned}$$

c.

$$\begin{aligned} x^2y^3 \times 2x^3y &= 2x^{2+3}y^{3+1} \\ &= 2x^5y^4 \qquad \text{The exponents with the same bases are added and the coefficient of 2 stays the same.} \end{aligned}$$

d.

$$a^2 \times b^3 \times y^2 = a^2b^3y^2 \qquad \text{This cannot be simplified because all the bases are different. All that has been done is to rewrite the expression without the explicit multiplication signs.}$$

e.

$$\begin{aligned} \left(-x^2y^4\right)\left(-3yx^3\right) &= 3x^{2+3}y^{4+1} \\ &= 3x^5y^5 \qquad \text{Be careful with the signs.} \end{aligned}$$

f.

$$\begin{aligned} &\left(-2a^3b^2\right)(-bxa)\left(a^2x^3b\right) \\ &= 2a^{3+1+2}b^{2+1+1}x^{1+3} \\ &= 2a^6b^4x^4 \qquad \text{The order of the variables does not matter. Just make sure to allocate the exponents correctly.} \end{aligned}$$

g.

$$\begin{aligned} 2^{2x} \times 2^{3x} &= 2^{2x+3x} \\ &= 2^{5x} \qquad \text{Even though variables appear as exponents, the same rule applies. Keep the base and add the exponents when multiplying powers with the same base.} \end{aligned}$$

h.

$$\begin{aligned} 3^{3a} \times 3^{2a+b} &= 3^{3a+2a+b} \\ &= 3^{5a+b} \qquad \text{The rules of algebra apply here, you cannot add unlike variables.} \end{aligned}$$

Back to exercise 2.2

Exercise 2.3

1. a.

$$\begin{aligned} \frac{6a^3b}{a} &= 6a^{3-1}b \\ &= 6a^2b \end{aligned}$$

b.

$$\begin{aligned} &\frac{3^2}{2^4} \qquad \text{Bases are different so you cannot simplify any further using exponent rules.} \\ &= \frac{9}{16} \qquad \text{You can write answer as a common fraction.} \end{aligned}$$

c.

$$\begin{aligned}\frac{xy^2 \times 2x^2y}{3xy} &= \frac{2x^{1+2}y^{2+1}}{3xy} \\ &= \frac{2x^3y^3}{3xy} \\ &= \frac{2x^{3-1}y^{3-1}}{3} \\ &= \frac{2}{3}x^2y^2\end{aligned}$$

Back to exercise 2.3

Exercise 2.4

1. a.

$$\begin{aligned}3^{12} \div 3^9 &= 3^{12-9} \\ &= 3^3 \\ &= 27\end{aligned}$$

b.

$$\begin{aligned}\frac{15x^{12}y^5}{12x^9y^3} &= \frac{5x^{12-9}y^{5-3}}{4} \\ &= \frac{5x^3y^2}{4}\end{aligned}$$

c.

$$\begin{aligned}\frac{2^{x+3}}{2^{2+x}} &= 2^{x+3-(2+x)} \\ &= 2^{x+3-2-x} \\ &= 2^{x+3-2-x} \\ &= 2^1 \\ &= 2\end{aligned}$$

Note: apply the same rule even though there are variables in the exponent.

Use brackets when subtracting the exponent so you remember to change the signs.

d.

$$\begin{aligned}\frac{3}{2^{-2}} &= 3 \cdot 2^2 \\ &= 12\end{aligned}$$

The sign of the exponent must change to a positive when you move it to the numerator.

e.

$$\begin{aligned}\frac{a^2x^{-2}}{a^{-1}x^3} &= \frac{a^{2-(-1)}x^{-2-(3)}}{1} \\ &= a^{2+1}x^{-2-(3)} \\ &= a^3x^{-5} \\ &= \frac{a^3}{x^5}\end{aligned}$$

Back to exercise 2.4

Unit 2: Assessment

1. a.

$$\begin{aligned}\left(3^{-2}a^{-6}\right)^0 \times 3^{-2} &= 1 \times \frac{1}{3^2} \\ &= \frac{1}{9}\end{aligned}$$

Remember: $a^0 = 1$ and $a^{-m} = \frac{1}{a^m}$

b.

$$\begin{aligned}-8x^2y^3 \div 6x^3y &= \frac{-8x^2y^3}{6x^3y} \\ &= \frac{-8x^{2-3}y^{3-1}}{6} \\ &= \frac{-4xy^2}{3}\end{aligned}$$

c.

$$\begin{aligned}\frac{(2x^2y)(-4xy^2)}{-2xy} &= \frac{(x^2y)(-4xy^2)}{-xy} \\ &= \frac{-4x^{2+1}y^{1+2}}{-xy} \\ &= 4x^{2+1-1}y^{1+2-1} \\ &= 4x^2y^2\end{aligned}$$

d.

$$\begin{aligned}\frac{2^{a-2}3^{a+3}}{2^{-2+a}} &= 2^{a-2-(-2+a)}3^{a+3} \\ &= 2^0 3^{a+3} \\ &= 3^{a+3}\end{aligned}$$

2. Alex is correct. Andy made the mistake of multiplying different bases together.

Back to the assessment

Media Attributions

Unit 3: Raising a power to a power, exponent laws

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Unit outcomes

By the end of this unit you will be able to:

- Raise a power to a power.
- Simplify exponential expressions by applying the exponent laws.

What you should know

Before you start this unit, make sure you can:

- Do all the work in Unit 2: Introduction to exponents

Here is a short self-assessment to make sure you have the skills you need to proceed with this unit.

Simplify:

1. $5x^0 \times \dfrac{2x}{x^2}$
2. $2^{2x}.3^x.2^{x-1}$

Solutions:

1.
$$\begin{aligned} 5x^0 \times \frac{2x}{x^2} &= 5(1) \times 2x^{1-2} \\ &= 10x^{-1} \\ &= \frac{10}{x} \end{aligned}$$
2.
$$\begin{aligned} 2^{2x}.3^x.2^{x-1} &= 2^{2x+x-1} \cdot 3^x \\ &= 2^{3x-1}3^x \end{aligned}$$

Introduction

So far we have worked with the product and quotient exponent rules and learnt the basic identities. You have also seen that exponents are used as a quick way to simplify expressions with repeated multiplication. By now you should be familiar with the following exponent rules and identities.

Take note!

Exponent rules and identities	Example
$a^n = a \times a \times a \ldots \times a\ [\text{n times}]$	$4 \times 4 \times 4 \times 4 \times 4 = 4^5$
$a^m \times a^n = a^{m+n}$	$x^2y^3 \times x^3y = x^5y^4$
$\frac{a^m}{a^n} = a^{m-n}$	$\frac{2^5}{2^2} = 2^3 = 8$
$a^0 = 1$	$(3^{-2}a^{-6})^0 = 1$
$a^{-m} = \frac{1}{a^m}\ ;\ \frac{1}{a^{-m}} = a^m$	$\frac{1}{x^{-2}} = x^2$

Raising a power to a power

You may have seen a base with an exponent raised to another exponent. This is called raising a power to a power. Look at $(2^2)^3$. This is an example of a power raised to another power. Think about what the expression means.

It helps to expand the brackets and then write the answer in exponential form.

From the exponential form, we can work out a rule for raising a power to a power. HINT: Look at the exponents you started with and the ones you end up with. What is the relationship between the two?

Let's work through this together:

In the expression $(2^2)^3$we see that the power 2^2 is raised to an exponent of three outside the bracket. This means that 2^2 is multiplied by itself three times. If we expand $(2^2)^3$ we get: $(2^2)^3 = 2^2 \times 2^2 \times 2^2$. By using the rule for multiplying powers with the same base and adding the exponents this becomes $2^2 \times 2^2 \times 2^2 = 2^{2+2+2} = 2^6$.

Is there a quick way to get from $(2^2)^3$ to 2^6? Yes.

We can see that the exponent six in the answer is simply the product of the exponents in the given expression so $(2^2)^3 = 2^{2\times3} = 2^6$.

When we raise a power to a power we multiply the exponents.

This is generalised as the following exponent law.

$(a^m)^n = a^{m\times n}$

This is called the exponent power rule.

Example 3.1

Simplify: $(x^2y^3)^3$

Solution

If we expand the expression, we get $(x^2y^3)^3 = (x^2y^3) \times (x^2y^3) \times (x^2y^3) = x^2 \times x^2 \times x^2 \times y^3 \times y^3 \times y^3$.

We can apply the product rule to multiply powers with the same base to get x^6y^9.

We can see that the exponent on each base is the product of the exponents from the original expression.

$(x^2y^3)^3 = x^{2\times 3} \times y^{3\times 3} = x^6y^9$

We conclude that when different bases multiplied within a bracket are raised to a power, each base must be raised to the power individually.

The above example is generalised as the following exponent law.

$(a^m b^n)^p = a^{mp} \cdot b^{np}$

This is called the power of a product rule.

Now, try to simplify $\left(\frac{a}{b}\right)^4$.

Example 3.2

Simplify: $\left(\frac{a}{b}\right)^4$

Solution

If we expand the expression, we get $\left(\frac{a}{b}\right)^4 = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} = \frac{a \times a \times a \times a}{b \times b \times b \times b}$.

We can write this in exponential form as $\frac{a^4}{b^4}$.

We can see that the exponent of each base is the product of the exponents from the original expression.

$\left(\frac{a}{b}\right)^4 = \left(\frac{a^1}{b^1}\right)^4 = \frac{a^4}{b^4}$

We can conclude that when different bases divided within a bracket are raised to a power, each base must be raised to the power individually.

The above example is generalised as the following exponent law.

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

This is called the power of a fraction rule.

The basic exponent laws can be applied to many different examples. Look at the examples below and see how each of them differs from the basic case each time.

Example 3.3

Simplify:

1. $2(a^2b)^3$
2. $(2y^2)^4$
3. $3(x^3)^{-2}$

Solutions

1.

$2(a^2b)^3$ The number 2 is not raised to the power of 3 since it is not inside the brackets.

$= 2a^6b^3$

2.

$(2y^2)^4$ Here the coefficient (number in front of a variable) 2 is also raised to the exponent 4 because it is inside the brackets.

$= 2^4y^8 = 16y^8$

3.

$3(x^3)^{-2}$ The coefficient is not raised to the exponent -2.

$= 3x^{3(-2)}$ Note that x^3 is raised to a negative exponent, be careful with the signs.

$= 3x^{-6}$

$= \frac{3}{x^6}$ Answers are generally left with positive exponents, so make the exponent positive by writing x^{-6} as $\frac{1}{x^6}$.

Exercise 3.1

Simplify each expression as far as possible and write answers with positive exponents:

1. $(xy)^{-2}$

2. $\left(-3xy^2\right)^2\left(-3x\right)^{-2}$

3. $\left(\dfrac{a}{b^{-2}c}\right)^3$

4. $2\left(-a^2b^4\right)^2\left(-4a^3b^6\right)\left(-16a^6b^8\right)^{-1}$

5. $\left(2^{-2}\right)^{2p+1}$

6. $\left(\dfrac{3q^{2x}}{q^{3x}y^{-2a-1}}\right)^2$

The full solutions are at the end of the unit.

Note

Do you need more practise raising a power to a power? Try this online activity called Powers of powers. You will need an internet connection.

Prime factorising and exponents

The word 'prime' in 'prime factorising' tells us that this type of factorising has something to do with prime numbers. Remember that a prime number is a number that has only two factors; the number one and itself. What are some examples of prime numbers?

The numbers $2; 3; 5; 17$ are examples of prime numbers. Prime factorisation is the process of breaking a number up into its prime factors.

For example, 12 broken down into its prime factors is $2 \times 2 \times 3$. Did you notice that there is repeated multiplication? That means we can use exponents to simplify the expression further: $2^2 \cdot 3$. This is a method you will often rely on to simplify powers.

Did you know?

Is 1 a prime number? Watch the video called "1 and Prime Numbers" about prime factorisation. It explains why 1 is not a prime number using the Fundamental Theorem of Arithmetic. It also explains more about prime numbers.

1 and Prime Numbers (Duration: 5.21)

Did you know?

You can use your calculator to prime factorise any number. The video called "How to do Prime Factorisation on a Casio FX-83GT PLUS" shows the calculator steps.

How to do Prime Factorisation on a Casio FX-83GT PLUS (Duration: 1.49)

Have a look at this next example of prime factorisation.

Example 3.4

Use prime factorisation to simplify: $(18)^{2x}$

Solution

Start by breaking 18 up into its prime factors.

$18 = 9 \times 2 = 3^2 \times 2$

So $18^{2x} = (3^2 \times 2)^{2x}$

Next, raise each of the prime factors to the power of $2x$.

$(3^2 \times 2)^{2x} = 3^{2\times 2x} \times 2^{1\times 2x}$

$3^{2\times 2x} \times 2^{1\times 2x} = 3^{4x} \cdot 2^{2x}$

Hence $18^{2x} = 3^{4x} \cdot 2^{2x}$

Take note!

You can use a calculator to find the prime factors of any number. Here are the steps for the Casio fx-82ZA PLUS calculator, shown below. If you have a different calculator check your manual for the steps.

To find the prime factors of 72:

Step 1: Enter 72 and press $=$

Step 2: Press SHIFT and FACT (the key that shows degrees and minutes °'")

The calculator will display the prime factors as $2^2 \times 3^2$

Use your calculator to factorise: $56; 112; 725$

When working with exponents, all the laws of operation for algebra apply.

Exercise 3.2

Simplify as far as possible:

1. $3^n \cdot (9)^{2n}$
2. $\dfrac{4^{2x}}{2^{3x}}$
3. $(8^n)^2$
4. $25^x \cdot 5$

The full solutions are at the end of the unit

Summary

In this unit you have learnt the following:

- The power rule.
- The power of a product rule.
- The power of a fraction rule.

  $(a^m)^n = a^{m \times n}; (a^m b^n)^p = a^{mp} \cdot b^{np}; \left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$
- How to raise a power to a negative exponent.
- How to use prime factorisation to simplify powers.

Unit 3: Assessment

Suggested time to complete: 20 minutes

1. Arrange the following from least to greatest:
   $2^3 - 2^{-1}; (2)^3 - (2)^2; 2^2 3^0 \div 2$
2. Calculate the following without using a calculator:
   a. $2^2 \times 3 \times 2^{-1}$
   b. $9^{-1} \times (3)^2 \times 2^{-2}$
   c. $6^a \times \dfrac{(2ab^4)^0}{2^a 3^a}$
3. Simplify:
   a. $(x^2)^{-2} \times 2(x^3)^2$
   b. $\dfrac{9(a^2)^{-2}}{3a^{-5}}$
   c. $\dfrac{(2^{x+2}3^x)^2}{6^x}$
   d. $\dfrac{3^n 9^{n-3}}{27^{n-1}}$

The full solutions are at the end of the unit.

Unit 3: Solutions

Exercise 3.1

1.
$$(xy)^{-2} = \frac{1}{(xy)^2} = \frac{1}{x^2y^2}$$

2.
$$\begin{aligned}
&(-3xy^2)^2(-3x)^{-2} && \text{Move } (-3x)^{-2} \text{ to the denominator to make the exponent positive.}\\
&= \frac{(-3xy^2)^2}{(-3x)^{+2}} && \text{Make the exponent of the second bracket positive.}\\
&= \frac{(-3)^2x^2y^4}{(-3)^2x^2} && \text{Apply the power of a product rule. The 3 must also be raised to the exponent of 2.}\\
&= \frac{9x^2y^4}{9x^2} = y^4
\end{aligned}$$

3.

$$\left(\frac{a}{b^{-2}c}\right)^3$$

$$= \left(\frac{ab^2}{c}\right)^3$$ Make the negative exponent positive by moving the power to the numerator.

$$= \frac{a^3b^6}{c^3}$$

4.

$$2\left(-a^2b^4\right)^2\left(-4a^3b^6\right)\left(-16a^6b^8\right)^{-1}$$

$$= \frac{2\left(-a^2b^4\right)^2\left(-4a^3b^6\right)}{\left(-16a^6b^8\right)^{+1}}$$ Move the bracket with the negative exponent to the denominator.

$$= \frac{2(a^4b^8)(-4a^3b^6)}{-16a^6b^8}$$ Raise the first bracket to the power of 2 before removing brackets and simplifying further.

$$= \frac{-8a^7b^{14}}{-16a^6b^8}$$

$$= \frac{-8a^{7-6}b^{14-8}}{-16}$$

$$= \frac{ab^6}{2}$$

5.

$$\left(2^{-2}\right)^{2p+1}$$

$$= \left(\frac{1}{2^2}\right)^{2p+1}$$ Variables are also multiplied if they form part of the exponent.

$$= \frac{1}{2^{4p+2}}$$

6.

$$\left(\frac{3q^{2x}}{q^{3x}y^{-2a-1}}\right)^2$$

$$= \left(3q^{2x-3x}y^{2a+1}\right)^2$$

$$= 3^2q^{-2x}y^{4a+2}$$ Each part of each exponents must be raised to the power of 2.

$$= \frac{9y^{4a+2}}{q^{2x}}$$

Back to exercise 3.1

Exercise 3.2

1.

$$3^n \cdot (9)^{2n}$$

$$= 3^n \cdot \left(3^2\right)^{2n}$$ Prime factorise 9.

$$= 3^n \cdot 3^{4n}$$

$$= 3^{5n}$$

2.

$$\frac{4^{2x}}{2^{3x}}$$
$$= \frac{(2^2)^{2x}}{2^{3x}}$$
$$= \frac{2^{4x}}{2^{3x}}$$
$$= 2^{4x-3x}$$
$$= 2^x$$

3.
$$(8^n)^2$$
$$= ((2^3)^n)^2 \qquad \text{Work from within the brackets and prime factorise 8 to } 2^3$$
$$= (2^{3n})^2$$
$$= 2^{6n}$$

4.
$$25^x \cdot 5$$
$$= (5^2)^x \cdot 5$$
$$= 5^{2x} \cdot 5^1$$
$$= 5^{2x+1}$$

Back to exercise 3.2

Unit 3: Assessment

1.
$$2^3 - 2^{-1} = 8 - \frac{1}{2} = 7\frac{1}{2}$$
$$(2)^3 - (2)^2 = 8 - 4 = 4$$
$$2^2 3^0 \div 2 = \frac{4 \times 1}{2} = 2$$
From least to greatest: $2^2 3^0 \div 2; (2)^3 - (2)^2; 2^3 - 2^{-1}$

2.

a.
$$2^2 \times 3 \times 2^{-1}$$
$$= 2^{2-1} \cdot 3$$
$$= 6$$

b.
$$9^{-1} \times (3)^2 \times 2^{-2}$$
$$= \frac{(3^2)^{-1} \times (3)^2}{2^2}$$
$$= \frac{3^{-2+2}}{2^2}$$
$$= \frac{3^0}{4} = \frac{1}{4}$$

c.

$$6^a \times \frac{(2ab^4)^0}{2^a 3^a}$$
$$= (3 \cdot 2)^a \times \frac{1}{2^a 3^a}$$
$$= \frac{3^a 2^a}{2^a 3^a}$$
$$= 1$$

3.

a.

$$(x^2)^{-2} \times 2(x^3)^2$$
$$= x^{-4} \times 2x^6$$
$$= 2x^{-4+6}$$
$$= 2x^2$$

b.

$$\frac{9(a^2)^{-2}}{3a^{-5}}$$
$$= \frac{9a^{-4}}{3a^{-5}}$$
$$= 3a^{-4-(-5)}$$
$$= 3a$$

c.

$$\frac{(2^{x+2}3^x)^2}{6^x}$$
$$= \frac{2^{2(x+2)}3^{2x}}{(2 \cdot 3)^x}$$
$$= \frac{2^{2x+4}3^{2x}}{2^x 3^x}$$
$$= 2^{2x+4-x}3^{2x-x}$$
$$= 2^{x+4}3^x$$

d.

$$\frac{3^n 9^{n-3}}{27^{n-1}}$$
$$= \frac{3^n (3^2)^{n-3}}{(3^3)^{n-1}}$$
$$= \frac{3^n 3^{2n-6}}{3^{3n-3}}$$
$$= 3^{n+2n-6-(3n-3)}$$
$$= 3^{3n-6-3n+3}$$
$$= 3^{-3} = \frac{1}{3^3} = \frac{1}{27}$$

Back to the assessment

Unit 4: Rational exponents

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Unit outcomes

By the end of this unit you will be able to:

- Write roots as rational exponents.

What you should know

Before you start this unit, make sure you can:

- Do all the work in Unit 2: Introduction to exponents
- Do all the work in Unit 3: Raising a power to a power, exponential laws

Here is a short self-assessment to make sure you have the skills you need to proceed with this unit.

Simplify:

1. $2^2 \times 3 \times 2^{-1}$
2. $6^a \times \dfrac{(2ab^4)^0}{2^a 3^a}$
3. $\dfrac{3^n 9^{n-3}}{27^{n-1}}$

Solutions

1. $$\begin{aligned}&2^2 \times 3 \times 2^{-1}\\ &= 2^{2-1} \cdot 3\\ &= 6\end{aligned}$$
2. $$\begin{aligned}&6^a \times \frac{(2ab^4)^0}{2^a 3^a}\\ &= (3 \cdot 2)^a \times \frac{1}{2^a 3^a}\\ &= \frac{3^a 2^a}{2^a 3^a}\\ &= 1\end{aligned}$$
3.

$$\begin{aligned}
&\frac{3^n 9^{n-3}}{27^{n-1}}\\
&= \frac{3^n\left(3^2\right)^{n-3}}{\left(3^3\right)^{n-1}}\\
&= \frac{3^n 3^{2n-6}}{3^{3n-3}}\\
&= 3^{n+2n-6-(3n-3)}\\
&= 3^{3n-6-3n+3}\\
&= 3^{-3} = \frac{1}{3^3} = \frac{1}{27}
\end{aligned}$$

Introduction

So far, we have worked with exponents that are integers (positive and negative whole numbers) and by now you know that the exponent shows how many times the base is multiplied by itself. However, exponents can be any rational number, which means they can be fractions too. So, what do fractional exponents represent?

We know that $2^3 = 2 \times 2 \times 2$. What do you think $8^{\frac{1}{3}}$ is equal to?

What about $9^{\frac{1}{2}}$? What do you think it means?

Use what you already know about exponents to simplify these powers with fractional exponents.

You can rewrite $8^{\frac{1}{3}}$ as $(2^3)^{\frac{1}{3}}$ using prime factorisation. Then, using the power rule $(a^m)^n = a^{m \times n}$, you will get $(2^3)^{\frac{1}{3}} = 2$. Similarly, you can show that $9^{\frac{1}{2}} = (3^2)^{\frac{1}{2}} = 3$.

We see that we can work out 'nice' answers to some bases raised to fractional exponents but this still does not answer the question 'what is a fractional exponent?' Will the answers always be 'nice' whole numbers?

Fractional exponents

Example 1

Think about this: $\sqrt[3]{8} = 2$ and $\sqrt{9} = 3$. How do we know this?

$2 \times 2 \times 2 = 8$, so 2 is the cube root of 8.

$3 \times 3 = 9$, so 3 is the square root of 9.

Let us look at the relationship between $8^{\frac{1}{3}}$ and $\sqrt[3]{8}$ and the relationship between $9^{\frac{1}{2}}$ and $\sqrt{9}$.

So far we have seen that $(8)^{\frac{1}{3}} = 2$ and $9^{\frac{1}{2}} = 3$ but you also know that $\sqrt[3]{8} = 2$ and $\sqrt{9} = 3$.

$(8)^{\frac{1}{3}} = 2 = \sqrt[3]{8}$. So these expressions are equivalent, hence $(8)^{\frac{1}{3}}$ means $\sqrt[3]{8}$.

$9^{\frac{1}{2}} = 3 = \sqrt{9}$. So $9^{\frac{1}{2}}$ is the same as $\sqrt{9}$.

It seems that fractional exponents describe the roots of numbers. But let us test this idea with a few more examples.

From what you've learnt, you can see that:

$\sqrt[3]{8} = (8)^{\frac{1}{3}}$

$\sqrt{9} = (9)^{\frac{1}{2}}$

$\sqrt[3]{27} = (27)^{\frac{1}{3}}$

$\sqrt[4]{16} = (16)^{\frac{1}{4}}$

In fact, by definition, a fractional exponent is equivalent to finding some root of a number.

$x^{\frac{1}{n}} = \text{n-th root of } x$

We generalise this as:

$x^{\frac{1}{n}} = \sqrt[n]{x} \text{ where } n \in \mathbb{N} \text{ and } x \in \mathbb{R}$

The root symbol has a special name; it is called a radical. Notice that the denominator of the fractional exponent is the same number that appears in the 'tail' of the radical symbol. Each part of a radical has its own name.

Take note!

The radical that you are most familiar with is the square root. With a square root there is no need to write the index of 2. It is understood that $\sqrt{a}$ means $\sqrt[2]{a}$.

Fractional to radical

Try the following examples to better understand how to convert between radical and exponential form. Remember that the term 'radicand' refers to the number inside the root sign.

Example 2

Convert the fractional exponents to radical form and then rewrite from radical form to exponential form:

1. $36^{\frac{1}{2}}$
2. $(125x)^{\frac{1}{5}}$
3. $\sqrt{15x}$
4. $\sqrt[4]{a^2+b^2}$
5. $\sqrt[3]{x^2}$

Solutions

1. $36^{\frac{1}{2}} = \sqrt{36}$ You do not need to write the numerator of the fraction '1' in the radicand, and you do not write the denominator '2' as the index of the radical.

2. $(125x)^{\frac{1}{5}}$. The brackets are important as they show us that the entire expression $(125x)$ makes up the radicand.

   $(125x)^{\frac{1}{5}} = \sqrt[5]{125x}$

3. $\sqrt{15x}$ It is important to use brackets to rewrite the radicand in exponential form.

   $\sqrt{(15x)} = (15x)^{\frac{1}{2}}$

4. $\sqrt[4]{a^2 + b^2} = \left(a^2 + b^2\right)^{\frac{1}{4}}$ Use brackets again to indicate that the entire sum makes up the radicand.

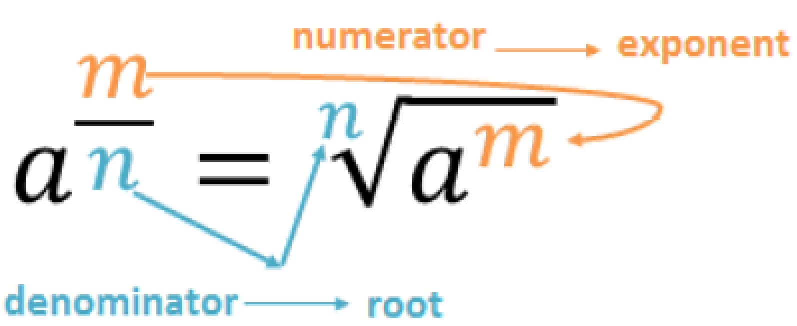
5. $\sqrt[3]{x^2}$. Here the radicand has an exponent. In fact, all the radicands in the previous examples have had exponents of 1. We have just not written them down.

   You know that $x^{\frac{1}{n}} = \sqrt[n]{x^1}$. This shows us that the exponent of the radicand '1' is the same as the numerator of the fractional exponent, and the index of the root is the same as the denominator. So, in $\sqrt[3]{x^2}$ the exponent of the radicand must be the same as the numerator of the fractional exponent and the index or root is the same as the denominator.

   $\sqrt[3]{x^2} = x^{\frac{2}{3}}$

Take note!

Example 4.2 shows that the number in the numerator of the fractional exponent becomes the exponent of the radicand and the number in the denominator of the fractional exponent becomes the root.



We can also prove the above by using the exponent law for raising a power to a power. $a^{\frac{m}{n}} = a^{\left(m \times \frac{1}{n}\right)} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m}$.

Here is another example for you to work through.

Example 3

Find the value of $4^{\frac{3}{2}}$ by converting to radical form. Without any further working out, find the value of $4^{-\frac{3}{2}}$.

Solution

$4^{\frac{3}{2}}$ in radical form is $\sqrt{4^3} = \sqrt{4 \times 4 \times 4} = \sqrt{64} = 8$.

$4^{-\frac{3}{2}} = \frac{1}{8}$

This is because you can rewrite the negative exponent as a positive exponent $4^{-\frac{3}{2}} = \frac{1}{4^{\frac{3}{2}}}$.

Irrational numbers

Some radicals or roots can be written as rational numbers (positive or negative whole numbers or fractions), for example $\sqrt{64} = 8$ or $\sqrt[3]{\frac{8}{27}} = \frac{2}{3}$.

But what about $\sqrt[3]{\frac{15}{45}}$ or $\sqrt{55}$?

$\sqrt[3]{\frac{15}{45}} = \sqrt[3]{\frac{1}{3}}$. There is no 'nice' answer to this cube root.

If you use your calculator to work out the answer, you will get a non-repeating decimal value otherwise known as an irrational number.

Similarly, $\sqrt{55}$ can only be estimated using a calculator.

Some radicals or roots cannot be written as rational numbers and it is best to leave them in radical form, for example $\sqrt{55}$ or $\sqrt[3]{\frac{1}{3}}$.

A calculator can only work out a rough approximation of their value. We call these irrational roots, surds.

Note

You will learn more about surds in an upcoming unit but if you would like to learn about the basics of surds now, you can watch the video called "What are surds?".

What are surds? (Duration: 4.20)

Complete this exercise to help you work through what you have learnt about rational exponents.

Exercise 4.1

Simplify:

1. $2a^{\frac{1}{3}} \times 3a^{-\frac{1}{3}}$
2. $(0.027)^{\frac{1}{3}}$
3. $(27)^{-\frac{1}{3}}$
4. $\left((-2)^2 a^4 b^{-6}\right)^{\frac{1}{2}}$
5. $(5x^2)^{\frac{1}{2}} \times (5x^4)^{\frac{1}{2}}$
6. $\sqrt[3]{x^2 y} \times x^{\frac{1}{3}} y^{\frac{2}{3}}$
7. $6(a^6 b^{12})^{\frac{1}{3}} \times (64a^4 b^8)^{\frac{1}{2}}$

The full solutions are at the end of the unit.

Note

If you would like more practise working with rational exponents, then click on this link.

Here is a useful list of the exponential laws and examples of them.

LAW	EXAMPLE
1. $a^m \cdot a^n = a^{m+n}$	$x^6 \cdot x^2 = x^{6+2} = x^8$
2. $\dfrac{a^m}{a^n} = a^{m-n}$	$\dfrac{x^6}{x^2} = x^{6-2} = x^4$
3. $(a^m)^n = a^{m \times n}$	$\left(x^6\right)^5 = x^{6\times5} = x^{30}$
4. $(a^m \cdot b^n)^p = a^{mp} \cdot b^{np}$	$\left(x^2 y\right)^7 = x^{14} \cdot y^7$
5. $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$	$\left(\dfrac{x}{y}\right)^4 = \dfrac{x^4}{y^4}$
6. $a^{\frac{m}{n}} = \sqrt[n]{a^m}$	$x^{\frac{2}{3}} = \sqrt[3]{x^2}$

Summary

In this unit you have learnt the following:

- The definition of a fractional exponent.
- How to convert from a fractional exponent to root form, also called radical form.
- How to convert from radical form to a fractional exponent.
- By now you should be able to understand and apply all of the following exponent rules:

$$a^m \times a^n = a^{m+n}$$

$$\frac{a^m}{a^n} = a^{m-n}$$

$$(a^m)^n = a^{m\times n}$$

$$(a^m b^n)^p = a^{mp} \cdot b^{np}$$

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

$$x^{\frac{1}{n}} = \sqrt[n]{x}$$

Unit 4: Assessment

Suggested time to complete: 20 minutes

Simplify:

1. $\dfrac{3^{2x}}{9}$
2. $\dfrac{(-1)^4}{(-2)^{-3}}$
3. $m^{-2t} \times \left(3m^t\right)^3$

4. $\left((a^{36})^{\frac{1}{2}}\right)^{\frac{1}{3}}$

5. $(3^{-1}+2^{-1})^{\frac{1}{2}}$ Write the answer in radical form.

6. $12(a^{10}b^{20})^{\frac{1}{5}} \times (125a^{12}b^{15})^{\frac{1}{3}}$

7. $\sqrt[3]{\frac{27}{1000}}$

8. $\sqrt[8]{\sqrt[4]{a^{32}b^{64}}}$

The full solutions are at the end of the unit.

Unit 4: Solutions

Exercise 4.1

1. $2a^{\frac{1}{3}} \times 3a^{-\frac{1}{3}} = 6a^0 = 6$

2.
$$(0.027)^{\frac{1}{3}}$$
$$0.027 = \frac{27}{1000} \qquad \text{Convert the base from a decimal to a fraction first.}$$
$$\therefore (0.027)^{\frac{1}{3}} = \left(\frac{27}{1000}\right)^{\frac{1}{3}}$$
$$= \left(\frac{3^3}{10^3}\right)^{\frac{1}{3}}$$
$$= \frac{3}{10} = 0.3$$

3.
$$(27)^{\frac{-1}{3}}$$
$$= (3^3)^{\frac{-1}{3}}$$
$$= 3^{-1}$$
$$= \frac{1}{3}$$

4.
$$((-2)^2a^4b^{-6})^{\frac{1}{2}}$$
$$= (-2)^{2\times\frac{1}{2}}a^{4\times\frac{1}{2}}b^{-6\times\frac{1}{2}}$$
$$= \frac{-2a^2}{b^3}$$

5.

$$(5x^2)^{\frac{1}{2}} \times (5x^4)^{\frac{1}{2}}$$
$$= 5^{\frac{1}{2}} x \times 5^{\frac{1}{2}} x^2$$
$$= 5x^3$$

6.

$$\sqrt[3]{x^2 y} \times x^{\frac{1}{3}} y^{\frac{2}{3}}$$
$$= (x^2 y)^{\frac{1}{3}} \times x^{\frac{1}{3}} y^{\frac{2}{3}} \qquad \text{Rewrite without the cube root in exponential form.}$$
$$= x^{\frac{2}{3}} y^{\frac{1}{3}} \times x^{\frac{1}{3}} y^{\frac{2}{3}}$$
$$= xy$$

7.

$$6(a^6 b^{12})^{\frac{1}{3}} \times (64a^4 b^8)^{\frac{1}{2}}$$
$$= 6(a^6)^{\frac{1}{3}} (b^{12})^{\frac{1}{3}} \times (2^6)^{\frac{1}{2}} (a^4)^{\frac{1}{2}} (b^8)^{\frac{1}{2}}$$
$$= 6a^2 b^4 \times 2^3 a^2 b^4$$
$$= 48a^4 b^8$$

Back to Exercise 4.1

Unit 4: Assessment

1. $\dfrac{3^{2x}}{9} = \dfrac{3^{2x}}{3^2} = 3^{2x-2}$

2.

$$\frac{(-1)^4}{(-2)^{-3}} = (-1)^4 (-2)^3$$
$$= (1)(-8)$$
$$= -8$$

3.

$$m^{-2t} \times (3m^t)^3$$
$$= m^{-2t} \times 3^3 m^{3t}$$
$$= 27m^t$$

4.

$$\left((a^{36})^{\frac{1}{2}} \right)^{\frac{1}{3}}$$
$$= (a^{18})^{\frac{1}{3}}$$
$$= a^6$$

5.

$$(3^{-1}+2^{-1})^{\frac{1}{2}}$$
$$=(\frac{1}{3}+\frac{1}{2})^{\frac{1}{2}}$$
$$=(\frac{1\times 2+1\times 3}{6})^{\frac{1}{2}}$$
$$=(\frac{5}{6})^{\frac{1}{2}}$$
$$=\sqrt{\frac{5}{6}}$$

6.

$$12(a^{10}b^{20})^{\frac{1}{5}}\times(125a^{12}b^{15})^{\frac{1}{3}}$$
$$=12(a^{10})^{\frac{1}{5}}(b^{20})^{\frac{1}{5}}\times(125)^{\frac{1}{3}}(a^{12})^{\frac{1}{3}}(b^{15})^{\frac{1}{3}}$$
$$=12a^2b^4\times 5a^4b^5$$
$$=60a^6b^9$$

7.

$$\sqrt[3]{\frac{27}{1000}}$$
$$=\sqrt[3]{\frac{3^3}{10^3}} \qquad \text{Rewrite the fraction in exponential form.}$$
$$=\frac{(3^3)^{\frac{1}{3}}}{(10^3)^{\frac{1}{3}}} \qquad \text{Convert from root to exponential form.}$$
$$=\frac{3}{10}$$

8.

$$\sqrt[8]{\sqrt[4]{a^{32}b^{64}}}$$
$$=\sqrt[8]{(a^{32})^{\frac{1}{4}}(b^{64})^{\frac{1}{4}}}$$
$$=\sqrt[8]{(a)^{\frac{32}{4}}(b)^{\frac{64}{4}}}$$
$$=\sqrt[8]{a^8b^{16}}$$
$$=a^{\frac{8}{8}}b^{\frac{16}{8}}$$
$$=ab^2$$

Back to Unit 4: Assessment

Media Attributions

Unit 5: Simplify surds

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Unit outcomes

By the end of this unit you will be able to:

- Rationalise fractions with surd denominators.
- Add, subtract, multiply and divide simple surds.

What you should know

Before you start this unit, make sure you can:

- Identify rational and irrational numbers. Work through Unit 1: Identify and work with rational and irrational numbers if you need to revise rational and irrational numbers.
- Apply all the exponent laws and write powers using rational exponents. If you need to revise exponent laws and rational exponents, go over Units 2, 3 and 4 within this topic.
- Simplify these expressions:

1. $2^{2x}.3^{x}.2^{x-1}$
2. $\sqrt[3]{x^2.y} \times x^{\frac{1}{3}}.y^{\frac{2}{3}}$
3. $\sqrt[8]{\sqrt[4]{a^{32}b^{64}}}$

Solutions

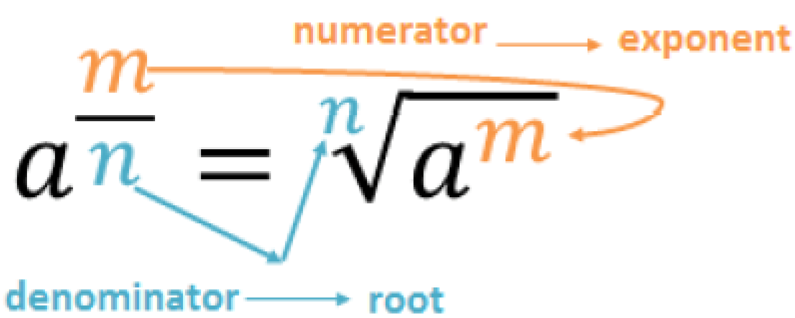
1. $$\begin{aligned} 2^{2x}.3^{x}.2^{x-1} &= 2^{2x+x-1}.3^{x} \\ &= 2^{3x-1}.3^{x} \end{aligned}$$
2. $$\begin{aligned} \sqrt[3]{x^2.y} \times x^{\frac{1}{3}}.y^{\frac{2}{3}} &= x^{\frac{2}{3}}\cdot y^{\frac{1}{3}} \times x^{\frac{1}{3}}.y^{\frac{2}{3}} \\ &= x^{\frac{1}{3}+\frac{2}{3}}.y^{\frac{1}{3}+\frac{2}{3}} \\ &= x.y \end{aligned}$$
3.

$$\sqrt[8]{\sqrt[4]{a^{32}b^{64}}} = \sqrt[8]{(a^{32})^{\frac{1}{4}}(b^{64})^{\frac{1}{4}}}$$
$$= \sqrt[8]{(a)^{\frac{32}{4}}(b)^{\frac{64}{4}}}$$
$$= \sqrt[8]{a^8b^{16}}$$
$$= a^{\frac{8}{8}}b^{\frac{16}{8}}$$
$$= ab^2$$

Introduction

We saw in Unit 4 of this topic that $a^{\frac{1}{n}} = \sqrt[n]{a}$. We also learnt that $\sqrt[n]{a}$ is called the nth root of a and is known as a radical or root. Let us recap what the parts that make up a radical are called.

Remember we write fractional exponents as roots. The number in the numerator of the fraction becomes the power of the base inside the root, and the number in the denominator of the fraction becomes the index.



Some radicals can be written as rational numbers (positive or negative whole numbers or fractions), for example $\sqrt{64} = 8$ or $\sqrt[3]{\frac{8}{27}} = \frac{2}{3}$.

But, some radicals cannot be written as rational numbers and we can only work out a rough approximation of their value. So it is best to leave them in radical form, for example $\sqrt{55}$ or $\sqrt[3]{\frac{1}{3}}$. We call these irrational roots, surds. In other words, surds are roots that cannot be reduced to a whole number or fraction.

Leaving a root in surd form is easier and more accurate than writing and rounding off the decimal value. However, there are methods we can use to simplify surds.

You will learn various ways to simplify surds in this unit.

Simplest surd form

The most basic way to simplify surds is to rewrite the radicand as a product of factors that can be further simplified. So, we need to find factors that are perfect nth roots in order for them to give 'nice' answers when we take them out of the radical sign. For example, $\sqrt{80}$ can be written as $\sqrt{8 \times 10}$ or $\sqrt{16 \times 5}$. Can you see that $\sqrt{16 \times 5}$ is the better option because we can take the $\sqrt{16} = 4$ so $\sqrt{16 \times 5} = 4\sqrt{5}$.

The product rule for simplifying roots is $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$.

Here is an example for you to work through.

Example 5.1

Simplify: $\sqrt[3]{54}$

Solution

First, write the radicand as a product of prime factors.

$\sqrt[3]{54} = \sqrt[3]{27 \times 2} = \sqrt[3]{3 \times 3 \times 3 \times 2} = \sqrt[3]{3^3 \cdot 2}$

Next, use $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$ to simplify further.

$\sqrt[3]{54} = \sqrt[3]{3^3} \times \sqrt[3]{2}$

Lastly, find the nth roots where possible.

$\sqrt[3]{3^3} \times \sqrt[3]{2} = 3 \times \sqrt[3]{2} = 3\sqrt[3]{2}$

Exercise 5.1

Simplify these radical expressions:

1. $\sqrt{300}$
2. $\sqrt{50a^6b^5}$

3. $\sqrt[3]{3} \times \sqrt[3]{9}$
4. $\sqrt{50a} \times \sqrt{2a}$
5. $\sqrt[3]{2} \times \sqrt{3}$

The full solutions are at the end of the unit.

The quotient rule for simplifying roots is $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

Just as we can rewrite the nth root of a product as a product of nth roots, so too can we rewrite the nth root of a quotient as a quotient of nth roots. We can separate the numerator and denominator of a fraction under a radical so that we can take their roots separately and vice versa.

The rule above is shown in Example 5.2 below.

Example 5.2

Simplify:

1. $\sqrt{\frac{5}{49}}$
2. $\frac{\sqrt{234a^{11}b}}{\sqrt{26x^7y}}$

Solutions

1. Rewrite this as a quotient of two radical expressions.
   $\sqrt{\frac{5}{49}} = \frac{\sqrt{5}}{\sqrt{49}}$
   Simplify the denominator. The numerator is already in its simplest or surd form.
   $= \frac{\sqrt{5}}{7}$
2. Combine the numerator and denominator into one radical expression.
   $\frac{\sqrt{234x^{11}y}}{\sqrt{26x^7y}} = \sqrt{\frac{234x^{11}y}{26x^7y}}$
   Simplify the fraction within the radicand and then simplify the root.
   $\sqrt{9x^{11-7}y^0} = \sqrt{9x^4}$
   $= 3x^2$

Adding and subtracting surds

We can add or subtract radical expressions only when they have the same radicand and the same index, for example $\sqrt{3} + 2\sqrt{3}$ is $3\sqrt{3}$. Adding and subtracting radicals is similar to collecting like terms. Just as we cannot simplify $x + y$ so too we cannot simplify $\sqrt{3} + \sqrt[3]{3}$. Even though they have the same radicand the indexes are different so we cannot add them.

Example 5.3

Simplify:

1. $5\sqrt{3} + 2\sqrt{3}$
2. $6\sqrt{20} - \sqrt{5}$
3. $\sqrt[3]{x} + 2\sqrt[3]{x}$

Solutions

1. $5\sqrt{3} + 2\sqrt{3} = 7\sqrt{3}$
2.
$$\begin{aligned} 6\sqrt{20} - \sqrt{5} &= 6\sqrt{4 \times 5} - \sqrt{5} \\ &= 6\sqrt{4} \times \sqrt{5} - \sqrt{5} \\ &= 6(2)\sqrt{5} - \sqrt{5} \\ &= 12\sqrt{5} - \sqrt{5} \\ &= 11\sqrt{5} \end{aligned}$$
3. $\sqrt[3]{x} + 2\sqrt[3]{x} = 3\sqrt[3]{x}$

Work through this exercise to assess your understanding of surds.

Exercise 5.2

Simplify:

1. $\sqrt{12} + \sqrt{3}$
2. $\sqrt{32} - \sqrt{98}$
3. $\left(\sqrt{20} - \sqrt{5}\right)^2$

The full solutions are at the end of the unit.

Note

If you would like to see a summary of simplifying surds you can watch the following videos online if you have an internet connection.

Multiplying and dividing surds (Duration: 02.08)

Adding and subtracting surds (Duration: 01.57)

Rationalising denominators

When an expression with radicals is written in simplest form, it will not contain any root in the denominator. We can remove radicals from the denominators of fractions by **rationalising the denominator**.

What happens to the value of $\sqrt{2}$ if we multiply it by $\frac{\sqrt{2}}{\sqrt{2}}$? Can you see that $\frac{\sqrt{2}}{\sqrt{2}}$ is the same as 1 ?

We know that multiplying by one does not change the value of an expression. So if we multiply $\sqrt{2}$ by $\frac{\sqrt{2}}{\sqrt{2}}$ its value will not change.

Let's see how this works:

$$\frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{\left(\sqrt{2}\right)^2}$$

$$= \frac{\sqrt{2}}{2}$$

Now there is a rational number in the denominator. This fraction is in simplest form as the denominator does not contain a radical. This is the technique we use to rationalise denominators.

Take note!

To remove radicals from the denominators of fractions, multiply by the form of 1 that will eliminate the radical.

For a denominator containing a single term, multiply by the radical in the denominator over itself.

For example, in $\frac{2}{a\sqrt{b}}$ the denominator is $a\sqrt{b}$, so we must multiply by $\frac{\sqrt{b}}{\sqrt{b}}$ to write the surd in its simplest form.

To rationalise a denominator containing the sum or difference of a rational and an irrational term, multiply the numerator and denominator by the conjugate of the denominator, which is found by changing the sign of the radical portion of the denominator.

Remember that the conjugate of an expression of the form $a+b\sqrt{c}$ is $a-b\sqrt{c}$. We find the conjugate by changing the sign of the radical portion of the expression.

For example, in $\frac{2}{a+b\sqrt{c}}$ the denominator is $a+b\sqrt{c}$, so we must multiply by $\frac{a-b\sqrt{c}}{a-b\sqrt{c}}$ to rationalise the denominator.

Example 5.4

Rewrite the following in simplest surd form:

1. $\frac{\sqrt{2}}{2\sqrt{3}}$
2. $\frac{3}{1-2\sqrt{5}}$
3. $\frac{4}{\sqrt{6}-2}$

Solutions

1. The radical in the denominator is $\sqrt{3}$. To simplify, multiply by $\frac{\sqrt{3}}{\sqrt{3}}$.

$$\begin{aligned}\frac{\sqrt{2}}{2\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} &= \frac{\sqrt{2}\times\sqrt{3}}{2\left(\sqrt{3}\right)^2}\\ &= \frac{\sqrt{2}\cdot\sqrt{3}}{2\cdot 3}\\ &= \frac{\sqrt{6}}{6}\end{aligned}$$

2. The denominator is $1-2\sqrt{5}$. Begin by finding the conjugate of the denominator by writing the denominator and changing the sign between the terms. The conjugate of $1-2\sqrt{5}$ is $1+2\sqrt{5}$.

   Therefore, we multiply the fraction by $\frac{1+2\sqrt{5}}{1+2\sqrt{5}}$. By multiplying the denominator by its conjugate, you will find that that it will always lead to the result of expanding a difference of two squares.

$$\frac{3}{1-2\sqrt{5}} \times \frac{1+2\sqrt{5}}{1+2\sqrt{5}} = \frac{3(1+2\sqrt{5})}{(1+2\sqrt{5})(1-2\sqrt{5})}$$
$$= \frac{3+6\sqrt{5}}{1-4\left(\sqrt{5}\right)^2}$$
$$= \frac{3+6\sqrt{5}}{1-4\,(5)}$$
$$= \frac{3+6\sqrt{5}}{-19}$$

3. .

$$\frac{4}{\sqrt{6}-2} \times \frac{4}{\sqrt{6}+2} = \frac{16}{(\sqrt{6}-2)(\sqrt{6}+2)}$$
$$= \frac{16}{(\sqrt{6})^2-(2)^2}$$
$$= \frac{16}{6-2}$$
$$= 4$$

Now, do this exercise to check your understanding of rationalising denominators.

Exercise 5.3

Simplify each expression:

1. $\dfrac{\sqrt{8}}{1-\sqrt{2}}$
2. $\dfrac{\sqrt{12}}{1-\sqrt{3x}}$
3. $\dfrac{\sqrt{27}}{2+2\sqrt{3}}$

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to add and subtract surds.
- How to multiply and divide surds.
- How to rationalise denominators to simplify surds.

Unit 5: Assessment

Suggested time to complete: 20 minutes

1. Simplify:
   a. $\dfrac{\sqrt{20}+\sqrt{5}+\sqrt{45}}{\sqrt{80}}$
   b. $\dfrac{\sqrt{20}+\sqrt{80}+\sqrt{125}-\sqrt{5}}{2\sqrt{5}+3\sqrt{5}}$
   c. $\dfrac{3\sqrt{12}+4\sqrt{75}+-\sqrt{27}}{\sqrt{75}}$
2. Simplify by rationalising the denominator:
   a. $\dfrac{3}{3-\sqrt{2}}$
   b. $\dfrac{3}{2\sqrt{7}-7}$
   c. $\dfrac{4}{2\sqrt{3}-3\sqrt{2}}$
3. Prove that: $\dfrac{\sqrt{125}+\sqrt{80}-\sqrt{20}}{\sqrt{180}+\sqrt{5}}=1$

The full solutions are at the end of the unit.

Unit 5: Solutions

Exercise 5.1

1.
$$\sqrt{300}=\sqrt{3\times 10^2}$$
$$=\sqrt{3}\times\sqrt{10^2} \qquad \text{Rewrite as product of radical expressions}$$
$$=10\sqrt{3}$$

2.
$$\sqrt{50a^6b^5}=\sqrt{25\times 2a^6b^5}$$
$$=\sqrt{25a^6}\times\sqrt{2b^5}$$
$$=5\mathrm{a}^3\sqrt{2b^5}$$

3.
$$\sqrt[3]{3}\times\sqrt[3]{9}=\sqrt[3]{3\times 9} \qquad \text{Express the product as a single radical expression.}$$
$$=\sqrt[3]{27}$$
$$=3$$

4.
$$\sqrt{50a}\times\sqrt{2a}=\sqrt{50a\times 2a}$$
$$=\sqrt{100a^2}$$
$$=10a$$

5.

$\sqrt[3]{2}\times\sqrt{3}$ Since the index of the roots are different, you cannot simplify any further.

Back to Exercise 5.1

Exercise 5.2

1.

$$\begin{aligned}&\sqrt{12}+\sqrt{3}=\sqrt{4\cdot 3}+\sqrt{3}\\&=2\sqrt{3}+\sqrt{3}\\&=3\sqrt{3}\end{aligned}$$

2.

$$\begin{aligned}&\sqrt{32}-\sqrt{98}=\sqrt{16\cdot 2}-\sqrt{49\cdot 2}\\&=4\sqrt{2}-7\sqrt{2}\\&=-3\sqrt{2}\end{aligned}$$

3.

$$\begin{aligned}&\left(\sqrt{20}-\sqrt{5}\right)^2=\left(\sqrt{4\cdot 5}-\sqrt{5}\right)^2\\&=\left(2\sqrt{5}-\sqrt{5}\right)^2\\&=\left(\sqrt{5}\right)^2\\&=5\end{aligned}$$

Back to Exercise 5.2

Exercise 5.3

1.

$$\begin{aligned}&\frac{\sqrt{8}}{1-\sqrt{2}}\times\frac{1+\sqrt{2}}{1+\sqrt{2}}=\frac{\sqrt{4\cdot 2}\left(1+\sqrt{2}\right)}{\left(1+\sqrt{2}\right)\left(1-\sqrt{2}\right)}\\&=\frac{2\sqrt{2}\left(1+\sqrt{2}\right)}{1-\left(\sqrt{2}\right)^2}\\&=\frac{2\sqrt{2}+2\left(\sqrt{2}\right)^2}{1-2}\\&=\frac{2\sqrt{2}+2(2)}{-1}\\&=-2\sqrt{2}-4\end{aligned}$$

2.

$$\begin{aligned}&\frac{\sqrt{12}}{1-\sqrt{3x}}\times\frac{1+\sqrt{3x}}{1+\sqrt{3x}}=\frac{\sqrt{4\cdot 3}(1+\sqrt{3x})}{(1-\sqrt{3x})(1+\sqrt{3x})} &&\text{Multiply by the conjugate}\\&=\frac{2\sqrt{3}(1-\sqrt{3x})}{1-(\sqrt{3x})^2}\\&=\frac{2\sqrt{3}-2\sqrt{3}(\sqrt{3x})}{1-3x}\\&=\frac{2\sqrt{3}-2(\sqrt{3})^2\sqrt{x}}{1-3x}\\&=\frac{2\sqrt{3}-6\sqrt{x}}{1-3x}\end{aligned}$$

3.

$$\frac{\sqrt{27}}{2+2\sqrt{3}} \times \frac{2-2\sqrt{3}}{2-2\sqrt{3}} = \frac{\sqrt{9\cdot 3}\left(2-2\sqrt{3}\right)}{\left(2+2\sqrt{3}\right)\left(2-2\sqrt{3}\right)}$$
$$= \frac{3\sqrt{3}\left(2-2\sqrt{3}\right)}{4-4(3)}$$
$$= \frac{6\sqrt{3}-6(3)}{-8} \qquad \text{Divide by common factor of 2}$$
$$= \frac{3\sqrt{3}-9}{-4}$$

Back to Exercise 5.3

Unit 5: Assessment

1.
   a.
$$\frac{\sqrt{20}+\sqrt{5}+\sqrt{45}}{\sqrt{80}} = \frac{\sqrt{4\cdot 5}+\sqrt{5}+\sqrt{9\cdot 5}}{\sqrt{16\cdot 5}}$$
$$= \frac{2\sqrt{5}+\sqrt{5}+3\sqrt{5}}{4\sqrt{5}}$$
$$= \frac{6\sqrt{5}}{4\sqrt{5}}$$
$$= \frac{6}{4}$$
   b.
$$\frac{\sqrt{20}+\sqrt{80}+\sqrt{125}-\sqrt{5}}{2\sqrt{5}+3\sqrt{5}} = \frac{\sqrt{4\cdot 5}+\sqrt{16\cdot 5}+\sqrt{25\cdot 5}-\sqrt{5}}{5\sqrt{5}}$$
$$= \frac{2\sqrt{5}+4\sqrt{5}+5\sqrt{5}-\sqrt{5}}{5\sqrt{5}}$$
$$= \frac{10\sqrt{5}}{5\sqrt{5}}$$
$$= 2$$
   c.
$$\frac{3\sqrt{12}+4\sqrt{75}-2\sqrt{27}}{\sqrt{75}} = \frac{3\sqrt{4\cdot 3}-4\sqrt{25\cdot 3}-2\sqrt{9\cdot 3}}{\sqrt{25\cdot 3}}$$
$$= \frac{6\sqrt{3}+20\sqrt{3}-6\sqrt{3}}{5\sqrt{3}}$$
$$= \frac{20\sqrt{3}}{5\sqrt{3}}$$
$$= 4$$
2.
   a.
$$\frac{3}{3-\sqrt{2}} \times \frac{3+\sqrt{2}}{3+\sqrt{2}} = \frac{3\left(3+\sqrt{2}\right)}{3-\left(\sqrt{2}\right)^2}$$
$$= \frac{9+3\sqrt{2}}{3-2}$$
$$= 9+3\sqrt{2}$$
   b.

$$\frac{3}{2\sqrt{7}-7} \times \frac{2\sqrt{7}+7}{2\sqrt{7}+7} = \frac{3\left(2\sqrt{7}+7\right)}{\left(2\sqrt{7}\right)^2-(7)^2}$$
$$= \frac{3\left(2\sqrt{7}+7\right)}{28-49}$$
$$= \frac{3\left(2\sqrt{7}+7\right)}{-21} \qquad \text{Divide by common factor of 3}$$
$$= \frac{2\sqrt{7}+7}{-7}$$

c.

$$\frac{4}{2\sqrt{3}-3\sqrt{2}} \times \frac{2\sqrt{3}+3\sqrt{2}}{2\sqrt{3}+3\sqrt{2}} = \frac{4\left(2\sqrt{3}+3\sqrt{2}\right)}{\left(2\sqrt{3}\right)^2-\left(3\sqrt{2}\right)^2}$$
$$= \frac{4\left(2\sqrt{3}+3\sqrt{2}\right)}{(4\cdot 3)-(9\cdot 2)}$$
$$= \frac{4\left(2\sqrt{3}+3\sqrt{2}\right)}{-6} \qquad \text{Divide by common factor of 2}$$
$$= \frac{2\left(2\sqrt{3}+3\sqrt{2}\right)}{-3}$$
$$= \frac{4\sqrt{3}+6\sqrt{2}}{-3}$$

3.

To prove that $\frac{\sqrt{125}+\sqrt{80}-\sqrt{20}}{\sqrt{180}+\sqrt{5}} = 1$:

$$\frac{\sqrt{125}+\sqrt{80}-\sqrt{20}}{\sqrt{180}+\sqrt{5}} = \frac{\sqrt{25\cdot 5}+\sqrt{16\cdot 5}-\sqrt{4\cdot 5}}{\sqrt{36\cdot 5}+\sqrt{5}}$$
$$= \frac{5\sqrt{5}+4\sqrt{5}-2\sqrt{5}}{6\sqrt{5}+\sqrt{5}}$$
$$= \frac{7\sqrt{5}}{7\sqrt{5}}$$
$$= 1$$
$$\therefore \frac{\sqrt{125}+\sqrt{80}-\sqrt{20}}{\sqrt{180}+\sqrt{5}} = 1$$

Back to Unit 5: Assessment

Media Attributions

Unit 6: Arithmetic sequences and series

NATASHIA BEARAM-EDMUNDS

Unit outcomes

By the end of this unit you will be able to:

- Identify and work with arithmetic progressions and sequences.
- Find the sum of an arithmetic sequence.

What you should know

Before you start this unit, make sure you can:

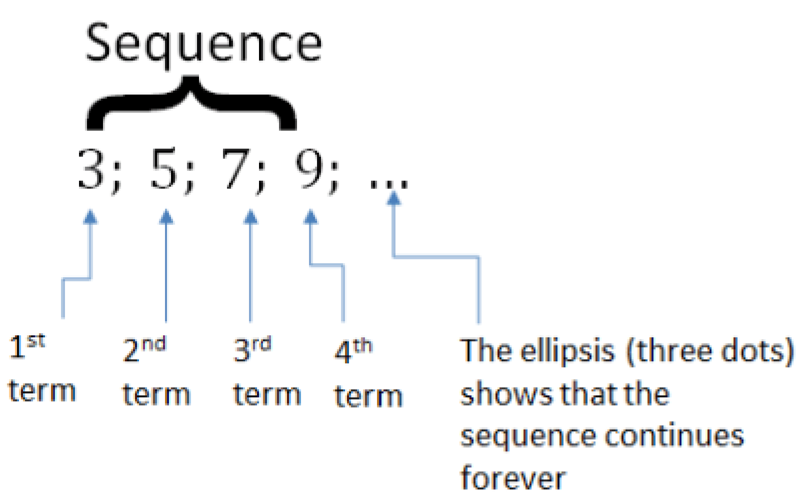
- Manipulate and simplify algebraic expressions. If you need to revise simplifying algebraic expressions you should revise Unit 1: Simplifying algebraic expressions in Topic 2.
- Solve algebraic equations. For more information on algebraic equations you can go to Topic 2, Subject Outcome 2.2 and work through all the units there.

Introduction

Mathematical patterns are all around us, from the incredible beauty of sunflowers to the intricate mystery and symmetry of galaxies. A list of numbers ordered according to a rule is called a number pattern or **sequence**. Number patterns are all about prediction. The ability to use patterns to predict the outcomes of situations is very useful.

An important application of number sequences, which you will use, is in financial calculations of loans and investments.

Each number which makes up a sequence is called a "term".



Look at the sequences below:

A. $3;\ 5;\ 7;\ 9;...$

B. $1;\ 8;\ 27;\ 64;...$

C. $13;\ 8;\ 3;\ -2;...$

Can you identify the pattern in each case?

Number sequences can have many different and interesting patterns. Let's examine the patterns of the listed sequences.

Sequence A: $3;\ 5;\ 7;\ 9;...$

The pattern is formed by adding 2 to the previous term. This type of number pattern is called a linear sequence. In a linear sequence the pattern increases (or decreases) by the same amount each time. The next five terms will be: $11\ ;\ 13;\ 15;\ 17;\ 19;...$

Sequence B: $1;\ 8;\ 27;\ 64;...$

This sequence is formed by cubing the natural numbers. So the next five terms will be: $125;\ 217;\ 343;\ 512;\ 729...$

Notice that this number pattern is not the same as the linear number pattern we saw in Sequence A, where the pattern increased by the same amount each time.

Sequence C: $13;\ 8;\ 3;\ -2;...$

There is a difference of -5 between consecutive terms. This is another example of a linear number pattern or sequence. The pattern is continued by adding -5 to the previous term. So, the next five terms will be: $-7;$ -12; -17; -22; $-27...$

A sequence can be infinite, as in the examples above, or it can be finite. For example, the sequence $\{13;\ 8;\ 3;\ -2\}$ is finite and contains only four terms.

As you can see, listing all of the terms in a sequence can be cumbersome and take a long time. For example, to find the 20th term would require listing all 20 terms. A more efficient way to determine a specific term is to write a formula to define the sequence. When a sequence follows a specific pattern we can write down the general formula to calculate any term.

One type of formula is an explicit formula, which defines the terms of a sequence using their position in the sequence. Explicit formulae are helpful if we want to find a specific term of a sequence without finding all of the previous terms. We can use the formula to find the nth term of the sequence, where n is any positive number.

In this unit we will use and discuss the arithmetic formula that gives you a quick way to find any term in a linear sequence and the position of any term in the sequence.

Did you know?

There are special patterns all around us from the number of petals on flowers to the structure of the human body. These patterns follow a famous growth pattern known as the Fibonacci sequence. The video called "The Fibonacci Sequence: Nature's Code" shows where the Fibonacci sequence can be seen in nature.

The Fibonacci Sequence: Nature's Code (Duration 02.46)

Arithmetic sequences

Below is an activity that will help you explore arithmetic sequences.

Activity 6.1: Explore arithmetic sequences

Time required: 20 minutes

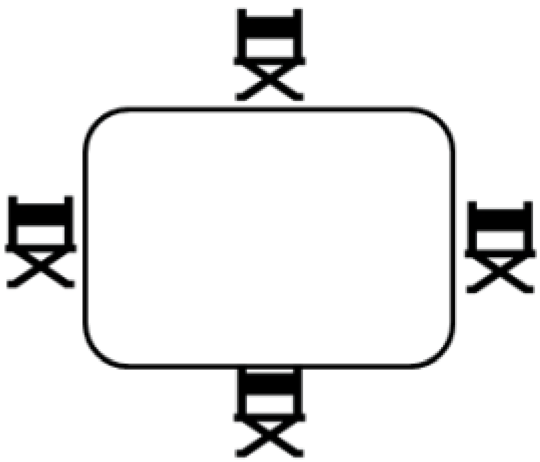
What you need:

- A pen or pencil
- A calculator
- Blank paper or a notebook

What to do:

Read the scenario and answer the questions that follow.

You and your family go to a restaurant to celebrate your birthday. The restaurant does not have a table that is long enough to seat everyone attending your birthday party. So, they have to join tables together. The restaurant only has tables that can seat four people at a time. As shown below.



1. Join two tables to the original table. How many people can be seated altogether now?

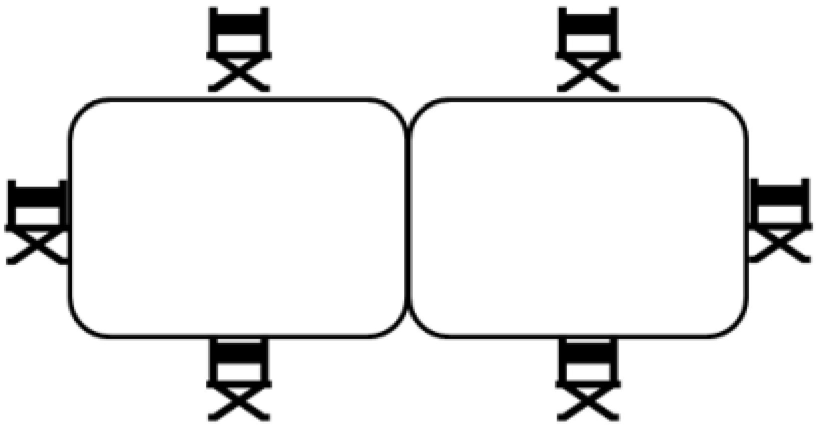
2. If the waiter joins five tables together, how many people can be seated?
3. Examine how the number of people sitting is related to the number of tables. Is there a pattern?
4. How many tables would you need to join together to seat 16 people?
5. Complete the table below to see if a pattern exists.

Number of tables (n)	1	2	3	4	5
Number of seated people					

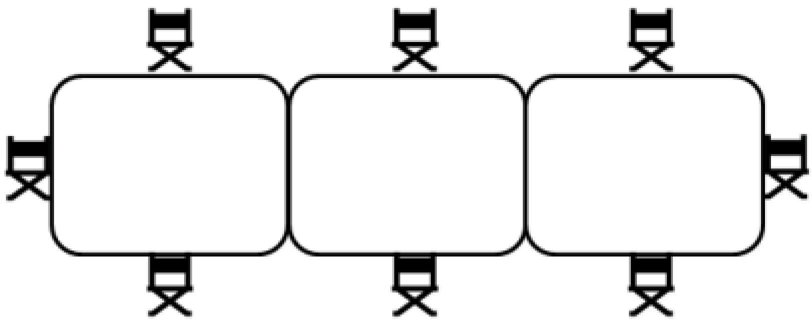
6. Describe the pattern in words.
7. Find a general rule or formula that relates the number of people to the number of tables.
8. Use your rule to find how many tables you need if there are 30 people coming to your birthday party.

What did you find?

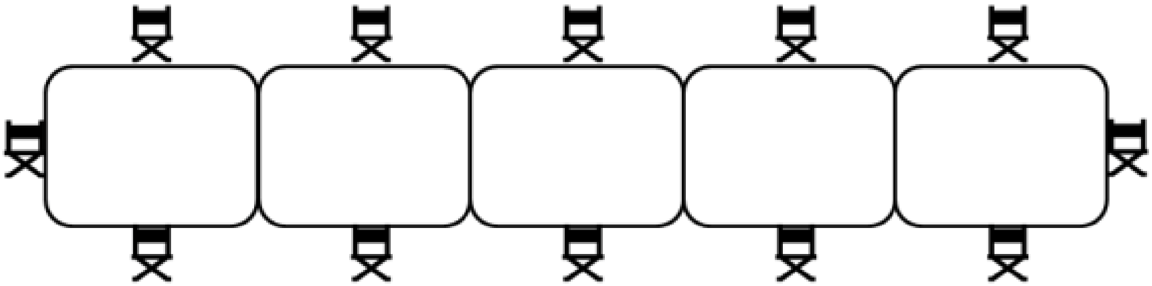
1. If another table is added to the original table, six people can be seated together.



If we join two tables to the original table there will be three tables in total, which can seat eight people.



2. Five tables joined can seat 12 people.



3. Two more people can be seated for each table added.
4. Since five tables can seat 12, if we have six tables then 14 people can be seated and by joining seven tables 16 people can be seated.
5. This is the completed table. We can confirm the pattern that two more people can be seated for each table added.

Number of tables (n)	1	2	3	4	5
Number of seated people	4	6	8	10	12

6. For each table added, the number of people increased by two. We can see that for three tables we can seat eight people, for four tables we can seat 10 people, and so on. So, the pattern formed is $4; 6; 8; 10; 12; \ldots$.
7. 1 table can seat 4 people.
2 tables can seat $4 + 1 \times 2$ people.
3 tables can seat $4 + 2 \times 2$ people.
4 tables can seat $4 + 3 \times 2$ people.We multiply 2 by one less than the table number and add 4. Therefore, for the nth table, we will have $4 + (n - 1)2$ people. This is called the nth term or general term. We represent the general term as T_n. So, we can write the general formula as $T_n = 4 + (n - 1) \times 2$. Note: This is not the only general formula that can be formed using the information in this activity. See if you can find another rule to represent the same pattern.
8. Using the formula $4 + 2(n - 1)$ we can solve for the number of tables (n).
$$4 + 2(n - 1) = 30$$
$$n - 1 = 26 \div 2$$
$$\therefore n = 13 + 1 = 14$$
You will agree that it is much quicker to use the rule to calculate the number of tables and people than to draw the tables each time.

Sequences like the one in Activity 6.1 are called arithmetic sequences or linear sequences. Arithmetic sequences are also called arithmetic progressions. We add a specific, constant number to each term in the sequence to find the next term in the sequence. The number we need to add to each term to create the next term is called the **common difference** and is represented by the letter d. The common difference is constant throughout an arithmetic sequence because the difference between any two successive terms is always the same.

Describing an arithmetic sequence

To describe terms in a number pattern we use the following notation: $T_1;\ T_2;\ T_3; \ldots T_n$

The subscript tells us the position of the term in the sequence.

T_1 is the first term of the sequence and is often represented using the letter a.

T_2 is the second term of the sequence.

T_3 is the third term of the sequence.

T_n is the nth term or general term of the sequence.

We know that if we find the relationship between the position of a term and its value, we can find a general formula which matches the pattern and any term in the sequence.

In Activity 6.1 we discovered that the general term of $4; 6; 8; 10; 12; \ldots$ is $T_n = 4 + (n-1) \times 2$.

T_1 is added to $(n-1)$ and multiplied by the common difference d to find any term in the sequence.

For sequences with a common difference, the general formula will always be: $T_n = a + (n-1)d$, where a is the first term and d is the common difference.

> The common difference is the difference between any two consecutive terms in an arithmetic sequence. The common difference is denoted by d.

A sequence is arithmetic if there is a constant difference between the terms in the sequence. To check if a sequence is arithmetic find the difference between consecutive terms. For a sequence to be arithmetic then $T_2 - T_1 = T_3 - T_2$.

Example 1

1. Write down the next three terms in the following sequence. Then decide if the sequence is arithmetic or not. If the sequence is arithmetic, write down the general term.
   $-1; \quad -4; \quad -7, \quad \ldots$
2. The general term of an arithmetic sequence is $T_n = 5 + (n-1)\,4$.
   a. Write down T_1.
   b. Find T_3 and T_{15}.
   c. Find the common difference.
   d. Which term has a value of 85?

Solution

1. We need to add -3 to each term to find the next term in the sequence. The next three terms are
   $-10; \quad -13; \quad -16, \quad \ldots$
   $T_2 - T_1 = -4 - (-1) = -3$
   $T_3 - T_2 = -7 - (-4) = -3$
   Therefore, this is an arithmetic sequence with a common difference of -3.
   The first term a has a value of -1.
   $$\begin{aligned} T_n &= a + (n-1)d \\ &= -1 + (n-1)(-3) \\ &= -1 - 3n + 3 \\ &= 2 - 3n \end{aligned}$$

2.

a. To find any term in the sequence, replace n with the term's position.
Replace n with 1
$T_n = 5 + (n - 1)\,4$
$T_1 = 5 + (1 - 1)\,4$
$= 5$

b. Replace n with 3
$T_3 = 5 + (3 - 1)\,4$
$= 13$
AND Replace n with 15
$T_{15} = 5 + (15 - 1)\,4$
$= 61$

c. You need the value of two consecutive terms, so you need to find T_2 to find the common difference. You are told the sequence is arithmetic so you can choose any two terms to work with.
We already know that $T_1 = 5$, so calculate T_2.
$T_2 = 5 + (2 - 1)\,4$
$= 9$
So to calculate the common difference:
$T_2 - T_1 = 9 - 5$
$= 4$
The common difference is 4

d. We are given the value of the term and need to calculate the term's position. In other words, we need to find n.
$5 + (n - 1)\,4 = 85$
$(n - 1)4 = 80$
$n - 1 = 20$
$n = 21$
Note: The position of a term in a sequence will always be a natural number. Therefore, n will never be a fraction or negative.

Take note!

It is important to note the difference between n and T_n. Think of n as a place holder indicating the position of the term in the sequence, while T_n is the value of the place held by n.

Work through this exercise to assess your understanding.

Exercise 6.1

1. List the first four terms of an arithmetic sequence with the following properties:

   a. $a = 16$ and $d = -2$

b. $a = -5$ and $d = 3$

2. Find the general term for the sequence: $120;\ 135;\ 150;\ldots$
3. Find the number of terms in the arithmetic sequence: $\{8;\ 1;\ \ -6;\ldots;-41\}$.
4. Find the general term of the arithmetic sequence with $T_2 = 11$ and $T_5 = 32$.

The full solutions are at the end of the unit.

Arithmetic series

Let's say you decide to save money and start immediately by saving $\mathrm{R}50$. If you increase your savings by $\mathrm{R}20$ every week, how much would you have at the end of three weeks if you do not put the money in a bank account?

It is a simple process of adding your savings amounts over the period to see how much you have altogether. Since the money is not invested in a bank account it will gain no interest and only grow by the amount you add each week. So, at the end of three weeks you will have $\mathrm{R}50 + \mathrm{R}70 + \mathrm{R}90 + \mathrm{R}110 = \mathrm{R}320$.

If you need to calculate the total amount saved after a few weeks, for example 33 weeks, it will be quite tedious to write out and add all of those amounts manually. Can you find a quicker way to calculate how much is saved in total over a period of time?

We could write the amount saved as a sequence of terms.

$50;\ 70;\ 90;\ 110;\ldots$

We see that this is an arithmetic sequence with a common difference of 20. But, now that we know this is an arithmetic sequence, is there a quicker way to find the sum of the terms? The sum of the terms of any sequence is called a series. There are two types of series; finite series and infinite series. In this unit, and in Maths Level 2, you will only deal with finite series.

The symbol S_n is used to show the sum of the first n terms of a finite sequence $S_n = T_1 + T_2 + T_3 + \ldots + T_n$.

For the sequence of savings $50;\ 70;\ 90;\ 110$ the sum of the first four terms is $S_4 = 50 + 70 + 90 + 110 = 320$. However, as we noted before, it will be very time-consuming to keep adding these terms together for n terms when n becomes large.

Since $S_n = T_1 + T_2 + T_3 + \ldots + T_n$ and we are dealing with an arithmetic sequence, it is possible to derive a general formula for the sum of n terms.

The general formula to find the sum of n terms in an arithmetic series is:
$S_n = \dfrac{n}{2}[2a + (n-1)d]$

You do not need to know how this formula is derived but you do need to know how to apply it.

To find the sum of your savings $50;\ 70;\ 90;\ 110;\ldots$ after 33 weeks, substitute the values $a = 50;\ n = 33;\ d = 20$ into the formula.

$$
\begin{aligned}
S_{33} &= \frac{33}{2}[2(33) + (33-1) \times 20] \\
&= \frac{33}{2}[706] \\
&= \mathrm{R}11\ 649
\end{aligned}
$$

This demonstrates how useful the general formula for determining an arithmetic series is, especially when the series has many terms.

Example 2

Find the sum of the series $20 + 15 + 10 + \ldots + (-50)$.

Solution

Step 1: Determine if the series is arithmetic.

$$T_3 - T_2 = 10 - 15 = -5$$
$$T_2 - T_1 = 15 - 20 = -5$$

Therefore, the series is arithmetic with $d = -5$ and $a = 20$.

Step 2: Calculate the value of n.

We also need to know the value of n in $S_n = \frac{n}{2}[2a + (n-1)d]$.

We can use $T_n = a + (n-1)d$ to find n.

The last term $T_n = -50$, $d = -5$ and $a = 20$.

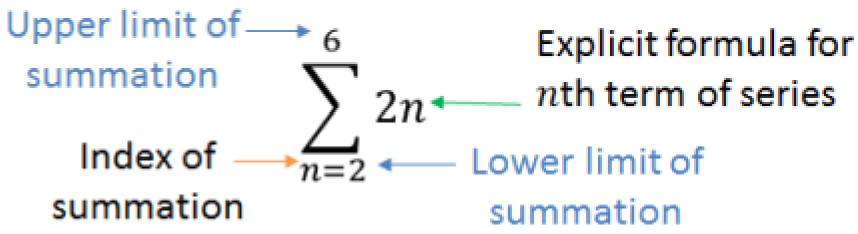
$$\begin{aligned} -50 &= 20 + (n-1)(-5) \\ -50 - 20 &= -5(n-1) \\ \frac{-70}{-5} &= n - 1 \\ 14 + 1 &= n \\ 15 &= n \end{aligned}$$

Step 3: Use $S_n = \frac{n}{2}[2a + (n-1)d]$ to find the sum.

$$\begin{aligned} S_{15} &= \frac{15}{2}[2(20) + (15-1)(-5)] \\ &= \frac{15}{2}[-30] \\ &= -225 \end{aligned}$$

Sigma notation

Sigma notation is a compact way to write the sum of a given number of terms in a series. The symbol Σ (sigma), which is the Greek capital letter S, is used to represent the sum.



The number of terms in the series = upper bound – lower bound + 1. So, the number of terms in this series is $6-2+1=5$.

If you were to list the terms of the series from $n=2$ up to and including $n=6$ you would get $4, 6, 8, 10, 12$. And if you count the number of terms you will have 5 terms in the series.

The given notation asks us to find the sum of the terms in the series $T_n = 2n$ for $n=2$ through to $n=6$.

Example 3

Find the value of the series $\sum_{k=2}^{15} 2k$.

Solution

Step 1: Find the first three terms of the series and determine if the series is arithmetic.

$$\begin{aligned} T_1 &= 2(1) = 2 \\ T_2 &= 2(2) = 4 \\ T_3 &= 2(3) = 6 \end{aligned}$$

We see that this is an arithmetic sequence.

$$\begin{aligned} &T_3 - T_2 = 6 - 4 = 2 \\ &T_2 - T_1 = 4 - 2 = 2 \\ &d = 2 \end{aligned}$$

Step 2: Work out the number of terms in the series.

Number of terms: $15 - 2 + 1 = 14$

Step 3: Work out the sum of the series using $S_n = \frac{n}{2}[2a + (n-1)d]$.

$a = 2;\ \ d = 2;\ \ n = 14$

$$\begin{aligned} S_{14} &= \frac{14}{2}[2(2) + (14-1)(2)] \\ &= 7[30] \\ &= 210 \end{aligned}$$

$$\therefore \sum_{k=2}^{15} 2k = 210$$

Now, do this exercise to check your understanding of arithmetic series and sigma notation.

Exercise 6.2

1. Find the sum of $13 + 21 + 29 + \ldots + 69$.
2. Jackson has minor surgery on his leg. A month after the surgery he can walk $1.5\ \text{km}$. Every month thereafter he walks an additional $0.5\ \text{km}$. After 5 weeks, what is the total number of kilometres he has walked?
3. Write the following series in sigma notation $31 + 24 + 17 + 10 + 3$.
4. Determine the value of k in the series $\sum_{n=1}^{k}(-2n) = -20$.
5. The sum to n terms of an arithmetic series is $S_n = \frac{n}{2}(7n + 15)$. How many terms of the series must be added to give a sum of 425?

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to determine if a sequence is arithmetic.
- How to find the common difference.
- The formula for n terms of an arithmetic sequence.
- How to find the sum of an arithmetic series by using the summing formula.
- The definition of sigma notation and how to interpret its meaning.

Unit 6: Assessment

Suggested time to complete: 40 minutes

1. Take a look at the sequence $2;\ 6;\ 10;\ 14; \ldots$.
   a. Is this an arithmetic sequence? Justify your answer by showing your calculations.
   b. Calculate T_{22}.
   c. Which term has a value of 322?
2. The 10^{th} term of an arithmetic sequence is 34 and the 16^{th} term is 52. Calculate the sum of the first 16 terms.
3. During her June holidays Siya plays a PlayStation game. She has no memory card and has to start the game from the beginning each time she plays. At the end of the first day she reaches stage 5 of the game. After restarting the game on the second day she reaches stage 9. At the end of the third day she reaches stage 13 and at the end of the fourth day she reaches stage 17.
   a. If she continues with this sequence, what stage will she reach at the end of the 22^{nd} day?
   b. If the game has 102 stages, on which day will she complete the game?
   c. If Siya spends 5 minutes of the first day reading and increases her reading time by 3 minutes each day, how many minutes would she spend on reading if her holiday is 36 days long.

4. To build a wall a builder finds that he will need 25 bricks in the bottom row, 30 bricks in the second row, 35 bricks in the third row, and so the pattern continues up to the 20^{th} row.
   a. How many bricks are there in the last row?
   b. How many bricks are there in the wall altogether?

The full solutions are at the end of the unit.

Unit 6: Solutions

Exercise 6.1

1.
   a. $a = 16$ and $d = -2$
      $T_1 = a = 16$
      $T_2 = 16 + 1 \times (-2) = 14$
      $T_3 = 16 + 2 \times (-2) = 16 - 4 = 12$
      $T_4 = 16 + 3 \times (-2) = 16 - 6 = 10$
      The first four terms are: $16;\ \ 14;\ \ 12;\ \ 10$
   b. $a = -5$ and $d = 3$
      The first four terms are: $-5;$ -2; $1;\ \ 4$.
2. For $120;\ \ 135;\ \ 150;\ldots\ a = 120;\ \ d = 135 - 120 = 150 - 135 = 15$
   $$\begin{aligned} T_n &= a + (n-1)d \\ &= 120 + (n-1)(15) \\ &= 120 + 15n - 15 \\ &= 105 + 15n \end{aligned}$$
3. In the sequence $\{8;\ \ 1;\ \ -6;\ldots;-41\}$, $a = 8;\ \ d = -7;\ \ \mathrm{T}_n = -41$
   We need to solve for n.
   $$\begin{aligned} \mathrm{T}_n &= -41 \\ a + (n-1)d &= -41 \\ 8 + (n-1)(-7) &= -41 \\ (n-1)(-7) &= -49 \\ n &= \frac{-49}{-7} + 1 \\ \therefore n &= 8 \end{aligned}$$
4. To find the general term of the arithmetic sequence with $T_2 = 11$ and $T_5 = 32$.
   $T_2 = 11$
   $\therefore a + d = 11 \text{ Eqn } 1$
   $T_5 = 32$
   $\therefore a + 4d = 32 \text{ Eqn } 2$
   Subtract Equation 1 from Equation 2 and solve simultaneously.
   $$\begin{array}{l} a + 4d = 32 \\ \underline{-a - d = -11} \\ \text{3d=21} \\ \therefore \text{d=7} \end{array}$$
   Substitute back into Equation 1 (or 2).
   $a + 7 = 11$
   $\therefore a = 4$

$\therefore T_n = 4 + 7(n-1)$
$= 7n - 3$

Back to Exercise 6.1

Exercise 6.2

1. Check for arithmetic progression.
   $13 + 21 + 29 + \ldots + 69$
   $T_3 - T_2 = 29 - 21 = 8$
   $T_2 - T_1 = 21 - 13 = 8$
   $\therefore d = 8$
   $a = 13$
   $T_n = 69$
   $13 + (n-1)8 = 69$
   $n = \left(\frac{69-13}{8}\right) + 1$
   $\therefore n = 8$
   $S_n = \frac{8}{2}[2(13) + (8-1)9]$
   $= 4(89)$
   $= 356$
2. Set up the sequence.
   $1.5;\ 2;\ 2.5;\ldots$
   Sequence is arithmetic with $d = 0.5$
   $S_5 = \frac{5}{2}[2(1.5) + (5-1)0.5]$
   $= \frac{5}{2}(5)$
   $= 12.5\ \text{km}$
   After 5 weeks, he has walked $12.5\ \text{km}$ in total.
3. First determine if the series is arithmetic: $31 + 24 + 17 + 10 + 3$
   $T_3 - T_2 = 17 - 24 = -7$
   $T_2 - T_1 = 24 - 31 = -7$
   $\therefore d = -7$
   Then determine the general formula of the series.
   $T_n = a + (n-1)d$
   $= 31 + (n-1)(-7)$
   $= 38 - 7n$
   Lastly, determine the sum of the series and write it in sigma notation.
   $S_n = 31 + 24 + 17 + 10 + 3 = 85$
   $\therefore \sum_{n=1}^{5}(-7n + 38) = 85$
4. $\sum_{n=1}^{k}(-2n) = -20$
   $T_1 = -2(1) = -2$
   $T_2 = -2(2) = -4$
   $T_3 = -2(3) = -6$
   The series is arithmetic with a common difference of -2.

$$
\begin{aligned}
S_k &= -20 \\
\frac{k}{2}[2(-2)+(k-1)(-2)] &= -20 \\
k\,[-2-2k] &= -40 && \text{multiply both sides by 2} \\
-2k-2k^2+40 &= 0 && \text{Divide by} -2 \text{ and rewrite in standard form} \\
k^2+k-20 &= 0 \\
(k-4)(k+5) &= 0 \\
k = 4 \text{ or } k &= -5 \text{ n/a} \\
\therefore k &= 4
\end{aligned}
$$

5.

$S_n = \frac{n}{2}(7n+15)$

$S_n = 425$

Solve for n.

$$
\begin{aligned}
\frac{n}{2}(7n+15) &= 425 \\
n(7n+15) &= 850 \\
7n^2+15n-850 &= 0 \\
(7n+85)(n-10) &= 0 \\
\therefore\ n &= 10
\end{aligned}
$$

Therefore, ten terms must be added to give a sum of 425.

Back to Exercise 6.2

Unit 6: Assessment

1.

a.

$2;\ 6;\ 10;\ 14;\ldots$

$T_3 - T_2 = 10 - 6 = 4$

$T_2 - T_1 = 6 - 2 = 4$

Sequence is arithmetic with $d = 4$

b.

$$
\begin{aligned}
T_{22} &= 2 + (22-1)4 \\
&= 86
\end{aligned}
$$

c.

$$
\begin{aligned}
T_n &= 2 + (n-1)4 \\
&= 4n - 2 \\
\therefore 4n - 2 &= 322 \\
4n &= 324 \\
n &= 81 \\
\therefore T_{81} &= 322
\end{aligned}
$$

2.

$T_{10} = 34;\ T_{16} = 52$

$a + 9d = 34$

$\therefore a = 34 - 9d$ Eqn 1

$a + 15d = 52$ Eqn 2

Substitute Equation 1 into Equation 2.

$$
\begin{aligned}
(34-9d) + 15d &= 52 \\
34 + 6d &= 52 \\
6d &= 18 \\
\therefore d &= 3
\end{aligned}
$$

Substitute back into Equation 1

$$\therefore a = 34 - 9(3) = 7$$

$$\begin{aligned} S_{16} &= \frac{16}{2}[2(7) + (15)(3)] \\ &= 8(59) \\ &= 472 \end{aligned}$$

3.

a.

$$5;\ 9;\ 13;\ 17;\ldots$$
$$13 - 9 = 4$$
$$9 - 5 = 4$$
$$d = 4$$
$$\begin{aligned} T_{22} &= 5 + (22 - 1)(4) \\ &= 89 \end{aligned}$$

She will reach stage 89 by end of the 22^{nd} day.

b.

$$\begin{aligned} T_n &= 102 \\ 5 + (n-1)(4) &= 102 \\ n &= \frac{(102-5)}{4} + 1 \\ &= 98 \end{aligned}$$

She will complete the game on day 98.

c.

$$5;\ 8;\ 11;\ldots$$
$$d = 3$$
$$\begin{aligned} S_{36} &= \frac{36}{2}[2(5) + (36 - 1)(3)] \\ &= 18(115) \\ &= 2070 \end{aligned}$$

She spends 2070 minutes reading.

4.

a. How many bricks are there in the last row?

$$25;\ 30;\ 35;\ldots$$
$$a = 25;\ d = 5$$
$$\begin{aligned} T_{20} &= 25 + (20 - 1)(5) \\ &= 120 \end{aligned}$$

There are 120 bricks in the last row.

b.

$$\begin{aligned} S_{20} &= \frac{20}{2}[2(25) + (19)(5)] \\ &= 10[145] \\ &= 1450 \end{aligned}$$

There are 1450 bricks in the wall.

Back to Unit 6: Assessment

Media Attributions

SUBJECT OUTCOME III

FUNCTIONS AND ALGEBRA: MANIPULATE AND SIMPLIFY ALGEBRAIC EXPRESSIONS

Subject outcome

Subject outcome 2.2: Manipulate and simplify algebraic expressions.

Learning outcomes

- Find products of two binomials.
- Find products of binomials with trinomials.
- Factorise by identifying and taking out the common factor.
- Factorise by grouping in pairs.
- Factorise the difference of two squares.
- Factorise trinomials.
- Simplify algebraic fractions with monomial denominators.

Unit 1 outcomes

By the end of this unit you will be able to:

- Find products of two binomials.
- Find products of binomials with trinomials.

Unit 2 outcomes

By the end of this unit you will be able to:

- Factorise by identifying and taking out the common factor.
- Factorise by grouping in pairs.
- Factorise the difference of two squares.
- Factorise trinomials.

Unit 3 outcomes

By the end of this unit you will be able to:

- Simplify algebraic fractions with monomial denominators.

Unit 1: Simplifying algebraic expressions

NATASHIA BEARAM-EDMUNDS

Unit outcomes

By the end of this unit you will be able to:

- Find products of two binomials.
- Find products of binomials with trinomials.

What you should know

Before you start this unit, make sure you can:

Work with real numbers and understand the basic exponent rules. Revise exponents in Subject outcome 1.2, Unit 2: Introduction to exponents.

Introduction

An algebraic expression is made up of constants and variables joined together by addition, subtraction, multiplication and division.

In the expression $3x + 2$, 2 is a **constant** because it does not vary and x is called a **variable** because its value can change. The 3 in front of the variable x is called the **coefficient** of x. The $3x$ and $+2$ are called the **terms** of the algebraic expression. Terms are separated by addition or subtraction. In this algebraic expression we have two terms.

We often simplify an algebraic expression to make it easier to work with or calculate, or to use it in some other way. To do so, we use the properties of real numbers.

Note

The following properties are true for real numbers a, b and c.

	Addition	Multiplication
Commutative property	$a+b=b+a$	$a\cdot b=b\cdot a$
Associative property	$a+(b+c)=(a+b)+c$	$a\,(bc)=(ab)\,c$
Distributive property	$a\cdot(b+c)=a\cdot b+a\cdot c$	
Identity property	There is a unique real number called the additive identity, 0 such that, for any real number a $a+0=a$.	There is a unique real number called the multiplicative identity, 1, such that, for any real number a $a\cdot 1=a$.
Inverse property	Every real number a has an additive inverse, or opposite, written as $-a$, such that $a+(-a)=0$.	Every non-zero real number a has a multiplicative inverse, or reciprocal, written as $\frac{1}{a}$, such that $a\cdot\frac{1}{a}=1$.

Products

Remember that a product is the result of multiplication. When you multiply brackets that contain terms you end up with algebraic expressions. Mathematical expressions are just like sentences and each part has a special name.

$2x^2-3x+4$ is called an expression and is made up of the following parts.
Terms: $2x^2, \ -3x, \ +4$
Variable: x
Constant: $+4$
Coefficients (numbers in front) of the variable: 2 and -3
Exponents: 2 and 1

The following are words used to describe specific expressions that you will come across often. You need to learn the definitions of these terms.

A **monomial** is an expression with one term, for example $2x$ or $3(xy)$. Remember that brackets do not separate terms.
A **binomial** is an expression with two terms, for example $x+y$ or $2a-b$.
A **trinomial** is an expression with three terms, for example $x+y+z$ or $2x^2-3x+4$.
You can remember it as 'mono' means one, 'bi' means two and 'tri' means three.

Multiplying a monomial and a binomial

In the next example we learn how to multiply a monomial by a binomial.

Example 1.1

Simplify:

1. $3x(x-10)$
2. $2a(a-2)-3(a^2-a)$

Solutions

1. Multiply the $3x$ by each term inside the bracket.
   $3x(x-10) = 3x \cdot x - 3x \cdot 10$
   Be careful with the signs. First multiply the signs, positive by positive and positive by negative, before multiplying the numbers.
   $$\begin{aligned} 3x(x-10) &= 3x \cdot x - 3x \cdot 10 \\ &= 3x^2 - 30x \end{aligned}$$
2. In this example we multiply two separate monomials by two binomials. After getting rid of the brackets, we need to add like terms together. Notice that a negative multiplied by a negative gives a positive answer.
   $$\begin{aligned} 2a(a-2)-3(a^2-a) &= 2a \cdot a - 2a \cdot 2 - 3 \cdot a^2 - 3(-a) \\ &= 2a^2 - 4a - 3a^2 + 3a \\ &= -a^2 - a \end{aligned}$$

Exercise 1.1

Simplify:

1. $2a(a\ +\ 3)$
2. $-6(t-1)+2(t+3)$
3. $(a-b)2$

The full solutions are at the end of the unit.

Multiply a binomial by a binomial

Here we multiply (or expand) two linear (highest power of the variable is one) binomials.
We can use the word FOIL (First Outers Inners Last) to help us remember that each term in the first bracket must be multiplied by each term in the second bracket.

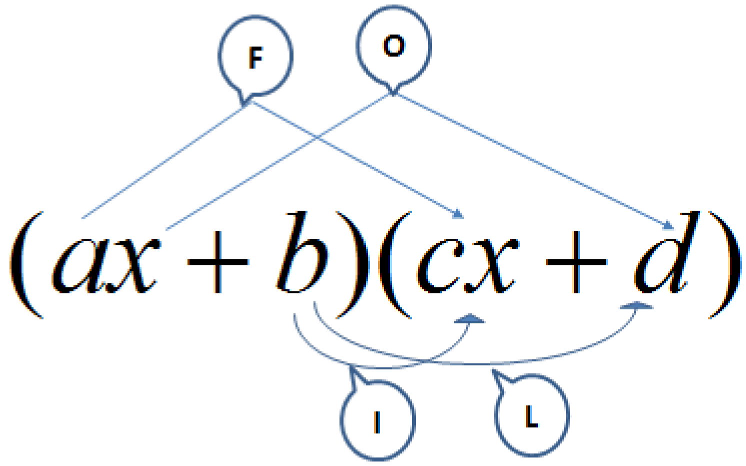


Figure 1: FOIL (First Outers Inners Last)

In general:

$$\begin{aligned}(ax+b)(cx+d) &= ax\cdot cx + ax\cdot d + b\cdot cx + b\cdot d\\ &= acx^2 + axd + bcx + bd\\ &= acx^2 + x(ad+bc) + bd\end{aligned}$$

When we multiply brackets we generally write the products with variables in alphabetical order, for example acd not dca. And we write terms in descending order of the first variable, for example x^2+2x+1 is written in descending powers of x.

Note

When you have an internet connection, watch the video called “Multiplying binomials” to learn more about FOIL.

Multiplying binomials (Duration: 05.47)

Activity 1.1: Practise using FOIL

Time required: 15 minutes

What you need:

- pen and paper

What to do:

1. Complete the table by finding the following products:

	Multiply brackets	Add like terms
a. $(ax+b)(ax+b)$		
b. $(a-b)(a-b)$		
c. $(x+2)(x+2)$		
d. $(ax-b)(ax+b)$		
e. $(a-b)(a+b)$		
f. $(x+2)(x-2)$		

2. In parts a) to c) what do you notice about the brackets before you multiply?
3. In parts a) to c) what happened to the first terms and last terms in the simplified expressions?
4. In parts a) to c) what do you notice about the middle term in the simplified expressions?
5. In parts d) to f) what do you notice about the brackets before you multiply?
6. In parts d) to f) what happened to the first terms and last terms in the simplified expressions?
7. In parts d) to f) what happened to the middle in the simplified expressions?

What did you find?

1.

	Multiply brackets	Add like terms
a. $(ax+b)(ax+b)$	$a^2x^2+axb+axb+b^2$	$a^2x^2+2axb+b^2$
b. $(a-b)(a-b)$	$a^2-ab-ab+b^2$	$a^2-2ab+b^2$
c. $(x+2)(x+2)$	$x^2+2x+2x+4$	x^2+4x+4
d. $(ax-b)(ax+b)$	$a^2x^2-abx+abx-b^2$	$a^2x^2-b^2$
e. $(a-b)(a+b)$	$a^2-ab+ab-b^2$	a^2-b^2
f. $(x+2)(x-2)$	$x^2-2x+2x-4$	x^2-4

2. The brackets are identical in a) to c).
3. In parts a) to c) the first terms have been squared and the last terms have been squared once the expression has been simplified.
4. In parts a) to c) the middle terms are repeated when you multiply the brackets out. So the final

answer to the middle term is twice the product of the first and second terms.

5. The brackets contain the same terms but with opposite signs.
6. In parts d) to f) the first terms and last terms of the brackets have been squared and are separated by a minus sign.
7. In parts d) to f) because the middle terms, when the brackets are multiplied out, are the same but with opposite signs they add up to zero when you simplify.

The product of two identical binomials is known as the square of the binomial and is written as:

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b)\\ &= a^2+2ab+b^2\end{aligned}$$

after simplifying and adding the two middle terms.

If the two terms are of the form $a+b$ and $a-b$, then their product is a^2-b^2. We simply square the first term, square the second term and separate with a minus sign. The middle term will always cancel out as you saw in Activity 1.1. This product gives us a difference of two squares.

Exercise 1.2

Simplify:

1. $(a-4)(a-4)$
2. $(x-y)(x+2y)$
3. $(c-2d)(c+2d)$
4. $2(a+2b)(a+2b)$
5. $3(x-3)^2+2(x+3)(x-3)$

The full solutions are at the end of the unit.

Multiply a binomial and a trinomial

You have already seen that a trinomial is an expression with three terms, for example ax^2+bx+c. Now we will learn how to find the product of a binomial and a trinomial.

To find the product of a binomial and a trinomial, multiply each term of the binomial by each term of the trinomial.

$$\begin{aligned}(a+b)(x+y+z) &= a(x+y+z)+b(x+y+z)\\ &= ax+ay+az+bx+by+bz\end{aligned}$$

Let's see how this works in an example.

Example 1.2

Find the product:

1. $(x-1)(x^2-2x+1)$
2. $(a-2b)(2a+b-3)$
3. $(-3-x)(2x^2+x-3)$
4. $(2y^2-y+3)(y-1)$

Solutions

1. Multiply each term in the first bracket by each term in the second bracket. Remember to multiply the signs carefully. Then collect all like terms.

$$\begin{aligned}(x-1)(x^2-2x+1) \quad &= x(x^2-2x+1)-1(x^2-2x+1)\\ &= x\cdot x^2 - x\cdot 2x + x\cdot 1 - 1\cdot x^2 - 1(-2x) - 1\cdot 1\\ &= x^3-2x^2+x-x^2+2x-1\\ &= x^3-3x^2+3x-1\end{aligned}$$

2.

$$\begin{aligned}(a-2b)(2a+b-3) \quad &= a(2a+b-3)-2b(2a+b-3)\\ &= 2a^2+ab-3a-4ab-2b^2+6b\\ &= 2a^2-3ab-3a-2b^2+6b\end{aligned}$$

3. Be careful with the negative signs in this example.

$$\begin{aligned}(-3-x)(2x^2+x-3) \quad &= -3(2x^2+x-3)-x(2x^2+x-3)\\ &= -6x^2-3x+9-2x^3-x^2+3x\\ &= -2x^3-7x^2+9\end{aligned}$$

4. Even though the trinomial is in the first bracket and the binomial in the second, the rules to multiply out the brackets stay the same. If you prefer you can use the commutative property of real numbers $a\cdot b = b\cdot a$ to rearrange the brackets before multiplying.

$$\begin{aligned}(2y^2-y+3)(y-1) \quad &= (y-1)(2y^2-y+3)\\ &= y(2y^2-y+3)-1(2y^2-y+3)\\ &= 2y^3-y^2+3y-2y^2+y-3\\ &= 2y^3-3y^2+4y-3\end{aligned}$$

Summary

In this unit you have learnt the following:

- How to use the properties of real numbers to work with algebraic expressions.
- How to identify the parts that make up an algebraic expression.
- How to multiply a monomial and a binomial.
- How to multiply a binomial and a binomial.
- How to multiply a binomial and a trinomial.

Assessment

Suggested time to complete: 35 minutes

1. Expand the following products:
   a. $2x(x-3)$
   b. $(a-2b)2b$
   c. $(-7+x)(7+x)$
   d. $-(2xy-2)(2xy-2)$
   e. $(2t-3)^2$
   f. $(-2y^2-4y+11)(3-y)$
2. Simplify:
   a. $(2x-3)^2-(x-2)^2$
   b. $2(3a+b)(3a-b)-(3a-b)^2$
   c. $(2a^2-a-1)(a^2+3a-2)$
   d. $(\frac{x}{3}-\frac{3}{x})(\frac{x}{4}+\frac{4}{x})$
   e. $\frac{1}{2}(10x-12y)+\frac{1}{3}(15x-18y)$

The full solutions are at the end of the unit.

Unit 1: Solutions

Exercise 1.1

1. $2a(a \ + \ 3)=2a^2+6a$
2. After multiplying brackets, simplify by adding like terms.
$$\begin{aligned} -6(t-1)+2(t+3) &= -6\cdot t-6(-1)+2\cdot t+2(3) \\ &= -6t+6+2t+6 \\ &= -4t+12 \end{aligned}$$
3. Even though the 2comes after the bracket, remember that multiplication is commutative.
$(a-b)2=2a-2b$

Back to Exercise 1.1

Exercise 1.2

1.
$$\begin{aligned} (a-4)(a-4) &= a^2-2(4a)+16 \\ &= a^2-8a+16 \end{aligned}$$

2.

$$\begin{aligned}(x-y)(x+2y) &= x^2+2xy-xy-2y^2\\ &= x^2+xy-2y^2\end{aligned}$$

3.

$$\begin{aligned}(c-2d)(c+2d) &= (c)^2-(2d)^2\\ &= c^2-4d^2\end{aligned}$$

4.

$$\begin{aligned}2(a+2b)(a+2b) &= 2(a^2+4ab+4b^2)\\ &= 2a^2+8ab+8b^2\end{aligned}$$

5.

$$\begin{aligned}3(x-3)^2+2(x+3)(x-3) &= 3(x^2-6x+9)+2(x^2-9)\\ &= 3x^2-18x+27+2x^2-18\\ &= 5x^2-18x+9\end{aligned}$$

Back to Exercise 1.2

Unit 1: Assessment

1.

a. $2x(x-3)=2x^2-6x$

b. $(a-2b)2b=2ab-4b^2$

c. $(-7+x)(7+x)=x^2-49$

d.

$$\begin{aligned}-(2xy-2)(2xy-2) &= -\left(4x^2y^2-8xy+4\right)\\ &= -4x^2y^2+8xy-4\end{aligned}$$

e. $(2t-3)^2=4t^2-12t+9$

f.

$$\begin{aligned}(-2y^2-4y+11)(3-y) &= -6y^2-12y+33+2y^3+4y^2-11y\\ &= 2y^3-2y^2-23y+33\end{aligned}$$

2.

a.

$$\begin{aligned}(2x-3)^2-(x-2)^2 &= 4x^2-12x+9-(x^2-4x+4)\\ &= 4x^2-12x+9-x^2+4x-4\\ &= 3x^2-8x+5\end{aligned}$$

b.

$$\begin{aligned}2(3a+b)(3a-b)-(3a-b)^2 &= 2(9a^2-b^2)-(9a^2-6ab+b^2)\\ &= 18a^2-2b^2-9a^2+6ab-b^2\\ &= 9a^2+6ab-3b^2\end{aligned}$$

c.

$$\begin{aligned}(2a^2-a-1)(a^2+3a-2) &= 2a^2(a^2+3a-2)-a(a^2+3a-2)-1(a^2+3a-2)\\ &= 2a^4+6a^3-4a^2-a^3-3a^2+2a-a^2-3a+2\\ &= 2a^4+5a^3-8a^2-a+2\end{aligned}$$

d.

$$
\begin{aligned}
\left(\frac{x}{3}-\frac{3}{x}\right)\left(\frac{x}{4}+\frac{4}{x}\right) \quad &= \frac{x}{3}\cdot\frac{x}{4}+\frac{x}{3}\cdot\frac{4}{x}-\frac{3}{x}\cdot\frac{x}{4}-\frac{3}{x}\cdot\frac{4}{x} \\
&= \frac{x^2}{12}+\frac{4x}{3x}-\frac{3x}{4x}-\frac{12}{x} \\
&= \frac{x^2}{12}+\frac{4}{3}-\frac{3}{4}-\frac{12}{x} \\
&= \frac{x^2}{12}+\frac{7}{12}-\frac{12}{x}
\end{aligned}
$$

e.

$$
\begin{aligned}
\frac{1}{2}(10x-12y)+\frac{1}{3}(15x-18y) \quad &= 5x-6y+5x-6y \\
&= 10x-12y
\end{aligned}
$$

Back to Unit 1: Assessment

Media Attributions

Unit 2: Factorisation

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Unit outcomes

By the end of this unit you will be able to:

- Factorise by identifying and taking out the common factor.
- Factorise by grouping in pairs.
- Factorise the difference of two squares.
- Factorise trinomials.

What you should know

Before you start this unit, make sure you can:

- Simplify algebraic expression by expanding brackets. Revise Subject outcome 2.2, Unit 1: Simplifying algebraic expressions for more on algebraic simplification.

Introduction

When we learn about fractions we are shown that the highest common factor (HCF) of two numbers is the largest number that divides evenly into both numbers. For example, 3 is the (HCF) of 9 and 15 because it is the largest number that divides exactly into both numbers. The HCF of polynomials works the same way.

Many polynomial expressions can be written in simpler forms by factorising. Finding the HCF is just one of the ways that we factorise expressions. In this unit, we will look at different methods to factorise polynomials.

Find the highest common factor

Factorisation reverses the process of multiplying and expanding brackets that we covered in Unit 1. For example, if we expand $3(x+2)$ we get $3x+6$. When we factorise, we start with $3x+6$ and end up with $3(x+2)$. The two expressions are exactly the same in value no matter what values we substitute for x.

Each part of a product is called a factor of the expression. If $c=ab$, then a and b are factors of c. Similarly, 3 and $x+2$ are factors of $3(x+2)$ since $3x+6=3(x+2)$. Factorising based on common factors relies on there being factors common to all the terms.

Let's factorise the expression $10a+5$. Both $10a$ and 5 have a common factor of 5:

$10a = 2 \times \underline{5} \times a$
$5 = 1 \times \underline{5}$

So we can write $10a + 5$ as $2 \times 5 \times a + 1 \times 5$. We can factorise by taking out the HCF of 5 from both terms and we will get:

$$\begin{aligned} 10a + 5 &= 5(2 \times a + 1) \\ &= 5(2a + 1) \end{aligned}$$

We have divided each term by the HCF of 5; this is called **factorising by taking out the highest common factor**.

When factorising a polynomial, our first step should always be to check for a HCF. Look for the HCF of the coefficients, and then look for the HCF of the variables and combine them to find the HCF of the expression. This is shown in the next example.

Example 2.1

Factorise:

1. $4x^3 + 2x^2 + 6x$
2. $3(a - b) - x(a - b)$
3. $2(x - y) - a(y - x)$

Solutions

1. First, find the HCF of the coefficients.
   The HCF of $4, 2$ and 6 is 2.
   $4 = \underline{2} \times 2$
   $2 = \underline{2} \times 1$
   $6 = \underline{2} \times 3$
   Next, find the HCF of the variables.
   $x^3 = \underline{x} \times x \times x$
   $x^2 = \underline{x} \times x$
   $x = \underline{x} \times 1$

   The HCF of x^3, x^2 and x is x. Note: the HCF of a set of expressions of the form x^n will always be the power with the smallest exponent.

   We combine the HCF's of both the coefficients and variables to get that the HCF of the entire expression is $2x$.

   Lastly, we divide each term by the HCF to factorise.
   $$\begin{aligned} 4x^3 + 2x^2 + 6x &= 2x\left(\frac{4x^3}{2x} + \frac{2x^2}{2x} + \frac{6x}{2x}\right) \\ &= 2x(2x^2 + x + 3) \end{aligned}$$

   Note: you can check your answer after factorising by finding the product or expanding the brackets. If you have factorised correctly you will end up with the original expression. Try and confirm this for yourself.
2. A common bracket can be taken out as the HCF provided the bracket is identical in both terms. There is a common factor bracket of $a - b$ in$3(a - b) - x(a - b)$. To take out the HCF, simply divide each term by the common bracket.

$$3(a-b)-x(a-b) \quad = (a-b)\left[\frac{3(a-b)}{(a-b)} - \frac{x(a-b)}{(a-b)}\right]$$
$$= (a-b)[3-x]$$

Factorising by finding a HCF which is a common bracket is called **grouping**.

3. In this example, the brackets look similar but the terms within have different signs. Use a 'switch around' method to find the common factor.
   $(y-x) = -1(x-y)$ Divide each term by negative one to switch around the terms.

$$\begin{aligned} 2(x-y)-a(y-x) &= 2(x-y)-a(-1)(x-y) \\ &= 2(x-y)+a(x-y) \\ &= (x-y)(2+a) \\ &= (x-y)(a+2) \end{aligned}$$

Note:$(2+a) = (a+2)$. There is no need to change signs outside the bracket when both terms are positive.

Exercise 2.1

Factorise:

1. $-ax^2+bx+x^3$
2. $(a-b)\,x+a-b$
3. $(a-b)\,x-a+b$
4. $-12a+24a^2-3$

The full solutions are at the end of the unit.

Factorising by grouping

Finding and taking out the HCF is the starting point of all factorising. We know that the factors of $2x+2$ are 2 and$(x+1)$. Similarly the factors of $3x^2+3x$ are $3x$ and $(x+1)$. If you have one expression that combines all of these terms, $2x+2+3x^2+3x$, there is no common factor to all four terms, but we can factorise as follows:

$$\begin{aligned} 2x+2+3x^2+3x &= (2x+2)+(3x^2+3x) \\ &= 2(x+1)+3x(x+1) \end{aligned}$$

Now there is a common factor of $(x+1)$.

By taking out the common bracket of $(x+1)$ we get $2(x+1)+3x(x+1) = (x+1)(2+3x)$. As you saw in Example 2.1, this is called factorising by grouping.

Example 2.2

Find the factors of: $6x + 3y + 2ax + ay$

Solution

Step 1: There are no factors common to all terms.

Step 2: Group terms with common factors together.

3 is a common factor of the first two terms and a is a common factor of the second two terms.

$6x + 3y + 2ax + ay = 3(2x + y) + a(2x + y)$

Step 3: Take out the common bracket as the HCF.

$$\begin{aligned} 3(2x+y) + a(2x+y) &= \frac{3(2x+y)}{(2x+y)} + \frac{a(2x+y)}{(2x+y)} \\ &= (2x+y)(3+a) \end{aligned}$$

Step 4: Write the final answer.

The factors of $6x + 3y + 2ax + ay$ are $(2x + y)$ and $(3 + a)$.

Note: you can get to the same answer by grouping $6x$ with $2ax$ and $3y$ with ay, try it yourself to confirm this.

Exercise 2.2

Factorise:

1. $x^2 - 6x + 5x - 30$
2. $6t - 15s + 2yt - 5ys$
3. $ab - a^2 - a + b$
4. $3ax + bx - 3ay - by - 9a - 3b$

The full solutions are at the end of the unit.

Difference of two squares

A difference of squares is a perfect square subtracted from another perfect square. A difference of squares can be rewritten as a product of binomials containing the same terms but opposite signs because the middle terms will cancel each other out if the two factors are multiplied.

Given a difference of squares, you can factorise it into binomials by:

- confirming that the first and last term are perfect squares and that the expression has one negative sign and one positive sign.
- write the factorised form as $(\sqrt{a^2} + \sqrt{b^2})(\sqrt{a^2} - \sqrt{b^2})$.

Note: We cannot factorise the sum of two squares.

Activity 2.1: Find factors for the difference of two squares

Time required: 10 minutes

What you need:

- pen and paper

What to do:

1. Complete the following table and see if you can find a pattern (rule), which you can use to predict the answers to the first column's calculations without needing to square the numbers.

(source: Sasol Inzalo Grade 9, Maths Book 2, Page 22 http://www.mstworkbooks.co.za/downloads/Maths2_Gr9_LB.pdf. CC BY-NC)

$3^2 - 2^2 =$	$3 + 2 =$	$3 - 2 =$	$(3 + 2)(3 - 2) =$
$4^2 - 3^2 =$	$4 + 3 =$	$4 - 3 =$	$(4 + 3)(4 - 3) =$
$5^2 - 4^2 =$	$5 + 4 =$	$5 - 4 =$	$(5 + 4)(5 - 4) =$
$6^2 - 5^2 =$			

2. Now predict the answers to each of the following without squaring. Check your answers. Does the rule that you discovered in Question 1 also hold for the following cases?
   a. $16^2 - 12^2$
   b. $27^2 - 18^2$
3. Write the rule you discovered in a formula using $a^2 - b^2$

What did you find?

1.

$3^2 - 2^2 = 5$	$3 + 2 = 5$	$3 - 2 = 1$	$(3 + 2)(3 - 2) = 5 \times 1 = 5$
$4^2 - 3^2 = 7$	$4 + 3 = 7$	$4 - 3 = 1$	$(4 + 3)(4 - 3) = 7 \times 1 = 7$
$5^2 - 4^2 = 9$	$5 + 4 = 9$	$5 - 4 = 1$	$(5 + 4)(5 - 4) = 9 \times 1 = 9$
$6^2 - 3^2 = 27$	$6 + 3 = 9$	$6 - 3 = 3$	$(6 + 3)(6 - 3) = 9 \times 3 = 27$

If you multiply the sum of the square roots by the difference of the square roots you end up with the same answer as the difference of the squared numbers in column 1. The sum and difference of the squares are the factors of the expressions in column 1.

2.
   a. $16^2 - 12^2 = (16 + 12)(16 - 12) = (28)(4) = 112$
   b. $27^2 - 18^2 = (27 + 18)(27 - 18) = 45 \times 9 = 405$
   Yes, the rule is the same in the above cases too.

3. A difference of two squares can be rewritten as two factors containing the same terms but opposite signs. Perfect squares $a^2 - b^2$ have factors of $(a+b)(a-b)$.

We can apply the rule for factorising a difference of squares to algebraic expressions too.

$16x^2 - 25$ is made up of two perfect squares because $16x^2 = (4x)^2$ and $25 = (5)^2$ and the terms are separated by a minus sign. So we can factorise and rewrite $16x^2 - 25$ as $(4x+5)(4x-5)$.

Exercise 2.3

Factorise:

1. $1 - x^2$
2. $8x^2 - 18y^2$
3. $-36 + t^2$
4. $2a(a^2 - 9) - 7(a^2 - 9)$

The full solutions are at the end of the unit.

Note

When you have access to an internet connection watch the video called "Factor a Sum or Difference of Cubes" which explains how to factorise the sum and difference of cubes.

Factor a Sum or Difference of Cubes (Duration: 03.18)

Factorising a trinomial

A quadratic expression is any expression where the variable has a highest power (or degree) of two. $ax^2 + bx$, x^2, $a^2 - b^2$ and $ax^2 + bx + c$ are all examples of quadratic expressions. We use the expression $ax^2 + bx + c$ so often that it has a special name. $ax^2 + bx + c$ is called a quadratic trinomial in standard form. Remember that a trinomial has three terms. Factorising using grouping or a difference of two squares was relatively simple. However, quadratic trinomials require a little more work to arrive at a general method.

Activity 2.2: Find factors of a quadratic trinomial

Time required: 10 minutes

What you need:

- pen and paper

What to do:

1. Complete the table by finding the product.

	Product	First term in the product	Two middle terms of product	Sum of the two middle terms	Product of the constant terms	Factors of the constant term
$(x+2)(x+3)$						
$(x+1)(x+6)$						
$(x-2)(x-3)$						
$(x-1)(x-6)$						

2. Which terms do you multiply together to find the x^2 in the product?
3. How do you find the co-efficient of the x term in the product?
4. Which terms do you multiply to find the constant term in the product?
5. What are the factors of x^2+5x+6?
6. What are the factors of x^2-5x+6?
7. What are the factors of x^2+7x+6?
8. What are the factors of x^2-7x+6?

What did you find?

1.

	Product	First term in the product	Two middle terms of product	Sum of the two middle terms	Product of the constant terms	Factors of the constant term
$(x+2)(x+3)$	x^2+5x+6	x^2	$2x$ and $3x$	$5x$	6	1 and 6, 2 and 3, -1 and -6, -2 and -3
$(x+1)(x+6)$	x^2+7x+6	x^2	x and $6x$	$7x$	6	1 and 6, 2 and 3, -1 and -6, -2 and -3
$(x-2)(x-3)$	x^2-5x+6	x^2	$-2x$ and $-3x$	$-5x$	6	1 and 6, 2 and 3, -1 and -6, -2 and -3
$(x-1)(x-6)$	x^2-7x+6	x^2	$-x$ and $-6x$	$-7x$	6	1 and 6, 2 and 3, -1 and -6, -2 and -3

2. We see that the x^2 term in the quadratic is the product of the x-terms in each bracket.
3. The middle term is the sum of two middle terms when the brackets are expanded.
4. The 6 in the quadratic is the product of the constant terms in the brackets.
5. Factors of x^2+5x+6 are $(x+2)(x+3)$.
6. Factors of x^2+7x+6 are $(x+1)(x+6)$.
7. Factors of x^2-5x+6 are $(x-2)(x-3)$.
8. Factors of x^2-7x+6 are $(x-1)(x-6)$.

The first method we will look at to factorise quadratic trinomials involves trial and error. As you saw in Activity 2.2 the signs of the terms in the brackets make a difference to the answers of the product. For example, even though both 2 and 3 and -2 and -3 are factors of 6, they give us different values for the middle term when the brackets are expanded. Hence $(x-2)(x-3)$ are factors of x^2-5x+6 but not factors of x^2+5x+6.

Note

For any quadratic trinomial ax^2+bx+c, if c is positive, then the factors of c must either both be positive or both be negative. If c is negative, it means only one of the factors of c is negative and the other one must be positive.

Once you have factorised, you can multiply out your brackets again just to make sure it really works. The following example uses the method of trial and error to factorise a quadratic trinomial.

Example 2.3

Factorise: $2x - 1 + 3x^2$

Solution

Step 1: Check that the quadratic is in the standard form of $ax^2 + bx + c$. If it is not, then rewrite it in standard form.

$2x - 1 + 3x^2 = 3x^2 + 2x - 1$

Step 2: Write down a set of factors for a and c.

The possible factors of $3x^2$ are $1x$ and $3x$.

The possible factors of -1 are 1 and -1.

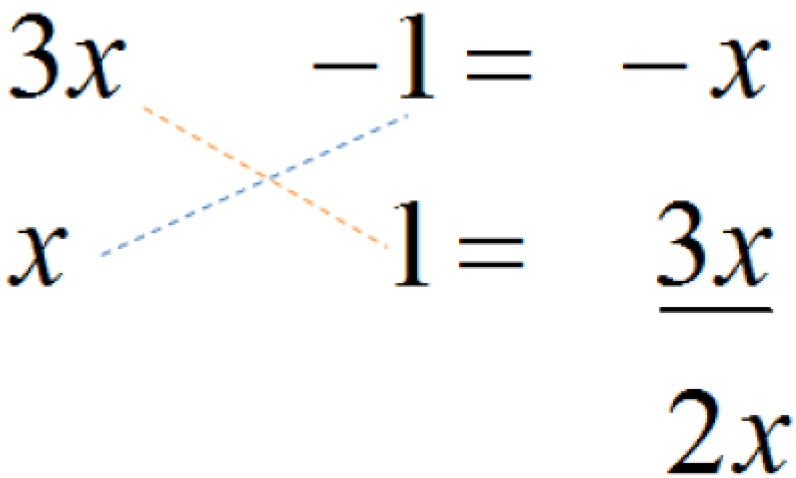
Step 3: Set up the brackets with terms for a and c arranged vertically to see which factors would add up to the middle term by cross multiplying. The factors are read horizontally.

Let's try factors of $(3x + 1)$ and $(x - 1)$. By cross multiplying we are working out what the middle term will be when the brackets are expanded.

$$\begin{array}{lrcl} 3x & +1 & = & x \\ x & -1 & = & \underline{-3x} \\ & & & -2x \end{array}$$

We see that the middle term would work out to be $-2x$ instead of $2x$. So these are not the correct factors of $3x^2 + 2x - 1$.

Let's try $(3x - 1)$ and $(x + 1)$.



We see that the factors of $(3x-1)$ and $(x+1)$ will give us a middle term of $2x$ when expanded and a constant term of -1. So these are the correct factors.

Step 4: Write the final answer.

$3x^2 + 2x - 1 = (3x-1)(x+1)$.

Exercise 2.4

Factorise by trial and error:

1. $x^2 + 7x - 8$
2. $2x^2 + 4x - 6$
3. $2y^2 + 5y - 3$

The full solutions are at the end of the unit.

Factorising quadratic trinomials by grouping

The method of factorising a quadratic trinomial by trial and error can take a while to master. Now, we will look at an alternative method to factorise trinomials. You may choose whichever method you find the easiest to use to factorise quadratic trinomials.

Example 2.4

Factorise: $2x^2 + x - 6$

Solution

Step 1: Write down the values of a, b and c.

$a = 2,\ \ b = 1,\ \ c = -6$

Step 2: Determine $a \cdot c$ and list the factors of $a \cdot c$. In this case $ac = -12$.

Factors of -12
$1, -12$
$-12, 1$
$2, -6$
$-6, 2$
$3, -4$
$-3, 4$

Step 3: Find p, q a pair of factors of ac with a sum of b.

Factors of -12	Sum of factors
$1, -12$	$1 - 12 = -11$
$-1, 12$	$+11$
$2, -6$	-4
$-2, 6$	$+4$
$3, -4$	-1
$-3, 4$	$+1$

For $b = 1$, $p = -3$ and $q = 4$.

Step 4: Rewrite the original expression as $ax^2 + px + qx + c$ and factorise by grouping in pairs.

$$2x^2 - 3x + 4x - 6 = (2x^2 - 3x) + (4x - 6)$$
$$= x(2x - 3) + 2(2x - 3)$$
$$= (2x - 3)(x + 2)$$

Step 5: Write the final answer.

$2x^2 + x - 6 = (2x - 3)(x + 2)$

Note: You can check your answer by multiplying the brackets.

Check your understanding of this method by trying the exercise below.

Exercise 2.5

Factorise the following by using the method of grouping:

1. $x^2 - 7x + 6$
2. $2x^2 + 9x + 9$
3. $5x^2 + 7x - 6$

The full solutions are at the end of the unit.

Summary

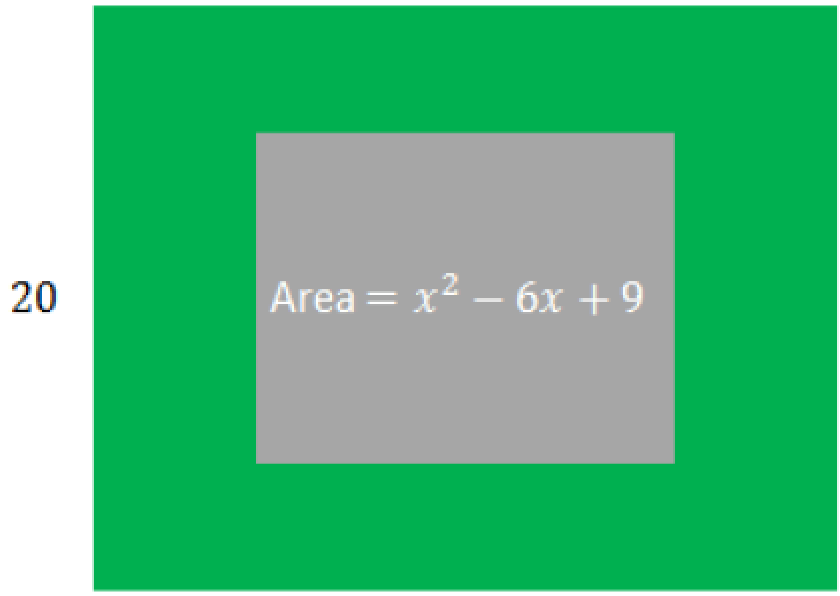
In this unit you have learnt the following:

- How to find the highest common factor of a polynomial.
- How to find the factors of a polynomial by grouping in pairs.
- How to factorise a difference of two squares.
- How to factorise a quadratic trinomial in two different ways.

Assessment

Suggested time to complete: 30 minutes

1. Factorise the following fully:
   a. $q^2y - 2q^3y$
   b. $6x^2yz + 78x^2y^2z - 36x^2yz^2$
   c. $3x - 6y - 3xy + 6y^2$
   d. $x^3 - 3x - 4x^2 + 12$
   e. $a^4 - 1$
   f. $(x + y)^2 - 16y^2$
   g. $2y^2 - y - 21$
   h. $6x^2 + xy - 12y^2$
   i. $2m^2 - 40m + 200$
2. A school is creating a pitch in the centre of the school grounds. The school ground is a square with side length 20 m as shown below. The pitch will be a square plot with an area of $x^2 - 6x + 9 \text{ m}^2$.



Find the length of the pitch by factorising.

The full solutions are at the end of the unit.

Unit 2: Solutions

Exercise 2.1

1.
$$\begin{aligned} -ax^2 + bx + x^3 &= x(-ax + b + x^2) \text{ We can write with descending powers of } x. \\ &= x(x^2 - ax + b) \end{aligned}$$

2. $a - b = (a - b)$

3. $-a + b = -1(a - b)$
$$\begin{aligned} (a-b)\,x - a + b &= (a-b)\,x - 1(a-b) \\ &= (a-b)(x-1) \end{aligned}$$

4.
$$\begin{aligned} -12a + 24a^2 - 3 &= 3(-4a + 8a^2 - 1) \\ &= 3(8a^2 - 4a - 1) \end{aligned}$$

Back to Exercise 2.1

Exercise 2.2

1.
$$\begin{aligned} x^2 - 6x + 5x - 30 &= x(x-6) + 5(x-6) \\ &= (x-6)(x+5) \end{aligned}$$

2.
$$\begin{aligned} 6t - 15s + 2yt - 5ys &= 3(2t - 5s) + y(2t - 5s) \\ &= (2t - 5s)(3 + y) \end{aligned}$$

3.

$$\begin{aligned} ab - a^2 - a + b &= a(b-a) + 1(b-a) \\ &= (b-a)(a+1) \end{aligned}$$

OR

$$\begin{aligned} ab - a^2 - a + b &= -a^2 - a + ab + b \\ &= -a(a+1) + b(a+1) \\ &= (a+1)(b-a) \end{aligned}$$

4. Here we need to group three sets of binomial together to find a common bracket.

$$\begin{aligned} 3ax + bx - 3ay - by - 9a - 3b &= (3ax + bx) + (-3ay - by) + (-9a - 3b) \\ &= x(3a+b) - y(3a+b) - 3(3a+b) \\ &= (a+b)(x-y-3) \end{aligned}$$

Back to Exercise 2.2

Exercise 2.3

1. $1 - x^2 = (1-x)(1+x)$
2. Remember to always check for the HCF first when factorising.

$$\begin{aligned} 8x^2 - 18y^2 &= 2(4x^2 - 9y^2) \\ &= 2(2x-3y)(2x+3y) \end{aligned}$$

3. We can rearrange $-36 + t^2$ as $t^2 - 36$.
$t^2 - 36 = (t-6)(t+6)$
4. Take out the common factor$(a^2 - 9)$.
$(a^2 - 9)(2a - 7)$
You must go one step further and fully factorise the expression. Factorise the difference of squares in the first bracket.
$(a^2 - 9)(2a - 7) = (a+3)(a-3)(2a-7)$

Back to Exercise 2.3

Exercise 2.4

1. $x^2 + 7x - 8 = (x+8)(x-1)$
2. Take out the HCF before factorising the trinomial.

$$\begin{aligned} 2x^2 + 4x - 6 &= 2(x^2 + 2x - 3) \\ &= 2(x+3)(x-1) \end{aligned}$$

3. $2y^2 + 5y - 3 = (2y-1)(y+3)$

Back to Exercise 2.4

Exercise 2.5

1.

$$\begin{aligned} &a = 1, b = -7, c = 6 \\ &ac = 6 \end{aligned}$$

Factors of 6	Sum of factors
$1, 6$	7
$-1, -6$	-7
$2, 3$	5
$-2, -3$	-5

$p = -1, q = -6$

$$\begin{aligned} x^2 - 7x + 6 &= x^2 - x - 6x + 6 \\ &= x(x-1) - 6(x-1) \\ &= (x-1)(x-6) \end{aligned}$$

2.

$$\begin{aligned} 2x^2 + 9x + 9 &= 2x^2 + 6x + 3x + 9 \\ &= 2x(x+3) + 3(x+3) \\ &= (2x+3)(x+3) \end{aligned}$$

3.

$$\begin{aligned} 5x^2 + 7x - 6 &= 5x^2 - 3x + 10x - 6 \\ &= x(5x-3) + 2(5x-3) \\ &= (5x-3)(x+2) \end{aligned}$$

Back to Exercise 2.5

Unit 2: Assessment

1.
   a. $q^2y - 2q^3y = q^2y(1-2q)$
   b. $6x^2yz + 78x^2y^2z - 36x^2yz^2 = 6x^2yz(1 + 13y - 6z)$
   c. $$\begin{aligned} 3x - 6y - 3xy + 6y^2 &= 3(x - xy - 2y + 2y^2) \\ &= 3\left[x(1-y) - 2y(1-y)\right] \\ &= 3(1-y)(x-2y) \end{aligned}$$
   d. $$\begin{aligned} x^3 - 3x - 4x^2 + 12 &= x(x^2-3) - 4(x^2-3) \\ &= (x^2-3)(x-4) \end{aligned}$$
   e. $$\begin{aligned} a^4 - 1 &= (a^2-1)(a^2+1) \\ &= (a-1)(a+1)(a^2+1) \end{aligned}$$
   f. $$\begin{aligned} (x+y)^2 - 16y^2 &= \left(\sqrt{(x+y)^2} - \sqrt{16y^2}\right)\left(\sqrt{(x+y)^2} + \sqrt{16y^2}\right) \\ &= (x+y-4y)(x+y+4y) \\ &= (x-3y)(x+5y) \end{aligned}$$
   g. $2y^2 - y - 21 = (2y-7)(y+3)$
   h. $6x^2 + xy - 12y^2 = (2x+3y)(3x-4y)$
   i. $$\begin{aligned} 2m^2 - 40m + 200 &= 2(m^2 - 20m + 100) \\ &= 2(m-10)(m-10) \end{aligned}$$

2. The pitch is a square therefore its lengths are equal.

$$
\begin{aligned}
\text{A} &= l \times l \\
l^2 &= x^2 - 6x + 9 \\
&= (x-3)(x-3) \\
&= (x-3)^2 \\
\therefore l &= x - 3
\end{aligned}
$$

Back to Unit 2: Assessment

Media Attributions

Unit 3: Simplification of algebraic fractions

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Unit outcomes

By the end of this unit you will be able to:

- Simplify algebraic fractions with monomial denominators.

What you should know

Before you start this unit, make sure you can:

- Simplify algebraic expressions and factorise. To revise this, go over Units 1 and 2 of this subject outcome.
- Add, subtract, multiply and divide fractions.

Try this short self-assessment to make sure you are ready for this unit.

- Calculate (show all working):

  1. $\dfrac{3}{20}+\dfrac{2}{5}$
  2. $\dfrac{5}{8}+\dfrac{2}{5}$
  3. $\dfrac{13}{15}-\dfrac{2}{5}$
  4. $\dfrac{12}{27}\div\dfrac{1}{9}$
  5. $\dfrac{2}{3}\left(\dfrac{3}{4}+\dfrac{7}{10}\right)$

- *Solutions*

  1. $$\begin{aligned}\frac{3}{20}+\frac{2}{5} &= \frac{3}{20}+\frac{2\times 4}{5\times 4} \quad \text{(the LCD is 20)}\\ &= \frac{3+8}{20}\\ &= \frac{11}{20}\end{aligned}$$
  2.

$$\begin{aligned}\frac{5}{8}+\frac{2}{5} &= \frac{(5\times 5)+(2\times 8)}{40} \quad \text{(the LCD is 40)}\\ &= \frac{25+16}{40}\\ &= \frac{41}{40}\end{aligned}$$

3.

$$\begin{aligned}\frac{13}{15}-\frac{2}{5} &= \frac{13-2\times 3}{15}\\ &= \frac{7}{15}\end{aligned}$$

4.

$$\begin{aligned}\frac{12}{27}\div\frac{1}{9} &= \frac{12}{27}\times\frac{9}{1}\\ &= \frac{12}{3}\\ &= 4\end{aligned}$$

5.

$$\begin{aligned}\frac{2}{3}\left(\frac{3}{4}+\frac{7}{10}\right) &= \left(\frac{2}{3}\times\frac{3}{4}\right)+\left(\frac{2}{3}\times\frac{7}{10}\right)\\ &= \frac{2}{4}+\frac{14}{30}\\ &= \frac{1}{2}+\frac{7}{15}\\ &= \frac{(1\times 15)+(7\times 2)}{30}\\ &= \frac{15+14}{30}\\ &= \frac{29}{30}\end{aligned}$$

Introduction

So far you would have worked with numerical fractions similar to the ones in the short assessment at the beginning of this unit. An algebraic fraction simply means there are variables in the numerator or the denominator of a fraction. We can apply the properties of numerical fractions to algebraic fractions.

A **rational expression** has a polynomial in the numerator and denominator. For example $\frac{x^2+2x}{x+2}$ is a rational expression with a binomial in the numerator and a binomial in the denominator. In this unit, the algebraic fractions you will learn to simplify will only have monomials (single terms) in the denominator. $\frac{1}{x}$, $\frac{5x}{y}$ and $\frac{3}{xy}$ are examples of algebraic fractions with monomial denominators.

Adding and subtracting simple algebraic fractions

Remember the following techniques for working with numerical fractions.

- $\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}$ $(b\neq 0)$

- $\frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}$ $(b \neq 0)$
- $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$ $(b \neq 0; d \neq 0)$
- $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$ $(b \neq 0; d \neq 0; c \neq 0)$

Let's see how we apply these to algebraic fractions.

Example 3.1

Add:

1. $\frac{5}{x} + \frac{3}{y}$
2. $\frac{2}{a} + \frac{a}{3}$
3. $\frac{6}{y} - \frac{5}{xy}$

Solutions

1. Just as we do with numerical fractions, we must find the lowest common denominator (LCD) to add fractions with different denominators. The LCD of $\frac{5}{x} + \frac{3}{y}$ is the product of the denominators $x \cdot y$. We then multiply each expression by the appropriate form of 1 to get xy as the denominator for each fraction.

$$\begin{aligned}\frac{5}{x} + \frac{3}{y} &= \frac{5}{x} \cdot \frac{y}{y} + \frac{3}{y} \cdot \frac{x}{x} \\ &= \frac{5y}{xy} + \frac{3x}{xy} \qquad \text{Now that the expressions have the same denominator, we simply add the numerators.} \\ &= \frac{5y + 3x}{xy}\end{aligned}$$

Note: Multiplying by $\frac{y}{y}$ or $\frac{x}{x}$ does not change the value of the original expression because any number divided by itself is 1 and multiplying an expression by 1 gives the original expression.

2. We have to rewrite the fractions so they share a common denominator before we are able to add. The LCD of this algebraic fraction is $3a$.

$$\begin{aligned}\frac{2}{a} + \frac{a}{3} &= \frac{2}{a} \cdot \frac{3}{3} + \frac{a}{3} \cdot \frac{a}{a} \\ &= \frac{6}{3a} + \frac{a^2}{3a} \\ &= \frac{6 + a^2}{3a}\end{aligned}$$

3.

$$\begin{aligned}\frac{6}{y}-\frac{5}{xy} &= \frac{6}{y}\cdot\frac{x}{x}-\frac{5}{xy}\\ &= \frac{6x}{xy}-\frac{5}{xy}\\ &= \frac{6x-5}{xy}\end{aligned}$$

Exercise 3.1

Simplify:

1. $\frac{a}{4}-\frac{a-2}{3}$
2. $\frac{5}{a}+\frac{3}{ab}$
3. $\frac{7}{3x}-\frac{1}{2x}$

The full solutions are at the end of the unit.

Multiplying and dividing simple algebraic fractions

Multiplication and division of algebraic fractions works in the same way as division and multiplication of other fractions.

Remember that $\frac{2}{x}\times\frac{x}{3}=\frac{2\cancel{x}}{3\cancel{x}}=\frac{2}{3}$ by cancelling the common factors. Another way of looking at this is to realise that $\frac{x}{x}=1$.

To simplify $\frac{1}{x}\div\frac{3}{x}$ multiply the first expression by the reciprocal of the second.

So we get, $\frac{1}{x}\times\frac{x}{3}$. Once the division expression has been rewritten as a multiplication expression, we can multiply as we did before.

$$\begin{aligned}\frac{1}{x}\times\frac{x}{3} &= \frac{\cancel{x}}{3\cancel{x}}\\ &= \frac{1}{3}\end{aligned}$$

Let's have a look at a few more examples.

Example 3.2

Simplify:

1. $\dfrac{1}{b} \times \dfrac{2}{b} + \dfrac{3}{b^2}$
2. $\dfrac{2y}{ab} \div \dfrac{4}{a}$
3. $\dfrac{y + \dfrac{1}{x}}{\dfrac{x}{y}}$

Solutions

1. Remember the order of operations of BODMAS you must multiply before you add.
$$\begin{aligned}\frac{1}{b} \times \frac{2}{b} + \frac{3}{b^2} &= \frac{1 \times 2}{b \times b} + \frac{3}{b^2} \\ &= \frac{2}{b^2} + \frac{3}{b^2} \\ &= \frac{5}{b^2}\end{aligned}$$
2. When we divide fractions, we tip the second expression around (find the reciprocal) and multiply.
$$\begin{aligned}\frac{2y}{ab} \div \frac{4}{a} &= \frac{\cancel{2}y}{\cancel{a}b} \times \frac{\cancel{a}}{2\cancel{4}} \text{ Now divide the common factors of } a \text{ and } 2 \\ &= \frac{y}{2b}\end{aligned}$$
3. Begin by combining the expressions in the numerator into one expression.
$$\begin{aligned}y + \frac{1}{x} &= y \cdot \frac{x}{x} + \frac{1}{x} \\ &= \frac{yx}{x} + \frac{1}{x} \\ &= \frac{xy + 1}{x}\end{aligned}$$
Now the numerator is a single rational expression and the denominator is a single rational expression.
$$\frac{\dfrac{xy + 1}{x}}{\dfrac{x}{y}}$$
We can rewrite the above expression as division, and then multiplication.
$$\begin{aligned}\frac{xy + 1}{x} \div \frac{x}{y} &= \frac{xy + 1}{x} \times \frac{y}{x} \\ &= \frac{y(xy + 1)}{x^2}\end{aligned}$$

As you can see from the example, algebraic fractions can be simplified by cancelling common factors in the numerator and denominator.

Exercise 3.2

Simplify:

1. $\dfrac{5x}{6y^2} \times \dfrac{3y}{15x}$
2. $\dfrac{x+2}{4z^2} \div \dfrac{1}{4z^3}$
3. $\dfrac{\dfrac{x}{y} - \dfrac{y}{x}}{y}$

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to define an algebraic fraction.
- How to simplify simple algebraic fractions with monomial denominators.
- How to add and subtract algebraic fractions with monomial denominators.
- How to multiply and divide algebraic fractions with monomial denominators.
- How to simplify complex fractions with monomial denominators.

Assessment

Suggested time to complete: 15 minutes

1. Simplify:
   a. $\dfrac{x}{y} + \dfrac{2}{x}$
   b. $\dfrac{a}{2b} - \dfrac{2b}{9a}$
   c. $\dfrac{1}{2} - \dfrac{b}{3a}$
2. Enrichment: Use your answers from b) and c) to find:
   $\dfrac{\dfrac{a}{2b} - \dfrac{2b}{9a}}{\dfrac{1}{2} - \dfrac{b}{3a}}$

The full solutions are at the end of the unit.

Unit 3: Solutions

Exercise 3.1

1.
$$\begin{aligned}\frac{a}{4}-\frac{a-2}{3} &= \frac{a\times 3-(a-2)\times 4}{12}\\ &= \frac{3a-4a+8}{12}\\ &= \frac{-a+8}{12}\end{aligned}$$

2.
$$\begin{aligned}\frac{5}{a}+\frac{3}{ab} &= \frac{5\cdot b+3}{ab} \text{ LCD is } ab \text{ so only the 5 is multiplied by } b\\ &= \frac{5b+3}{ab}\end{aligned}$$

3.
$$\begin{aligned}\frac{7}{3x}-\frac{1}{2x} &= \frac{7}{3x}\cdot\frac{2}{2}-\frac{1}{2x}\cdot\frac{3}{3} \quad \text{LCD is } 6x.\ 3x\times 2=6x \text{ and } 2x\times 3 \quad =6x\\ &= \frac{14}{6x}-\frac{3}{6x}\\ &= \frac{14-3}{6x}\\ &= \frac{11}{6x}\end{aligned}$$

Back to Exercise 3.1

Exercise 3.2

1.
$$\begin{aligned}\frac{5x}{6y^2}\times\frac{3y}{15x} &= \frac{\cancel{5}\,\cancel{x}}{2y}\times\frac{\cancel{3}\,\cancel{y}}{3\,\cancel{x}}\\ &= \frac{1}{6y}\end{aligned}$$

2.
$$\begin{aligned}\frac{x+2}{4z^2}\div\frac{1}{4z^3} &= \frac{x+2}{4z^2}\times\frac{4z^3}{1}\\ &= \frac{x+2}{\cancel{4}\,\cancel{z}^{\cancel{2}}}\times\frac{\cancel{4}\,z^{\cancel{3}}}{1}\\ &= z(x+2)\end{aligned}$$

3.
$$\frac{\frac{x}{y}-\frac{y}{x}}{y}=\frac{x}{y}-\frac{y}{x}\div\frac{1}{y}$$

Work with the numerator first.

$$\begin{aligned}\frac{x}{y}-\frac{y}{x} &= \frac{x\cdot x-y\cdot y}{xy}\\ &= \frac{x^2-y^2}{xy}\end{aligned}$$

Then bring back the divisor.

$$\frac{x^2-y^2}{xy} \div \frac{1}{y} = \frac{x^2-y^2}{x\cancel{y}} \times \frac{\cancel{y}}{1}$$
$$= \frac{x^2-y^2}{x}$$

Back to Exercise 3.2

Assessment

Simplify:

1.
$$\frac{x}{y} + \frac{2}{x} = \frac{x \cdot x + 2 \cdot y}{xy}$$
$$= \frac{x^2+2y}{xy}$$

a.
$$\frac{a}{2b} - \frac{2b}{9a} = \frac{9 \cdot a \cdot a}{18ab} - \frac{2b \cdot (2b)}{18ab}$$
$$= \frac{9a^2-4b^2}{18ab}$$

b. $\frac{1}{2} - \frac{b}{3a} = \frac{3a-2b}{6a}$

2. From b) and c) we get:
$$\frac{\frac{a}{2b} - \frac{2b}{9a}}{\frac{3}{2} - \frac{b}{3a}} = \frac{9a^2-4b^2}{18ab} \div \frac{3a-2b}{6a}$$
Next, we multiply by the reciprocal:
$$\frac{9a^2-4b^2}{18ab} \times \frac{6a}{3a-2b}$$
To simplify even further we must factorise the numerator of the first expression using a difference of two squares.
$$\frac{9a^2-4b^2}{18ab} \times \frac{6a}{3a-2b} = \frac{(3a-2b)(3a+2b)}{18ab} \times \frac{6a}{(3a-2b)}$$
$$= \frac{(3a+2b)}{3\ \cancel{1}\ \cancel{8}\ \cancel{a}\ b} \times \frac{\cancel{6}\ \cancel{a}}{1} \quad \text{Factors of } (3a-2b) \text{ cancel}$$
$$= \frac{3a+2b}{3b}$$

Back to Unit 3: Assessment

SUBJECT OUTCOME IV

FUNCTIONS AND ALGEBRA: SOLVE ALGEBRAIC EQUATIONS AND INEQUALITIES

Subject outcome 2.3

Solve algebraic equations and inequalities

Learning outcomes

- Solve linear equations.
- Solve quadratic equations by factorisation.
- Solve exponential equations of the form $ka^x = m$ (where x is an integer) by using the laws of exponents.
- Solve inequalities in one variable and represent the solution in set builder notation, interval notation and on a number line.
- Solve simultaneous equations with two unknowns algebraically and graphically, where both equations are linear.

Unit outcomes: Unit 1: Solve linear and quadratic equations

By the end of this unit you will be able to:

- Solve equations with a single variable which are called linear equations
- Solve equations with a single variable that is squared (quadratic equations) by factorisation.

Unit outcomes: Unit 2: Solve exponential and literal equations

By the end of this unit you will be able to:

- Solve exponential equations of the form $ka^{x+p} = m$ by using the laws of exponents.
- Solve literal equations.

Unit outcomes: Unit 3: Solve algebraic inequalities

By the end of this unit you will be able to:

- Solve linear inequalities with a single unknown or variable.
- Represent the solution to linear inequalities:
  - In set builder notation
  - In interval notation
  - On the number line.

Unit outcomes: Unit 4: Solve simultaneous equations

By the end of this unit you will be able to:

- Solve systems of simultaneous equations where both equations are linear equations algebraically by means of:
  - Elimination
  - Substitution.
- Solve systems of simultaneous equations where both equations are linear equations graphically by finding the point of intersection of the functions.

Unit 1: Solve linear and quadratic equations

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Unit outcomes

By the end of this unit you will be able to:

- Solve equations with a single variable which are called linear equations
- Solve equations with a single variable that is squared (quadratic equations) by factorisation.

What you should know

Before you start this unit, make sure you can:

- Simplify algebraic expressions. Work through Subject outcome 2.2, Unit 1 if you need to revise this.
- Factorise. Work through Subject outcome 2.2, Unit 2 if you need to revise this.
- Simplify algebraic fractions. Work through Subject outcome 2.2, Unit 3 if you need to revise this.

Here is a short self-assessment to make sure you have the skills you need to proceed with this unit.

1. Simplify $\left(\frac{x}{3}-\frac{3}{x}\right)\left(\frac{x}{4}+\frac{4}{x}\right)$
2. Factorise $6x^2+xy-12y^2$
3. Simplify $\frac{a}{2b}-\frac{2b}{9a}$

Solutions

1.
$$\begin{aligned}\left(\frac{x}{3}-\frac{3}{x}\right)\left(\frac{x}{4}+\frac{4}{x}\right) &= \frac{x}{3}\times\frac{x}{4}+\frac{x}{3}\times\frac{4}{x}-\frac{3}{x}\times\frac{x}{4}-\frac{3}{x}\times\frac{4}{x}\\ &= \frac{x^2}{12}+\frac{4x}{3x}-\frac{3x}{4x}-\frac{12}{x^2}\\ &= \frac{x^2}{12}+\frac{4}{3}-\frac{3}{4}-\frac{12}{x^2}\\ &= \frac{x^2}{12}+\frac{7}{12}-\frac{12}{x^2}\end{aligned}$$
2. $6x^2+xy-12y^2=(2x+3y)(3x-4y)$
3.

$$
\begin{aligned}
\frac{a}{2b}-\frac{2b}{9a} &= \frac{9a\times a}{9a\times 2b}-\frac{2b\times 2b}{2b\times 9a}\\
&= \frac{9a^2}{18ab}-\frac{4b^2}{18ab}\\
&= \frac{9a^2-4b^2}{18ab}\\
&= \frac{(3a+2b)(3a-2b)}{18ab}
\end{aligned}
$$

Introduction

Equations are at the heart of almost all Mathematics. You might even say that Mathematics is all about solving equations. We use equations to find the value (or values) of some unknown quantity or quantities.

Some equations are simple to solve like $x+1=3$. Some equations are famously complicated and difficult to solve like this one from the theory of General Relativity $G_{\mu v}=\frac{8\pi G}{c^4}T_{\mu v}$. Some equations seem to have taken on a life of their own like $E=mc^2$.

You may already be familiar with how to solve equations, but just in case you are not, this unit is going to start from the very beginning by asking, 'what is an equation?'.

Solving simple linear equations

Before we start solving equations, let's pause and think for a moment about what an equation is. What does the word 'equal' mean? What does it mean if we put two expressions on either side of an equal sign?

In an equation, two mathematical expressions are equal to each other. In this way, we are told that the value of whatever is on the left-hand side of the equal sign is equal in value to whatever is on the right-hand side of the equal sign.

We can think of the left-hand side and the right-hand side of the equation as being balanced (see Figure 1).

Figure 1: Balanced scales

Now, if we have the expression $2x+3$ on the left-hand side and the expression 5 on the right-hand side, the equation would be $2x+3=5$ and the equation scale would look like Figure 2. This is called a linear equation because the exponent on the x is 1.

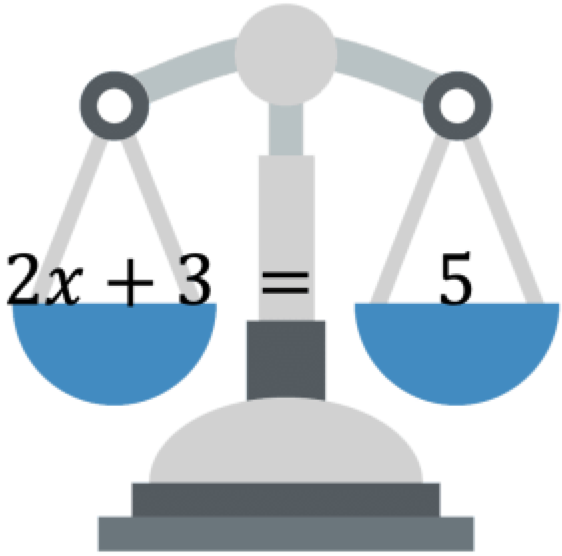


Figure 2: Equation scale for $2x + 3 = 5$

There is one basic rule that always applies to all equations. **We must keep the scales balanced.** We can add, subtract, divide and multiply whatever we like to one side of the equation so long as we perform the exact same operation to the other side of the equation.

Now, how would we solve for x in $2x + 3 = 5$?

In this case, we need to get the variable, x, all alone on the left-hand side (LHS) of the equation. The first thing we can do is to subtract 3 from the LHS. But whatever we do to the LHS we have to do to the right-hand side (RHS).

So, our equation becomes $2x + 3 - 3 = 5 - 3$. We can simplify both sides of the equation to get $2x = 2$.

Now, we have x multiplied by 2 on the LHS so we can divide the LHS by 2 to get x on its own. But whatever we do to the LHS, we have to do to the RHS, so our equation becomes $\frac{2x}{2} = \frac{2}{2}$ which we can simplify to get $x = 1$. And we have solved for x.

Generally, this is how you should set all your equations out. You don't always have to get the variables onto the left-hand side though. You can also get them onto the right-hand side.

$$\begin{aligned} 2x + 3 &= 5 \\ \therefore 2x + 3 - 3 &= 5 - 3 \\ \therefore 2x &= 2 \\ \therefore \frac{2x}{2} &= \frac{2}{2} \\ \therefore x &= 1 \end{aligned}$$

The $\therefore$ symbol means 'therefore'. Can you see how we have just constructed a logical Mathematical argument? As you get better at solving equations, you will be able to leave out some of these steps.

Example 1.1

Solve for x:

1. $4x + 5 = 17$
2. $3(x + 2) = -4(x + 9)$

Solutions

1.
$$\begin{aligned} 4x + 5 &= 17 \\ \therefore 4x + 5 - 5 &= 17 - 5 && \text{We subtract 5 from both sides} \\ \therefore 4x &= 12 \\ \therefore \frac{4x}{4} &= \frac{12}{4} && \text{We divide both sides by 4} \\ \therefore x &= 3 \end{aligned}$$

   You should always check that you have solved the equation correctly by substituting your answer into the original equation to see if the LHS is equal to the RHS.

$$\begin{aligned} \text{LHS: } 4(3) + 5 = 12 + 5 &= 17 \\ \text{RHS: } &17 \end{aligned}$$

   The LHS = the RHS. Therefore, the solution is correct.

2.
$$\begin{aligned} 3(x + 2) &= -4(x + 9) && \text{First multiply out the brackets} \\ \therefore 3x + 6 &= -4x - 36 \\ \therefore 3x + 4x + 6 &= -4x + 4x - 36 && \text{Now we add } 4x \text{ to both sides} \\ \therefore 7x + 6 &= -36 \\ \therefore 7x + 6 - 6 &= -36 - 6 && \text{Now we subtract 6 from both sides} \\ \therefore 7x &= -42 \\ \therefore \frac{7x}{7} &= \frac{-42}{7} && \text{Divide both sides by 7} \\ \therefore x &= -6 \end{aligned}$$

   Check that the solution is correct.

$$\begin{aligned} \text{LHS: } & 3(-6 + 2) = 3(-4) = -12 \\ \text{RHS: } & -4(-6 + 9) = -4(3) = -12 \end{aligned}$$

   The LHS = the RHS, so the solution is correct.

Remember, both of these equations we have just solved are called **linear equations** because the unknown, x, has an exponent of 1.

Note

When solving equations, whatever you do to the one side of the equation, you must also do to the other side of the equation. This keeps the equation balanced and the equality between the LHS and the RHS.

Exercise 1.1

Solve for the unknown in the following equations.

1. $2x - 6 = 8$
2. $4 - 2y = 6$
3. $-5x - 4 = 11 + 2x + 3$
4. $3(x + 7) = -2(x - 7)$
5. $7a - (3a + 10) = 6 - 2(12 - 9a)$

The full solutions are at the end of the unit.

Solving linear equations with fractions

All the equations we have solved so far have had no fractions in the original equation. When there is a fraction, we need to 'undo' it by multiplying both sides of the equation by the lowest common denominator so that we can 'cancel the denominator' or create a denominator of 1. Look at the next example to see what we mean.

Example 1.2

Solve for x:

1. $\dfrac{x+2}{3} = 5$
2. $\dfrac{4x-2}{3} = \dfrac{3x-1}{2}$
3. $\dfrac{5x}{3} - 2 = \dfrac{3x}{5}$

Solutions

1. We need to multiply both sides of the equation by 3 so that we can create a denominator of 1 on the LHS.

$$\begin{aligned} \frac{x+2}{3} &= 5 \\ \therefore 3 \times \left(\frac{x+2}{3}\right) &= 3 \times 5 \quad \text{Multiply both sides by } 3 \\ \therefore x + 2 &= 15 \\ \therefore x &= 15 - 2 \quad \text{Subtract } 2 \text{ from both sides} \\ \therefore x &= 13 \end{aligned}$$

Check the solution:

$$\begin{aligned} \text{LHS: } \frac{13+2}{3} &= \frac{15}{3} = 5 \\ \text{RHS: } 5 \end{aligned}$$

The LHS = the RHS, so the solution is correct.

2. In this equation, we have two denominators. The best way to deal with both denominators at the same time, is to multiply both sides of the equation by the lowest common denominator (LCD). In this case, the lowest common denominator is 6.

$$\begin{aligned}
\frac{4x-2}{3} &= \frac{3x-1}{2} \\
\therefore 6 \times \left(\frac{4x-2}{3}\right) &= 6 \times \left(\frac{3x-1}{2}\right) \quad \text{Multiply both sides by the LCD} \\
\therefore 2(4x-2) &= 3(3x-1) \quad \text{Simplify both sides: } \frac{6}{3} = 2 \text{ and } \frac{6}{2} = 3 \\
\therefore 8x-4 &= 9x-3 \quad \text{Multiply out the brackets} \\
\therefore 8x-9x-4 &= -3 \quad \text{Subtract } 9x \text{ from both sides} \\
\therefore -x &= -3+4 \quad \text{Add 4 to both sides} \\
\therefore -x &= 1 \quad \text{Multiply both sides by } -1 \\
\therefore x &= -1
\end{aligned}$$

Check the solution:

$$\begin{aligned}
\text{LHS: } &\frac{4 \times (-1) - 2}{3} = \frac{-6}{3} = -2 \\
\text{RHS: } &\frac{3 \times (-1) - 1}{2} = \frac{-4}{2} = -2
\end{aligned}$$

The LHS = the RHS, so the solution is correct.

3. In this question, the LCD is 15 so we need to multiply both sides of the equation by 15.

$$\begin{aligned}
\frac{5x}{3} - 2 &= \frac{3x}{5} \\
\therefore 15 \times \left(\frac{5x}{3} - 2\right) &= 15 \times \left(\frac{3x}{5}\right) \quad \text{Multiply both sides by the LCD of 15} \\
\therefore 15 \times \left(\frac{5x}{3}\right) - 30 &= 15 \times \left(\frac{3x}{5}\right) \quad \text{Simplify} \\
\therefore 5(5x) - 30 &= 3(3x) \\
\therefore 25x - 30 &= 9x \\
\therefore 25x - 9x - 30 &= 0 \quad \text{Subtract } 9x \text{ from both sides} \\
\therefore 16x &= 30 \quad \text{Add 30 to both sides} \\
\therefore \frac{16x}{16} &= \frac{30}{16} \quad \text{Divide both sides by 16} \\
\therefore x &= \frac{30}{16} = \frac{15}{8}
\end{aligned}$$

Check the solution:

$$\text{LHS: } \frac{5 \times \frac{15}{8}}{3} - 2 = \frac{\frac{75}{8}}{3} - 2 = \left(\frac{75}{8} \times \frac{1}{3}\right) - 2 = \frac{75}{24} - \frac{48}{24} = \frac{27}{24} = \frac{9}{8}$$

$$\text{RHS: } \frac{3 \times \frac{15}{8}}{5} = \frac{\frac{45}{8}}{5} = \frac{45}{8} \times \frac{1}{5} = \frac{45}{40} = \frac{9}{8}$$

The LHS = the RHS, so the solution is correct.

Exercise 1.2

Solve for the unknown in each of the following equations.

1. $\dfrac{x+2}{4} - \dfrac{x-6}{3} = 12$
2. $\dfrac{2-5a}{3} - 6 = \dfrac{4a}{3} + 2 - a$
3. $3 - \dfrac{y-2}{4} = 4$
4. $\dfrac{1}{5}(x-1) = \dfrac{1}{3}(x-2) + 3$

The full solutions are at the end of the unit.

You should feel confident about solving linear equations with fractions in them. However, in all the equations we have solved thus far, the denominators had only a single term in them, and they were all constants. How would we solve an equation like $\dfrac{4}{x} = \dfrac{2}{x-2}$ where we have a denominator with more than a single term and a variable? Have a look at the next example to see the solution.

Example 1.3

Solve: $\dfrac{4}{x} = \dfrac{2}{x-2}$

Solution

As before, we need to multiply both sides of the equation by the LCD. In this case, the LCD is $x(x-2)$. But before we do, there is something important to note. If we multiply both sides by $x(x-2)$, we need to make sure that we are not multiplying both sides by zero. This means that $x \neq 0$ and $x \neq 2$. We call these **restrictions**. If the solution to the equation turns out to be 0 or 2, we have to disregard it and say that there is no solution to the equation. This is just another way of saying that neither of the denominators in the original equation are allowed to be zero because we know that we cannot divide anything by zero. This is **undefined**.

$$\begin{aligned}
\frac{4}{x} &= \frac{2}{x-2} && x \neq 0; x \neq 2 \\
\therefore x(x-2) \times \left(\frac{4}{x}\right) &= x(x-2) \times \left(\frac{2}{x-2}\right) && \text{Multiply by the LCD of } x(x-2) \\
\therefore (x-2) \times 4 &= x \times 2 && \text{Simplify} \\
\therefore 4x - 8 &= 2x && \text{Subtract } 2x \text{ from both sides and add 8 to both sides} \\
\therefore 2x &= 8 && \text{Divide both sides by 2} \\
\therefore x &= 4
\end{aligned}$$

The solution does not violate either of our restrictions so we can accept it and check that it is correct.

Check the solution:

$$\text{LHS: } \frac{4}{4} = 1$$

$$\text{RHS: } \frac{2}{4-2} = \frac{2}{2} = 1$$

The LHS = the RHS, so the solution is correct.

Exercise 1.3

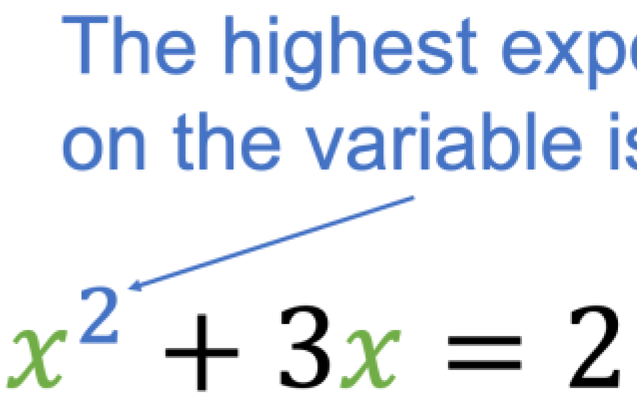
Solve the following equations, noting any restrictions.

1. $5 - \frac{7}{q} = \frac{2(q+4)}{q}$
2. $\frac{5}{2x} + \frac{1}{6x} = \frac{3}{x} + 2$
3. $\frac{2}{(b-5)} - \frac{4}{(b+5)} = \frac{3}{(b+5)}$

The full solutions are at the end of the unit.

Solving quadratic equations by factorisation

Besides linear equations, the other most common type of equation that you will have to solve is a **quadratic equation**. In quadratic equations, the biggest exponent on the unknown is 2.



$x^2 - 9 = 0$ and $x^2 + 4x = -3$ are more examples of quadratic equations. The highest power on the unknown in each case is 2. The standard form of a quadratic equation is $ax^2 + bx + c = 0$. As you will see in Example 1.4, quadratic equations must be written in this form before you solve.

All linear equations will have one solution at most. As we will see, quadratic equations have two solutions at most. We sometimes refer to the solutions of an equation as its **roots**. Linear equations have one root. Quadratic equations have two roots.

There are several methods we can use to solve quadratic equations but, in this unit, we will only look at solving quadratic equations using factorisation. When we use factorisation to solve any quadratic function, we have to rely on a law called the **zero product property**.

Think about this equation: $x(x-3)=0$. In order for the LHS to be equal to the RHS one or both of the factors on the LHS **have to be equal to zero**. In other words, $x=0$ or $(x-3)=0$ or they are both equal to zero. We can use this fact to solve this equation. Either:

$x=0$

or

$$\begin{aligned} x-3&=0 \\ \therefore x&=3 \end{aligned}$$

In other words, $x=0$ or $x=3$ are both solutions to the equation.

The zero product property states that if $A \times B = 0$ then $A=0$ or $B=0$ or both $A=0$ and $B=0$

Example 1.4

Solve for x, noting any restrictions:

1. $x^2-9=0$
2. $x^2-2x=8$
3. $-4x^2=-3x$
4. $2x^2=15-7x$
5. $3x=\dfrac{54}{2x}$
6. $\dfrac{3(x^2+1)+10x}{3x+1}=1$

Solutions

1. The first thing we always need to do when solving a quadratic equation by factorising (in other words using the zero product property) is to get the one side of the equation equal to zero. The equation is already in standard form, so we can start by factorising the other side of the equation.

$$\begin{aligned} x^2-9&=0 && \text{Factorise the difference of 2 squares} \\ \therefore (x+3)(x-3)&=0 && \text{Apply the zero product law} \\ \therefore x+3=0 &\text{ or } x-3=0 \\ \therefore x=-3 &\text{ or } x=3 \end{aligned}$$

It is a good idea to check both solutions:

$$\begin{aligned} x&=-3 \\ \text{LHS: } (-3)^2-9=9-9&=0 \\ \text{RHS: } &0 \end{aligned}$$

The LHS = the RHS, so the solution is correct.

$$\begin{aligned} x &= 3 \\ \text{LHS: } (3)^2 - 9 = 9 - 9 &= 0 \\ \text{RHS: } &0 \end{aligned}$$

The LHS = the RHS, so the solution is correct.

2. In this case, we first need to arrange the equation so that we have zero on the one side. We do this by subtracting 8 from both sides.

$$\begin{aligned} x^2 - 2x &= 8 && \text{Subtract 8 from both sides to get the RHS} = 0 \\ \therefore x^2 - 2x - 8 &= 0 && \text{Factorise the trinomial on the LHS} \\ \therefore (x-4)(x+2) &= 0 && \text{Apply the zero product law} \\ \therefore x - 4 = 0 &\text{ or } x + 2 = 0 \\ \therefore x = 4 &\text{ or } x = -2 \end{aligned}$$

Check both solutions:

$$\begin{aligned} x &= 4 \\ \text{LHS: } 4^2 - 2 \times 4 &= 8 \\ \text{RHS: } &8 \end{aligned}$$ The LHS = the RHS, so the solution is correct.

$$\begin{aligned} x &= -2 \\ \text{LHS: } (-2)^2 - 2(-2) = 4 + 4 &= 8 \\ \text{RHS: } &8 \end{aligned}$$

The LHS = the RHS, so the solution is correct.

3. Start by re-arranging the equation so that one side is equal to zero.

$$\begin{aligned} -4x^2 &= -3x && \text{Add } 3x \text{ to both sides to get 0 on one side} \\ \therefore -4x^2 + 3x &= 0 && \text{Multiply both sides by -1} \\ \therefore 4x^2 - 3x &= 0 && \text{Factorise the LHS by taking out a common factor} \\ \therefore x(4x-3) &= 0 && \text{Apply the zero product law} \\ \therefore x = 0 &\text{ or } 4x - 3 = 0 \\ \therefore x = 0 &\text{ or } 4x = 3 \\ \therefore x = 0 &\text{ or } x = \frac{3}{4} \end{aligned}$$

Don't forget to check both solutions.

4. Once again, we need to start by re-arranging the equation so that one side is equal to zero.

$$\begin{aligned} 2x^2 &= 15 - 7x && \text{Add } 7x \text{ to and subtract 15 from both sides} \\ \therefore 2x^2 + 7x - 15 &= 0 && \text{Factorise the trinomial} \\ \therefore (2x-3)(x+5) &= 0 && \text{Apply the zero product law} \\ \therefore 2x - 3 = 0 &\text{ or } x + 5 = 0 \\ \therefore 2x = 3 &\text{ or } x = -5 \\ \therefore x = \frac{3}{2} &\text{ or } x = -5 \end{aligned}$$

Don't forget to check both solutions.

5. $3x = \frac{54}{2x}, x \neq 0$

We know that we may never divide by zero. Therefore, we cannot allow the unknown in the denominator to assume any value that makes the denominator zero. This means that $2x \neq 0$ and therefore $x \neq 0$.

$$\therefore 6x^2 = 54$$
$$\therefore x^2 = 9$$
$$\therefore x = \pm\sqrt{9}$$
$$\therefore x = \pm 3$$

Or

$$\begin{aligned} 3x &= \frac{54}{2x}, x \neq 0 \\ \therefore 6x^2 &= 54 \\ \therefore 6x^2 - 54 &= 0 \\ \therefore x^2 - 9 &= 0 \\ \therefore (x+3)(x-3) &= 0 \\ \therefore x &= \pm 3 \end{aligned}$$

Don't forget to check the solution.

6. .

$$\begin{aligned} \frac{3(x^2+1)+10x}{3x+1} &= 1, \quad x \neq -\frac{1}{3} \\ \therefore 3(x^2+1)+10x &= 3x+1 \\ \therefore 3x^2+3+10x-3x-1 &= 0 \\ \therefore 3x^2+7x+2 &= 0 \\ \therefore (3x+1)(x+2) &= 0 \\ \therefore 3x+1 = 0 &\text{ or } x+2=0 \\ \therefore 3x = -1 &\text{ or } x = -2 \\ \therefore \cancel{x = -\frac{1}{3}} &\text{ or } x = -2 \end{aligned}$$

We have to disallow the solution $x = -\frac{1}{3}$ because of the restriction.

Don't forget to check the solution.

Note

To solve a quadratic equation by factorisation, follow these steps:

1. Get one side of the equation equal to zero (write the quadratic in standard form).
2. Factorise the other side of the equation.
3. Apply the zero product property to set each factor equal to zero.

4. Solve for x in each factor.
5. Check your solutions

Exercise 1.4

Solve the following quadratic equations, noting any restrictions.

1. $-x^2 - 7x = 12$
2. $x^2 - 49 = 0$
3. $x^2 + 6x - 14 = 13$
4. $3x^2 - 18x - 48 = 0$
5. $2x(x+1) - (x-3) = 6$
6. $(x+1)^2 = (2x+3)(x+1)$
7. $(x+2) = \dfrac{6x-12}{x-2}$
8. $\dfrac{4x}{4x+3} = -\dfrac{3}{x}$

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- What an equation is, and that you need to keep the equation balanced by always performing the same operation to both sides of the equal sign.
- How to solve simple linear equations.
- How to solve linear equations with fractions by multiplying both sides of the equation by the LCD.
- How to solve quadratic equations by factorisation and applying the zero product property.

Unit 1: Assessment

Suggested time to complete: 15 minutes

Solve for the unknown in each case:

1. $-7x = 8(1-x)$
2. $1 = \dfrac{3a-4}{2a+6}$

3. $\frac{1}{4}(x-1)-1\frac{1}{2}(3x+2)=0$
4. $\frac{-2}{(x^2-9)}+\frac{4}{(x+3)}=\frac{6}{(x-3)}$
5. $x^2-3=-2x$
6. $2(9c-4)=9c^2$
7. $\frac{3b}{b+2}+1=\frac{4}{b+1}$

The full solutions are at the end of the unit.

Unit 1: Solutions

Exercise 1.1

1.
$$\begin{aligned} 2x-6&=8\\ \therefore 2x-6+6&=8+6\\ \therefore 2x&=14\\ \therefore \frac{2x}{2}&=\frac{14}{2}\\ \therefore x&=7 \end{aligned}$$

2.
$$\begin{aligned} 4-2y&=6\\ \therefore 4-4-2y&=6-4\\ \therefore -2y&=2\\ \therefore \frac{-2y}{-2}&=\frac{2}{-2}\\ \therefore y&=-1 \end{aligned}$$

3.
$$\begin{aligned} -5x-4&=11+2x+3\\ \therefore -5x-4+4&=11+2x+3+4\\ \therefore -5x&=2x+18\\ \therefore -5x-2x&=2x-2x+18\\ \therefore -7x&=18\\ \therefore \frac{-7x}{-7}&=\frac{18}{-7}\\ \therefore x&=-\frac{18}{7} \end{aligned}$$

4.
$$\begin{aligned} 3(x+7)&=-2(x-7)\\ \therefore 3x+21&=-2x+14\\ \therefore 3x+21-21&=-2x+14-21\\ \therefore 3x+2x&=-2x+2x-7\\ \therefore 5x&=-7\\ \therefore x&=-\frac{7}{5} \end{aligned}$$

5.

$$
\begin{aligned}
7a-(3a+10)&=6-2(12-9a)\\
\therefore 7a-3a-10&=6-24+18a\\
\therefore 4a-10&=-18+18a\\
\therefore 4a-18a&=-18+10\\
\therefore -14a&=-8\\
\therefore a&=\frac{-8}{-14}=\frac{4}{7}
\end{aligned}
$$

Back to Exercise 1.1

Exercise 1.2

1.
$$
\begin{aligned}
\frac{x+2}{4}-\frac{x-6}{3}&=12\\
\therefore 12\times\left(\frac{x+2}{4}\right)-12\times\left(\frac{x-6}{3}\right)&=12\times 12\\
\therefore 3(x+2)-4(x-6)&=144\\
\therefore 3x+6-4x+24&=144\\
\therefore -x&=114\\
\therefore x&=-114
\end{aligned}
$$

2.
$$
\begin{aligned}
\frac{2-5a}{3}-6&=\frac{4a}{3}+2-a\\
\therefore 3\left(\frac{2-5a}{3}\right)-3(6)&=3\left(\frac{4a}{3}\right)+3(2-a)\\
\therefore 2-5a-18&=4a+6-3a\\
\therefore -6a&=22\\
\therefore a&=-\frac{22}{6}=-\frac{11}{3}
\end{aligned}
$$

3.
$$
\begin{aligned}
3-\frac{y-2}{4}&=4\\
\therefore 4(3)-4\left(\frac{y-2}{4}\right)&=4(4)\\
\therefore 12-y+2&=16\\
\therefore -y&=2\\
\therefore y&=-2
\end{aligned}
$$

4.
$$
\begin{aligned}
\frac{1}{5}(x-1)&=\frac{1}{3}(x-2)+3\\
\therefore 15\left(\frac{1}{5}(x-1)\right)&=15\left(\frac{1}{3}(x-2)\right)+15(3)\\
\therefore 3(x-1)&=5(x-2)+45\\
\therefore 3x-3&=5x-10+45\\
\therefore -2x&=38\\
\therefore x&=-19
\end{aligned}
$$

Back to Exercise 1.2

Exercise 1.3

1.

$$\begin{aligned} 5-\frac{7}{q} &= \frac{2(q+4)}{q} \qquad q \neq 0 \\ \therefore 5q-7 &= 2(q+4) \\ \therefore 5q-7 &= 2q+8 \\ \therefore 3q &= 15 \\ \therefore q &= 5 \end{aligned}$$

2.

$$\begin{aligned} \frac{5}{2x}+\frac{1}{6x} &= \frac{3}{x}+2 \qquad x \neq 0 \\ \therefore 3(5)+1 &= 6(3)+12x \qquad \text{Multiply both sides by the LCD of } 6x \\ \therefore 16 &= 18+12x \\ \therefore 12x &= -2 \\ \therefore x &= -\frac{1}{6} \end{aligned}$$

3.

$$\begin{aligned} \frac{2}{(b-5)}-\frac{4}{(b+5)} &= \frac{3}{(b+5)} \qquad b \neq 5; b \neq -5 \\ \therefore 2(b+5)-4(b-5) &= 3(b-5) \\ \therefore 2b+10-4b+20 &= 3b-15 \\ \therefore -5b &= -45 \\ \therefore b &= 9 \end{aligned}$$

Back to Exercise 1.3

Exercise 1.4

1.

$$\begin{aligned} -x^2-7x &= 12 \\ \therefore -x^2-7x-12 &= 0 \\ \therefore x^2+7x+12 &= 0 \\ \therefore (x+3)(x+4) &= 0 \\ \therefore x+3 = 0 \text{ or } x+4 &= 0 \\ \therefore x = -3 \text{ or } x &= -4 \end{aligned}$$

2.

$$\begin{aligned} x^2-49 &= 0 \\ \therefore (x+7)(x-7) &= 0 \\ \therefore x+7 = 0 \text{ or } x-7 &= 0 \\ \therefore x = -7 \text{ or } x &= 7 \end{aligned}$$

3.

$$\begin{aligned} x^2+6x-14 &= 13 \\ \therefore x^2+6x-27 &= 0 \\ \therefore (x+9)(x-3) &= 0 \\ \therefore x = -9 \text{ or } x &= 3 \end{aligned}$$

4.

$$
\begin{aligned}
3x^2 - 18x - 48 &= 0 \\
\therefore 3(x^2 - 6x - 16) &= 0 \\
\therefore x^2 - 6x - 16 &= 0 \\
\therefore (x-8)(x+2) &= 0 \\
\therefore x &= 8 \text{ or } x = -2
\end{aligned}
$$

5.

$$
\begin{aligned}
l2x(x+1) - (x-3) &= 6 \\
\therefore 2x^2 + 2x - x + 3 &= 6 \\
\therefore 2x^2 + x - 3 &= 0 \\
\therefore (2x+3)(x-1) &= 0 \\
\therefore x &= -\frac{3}{2} \text{ or } x = 1
\end{aligned}
$$

6.

$$
\begin{aligned}
(x+1)^2 &= (2x+3)(x+1) \\
\therefore x^2 + 2x + 1 &= 2x^2 + 5x + 3 \\
\therefore -x^2 - 3x - 2 &= 0 \\
\therefore x^2 + 3x + 2 &= 0 \\
\therefore (x+2)(x+1) &= 0 \\
\therefore x &= -2 \text{ or } x = -1
\end{aligned}
$$

Note: You cannot divide both sides of the equation by the $(x+1)$ factor as this has the effect of changing the equation into a linear equation and, therefore, removing one of the solutions. Which solution (or root) does this remove?

7.

$$
\begin{aligned}
(x+2) &= \frac{6x-12}{x-2}, \ x \neq 2 \\
\therefore (x+2)(x-2) &= 6x - 12 \\
\therefore x^2 - 4 &= 6x - 12 \\
\therefore x^2 - 6x + 8 &= 0 \\
\therefore (x-2)(x-4) &= 0 \\
\therefore \ \cancel{x=2} & \text{ or } x = 4
\end{aligned}
$$

8.

$$
\begin{aligned}
\frac{4x}{4x+3} &= -\frac{3}{x}, \ x \neq 0, \ x \neq -1 \\
\therefore 4x^2 &= -3(4x+3) \\
\therefore 4x^2 &= -12x - 9 \\
\therefore 4x^2 + 12x + 9 &= 0 \\
\therefore (2x+3)(2x+3) &= 0 \\
\therefore x &= -\frac{3}{2} \text{ or } x = -\frac{3}{2}
\end{aligned}
$$

Both solutions are the same. Therefore, we say that there is only one root (or answer) for the equation and it is $x = -\frac{3}{2}$.

Back to Exercise 1.4

Unit 1: Assessment

1.

$$
\begin{aligned}
-7x &= 8(1-x) \\
\therefore -7x &= 8 - 8x \\
\therefore x &= 8
\end{aligned}
$$

2.

$$
\begin{aligned}
1 &= \frac{3a-4}{2a+6} \qquad a \neq -3 \\
\therefore 2a+6 &= 3a-4 \\
\therefore -a &= -10 \\
\therefore a &= 10
\end{aligned}
$$

3.

$$
\begin{aligned}
\frac{1}{4}(x-1) - 1\frac{1}{2}(3x+2) &= 0 \\
\therefore \frac{1}{4}(x-1) - \frac{3}{2}(3x+2) &= 0 \\
\therefore (x-1) - 2 \times 3(3x+2) &= 0 \\
\therefore x - 1 - 18x - 12 &= 0 \\
\therefore -17x &= 13 \\
\therefore x &= -\frac{13}{17}
\end{aligned}
$$

4.

$$
\begin{aligned}
\frac{-2}{(x^2-9)} + \frac{4}{(x+3)} &= \frac{6}{(x-3)} \qquad x \neq \pm 3 \\
\therefore \frac{-2}{(x+3)(x-3)} + \frac{4}{(x+3)} &= \frac{6}{(x-3)} \\
\therefore -2 + 4(x-3) &= 6(x+3) \\
\therefore -2 + 4x - 12 &= 6x + 18 \\
\therefore -2x &= 32 \\
\therefore x &= -16
\end{aligned}
$$

5.

$$
\begin{aligned}
x^2 - 3 &= -2x \\
\therefore x^2 + 2x - 3 &= 0 \\
\therefore (x+3)(x-1) &= 0 \\
\therefore x = -3 &\text{ or } x = 1
\end{aligned}
$$

6.

$$
\begin{aligned}
2(9c-4) &= 9c^2 \\
\therefore 18c - 8 &= 9c^2 \\
\therefore 0 &= 9c^2 - 18c + 8 \\
\therefore 0 &= (3c-4)(3c-2) \\
\therefore 3c - 4 = 0 &\text{ or } 3c - 2 = 0 \\
\therefore c = \frac{4}{3} &\text{ or } c = \frac{2}{3}
\end{aligned}
$$

7.

$$
\begin{aligned}
\frac{3b}{b+2} + 1 &= \frac{4}{b+1} \qquad b \neq -2; b \neq -1 \\
\therefore 3b(b+1) + (b+2)(b+1) &= 4(b+2) \\
\therefore 3b^2 + 3b + b^2 + 3b + 2 &= 4b + 8 \\
\therefore 4b^2 + 2b - 6 &= 0 \\
\therefore 2(2b^2 + b - 3) &= 0 \\
\therefore 2b^2 + b - 3 &= 0 \\
\therefore (2b+3)(b-1) &= 0 \\
\therefore b = -\frac{3}{2} &\text{ or } b = 1
\end{aligned}
$$

Back to Unit 1: Assessment

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Unit 2: Solve exponential and literal equations

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Solve exponential equations of the form $ka^{x+p} = m$ by using the laws of exponents.
- Solve literal equations.

What you should know

Before you start this unit, make sure you can:

- Simplify expressions containing exponents using the exponent laws. If you need to revise the exponent laws see Subject outcome 1.2, Units 2 and 3.
- Solve linear equations. If you need to revise solving linear equations, see Unit 1 in this subject outcome.

Here is a short self-assessment to make sure you have the skills you need to proceed with this unit.

1. Simplify the following:
   a. $2^x.2^3$
   b. $\left(3^3\right)^{3x+4}$
   c. $4^{2x-3} \times 4^{3x+1}$
2. Solve for x:
   a. $\dfrac{4}{x} + 12 = 4$
   b. $\dfrac{3x}{2} + \dfrac{5x}{3} = 5$

Solutions

1.
   a. $2^x.2^3 = 2^{x+3}$
   b. $\left(3^3\right)^{3x+4} = 3^{9x+12}$
   c. $4^{2x-3} \times 4^{3x+1} = 4^{(2x-3)+(3x+1)} = 4^{5x-2}$
2. Solve for x:
   a.

$$\frac{4}{x} + 12 = 4$$
$$\therefore \frac{4}{x} = -8$$
$$\therefore -8x = 4$$
$$\therefore x = -\frac{1}{2}$$

b.

$$\frac{3x}{2} + \frac{5x}{3} = 5 \quad \text{Multiply both sides by the LCD of 6}$$
$$\therefore 9x + 10x = 30$$
$$\therefore 19x = 30$$
$$\therefore x = \frac{30}{19}$$

Introduction

At the start of the previous unit, we said that equations are at the heart of almost all Mathematics. In that unit we learnt how to solve **linear** and **quadratic** equations. Linear equations are equations where the exponent on the unknown is 1. Linear equations only ever have one solution. An example of a linear equation is $4x + 5 = 17$. The exponent on the unknown x is 1.

We also learnt how to solve quadratic equations. These are equations where the **highest** power on the unknown is 2 and they have up to two solutions. An example of a quadratic equation is $x^2 + 4x = -3$. Here the highest exponent on the x is 2.

In principle, we can also solve equations where the highest exponent on the unknown is 3 (cubic equations), 4, 5 or indeed any natural number n. Together, these are called **polynomial** equations. However, in all these cases, the unknown is in the base not the **exponent**.

In this unit, we are going to solve a different kind of equation, where the unknown is in the exponent. These are called **exponential** equations.

Example 2.1

Solve for x: $3^x - 27 = 0$.

Solution:

$3^x - 27 = 0$

As usual, the first step in solving an equation is to isolate the unknown. In this case, we have to isolate the term with the unknown in it, 3^x. We add 27 to both sides of the equation.

$\therefore 3^x = 27$

Now you might already be able to see what the solution is. The general principle when solving exponential equations is to get the bases on both sides of the equation the same. We know that $27 = 3^3$.

$\therefore 3^x = 3^3$

Now we can equate the exponents on both sides of the equation to solve for the unknown.
$\therefore x = 3$

Here are three more examples for you to work through.

Example 2.2

Solve for x: $3^{x-2} - 45 = 36$.

Solution:

$3^{x-2} - 45 = 36$

Once again, we need to isolate the terms with the unknown by adding 45 to both sides.

$\therefore 3^{x-2} = 36 + 45$

$\therefore 3^{x-2} = 81$

Next, we need to get the bases on both sides the same. We know that $81 = 3^4$.

$\therefore 3^{x-2} = 3^4$

Now we can equate the exponents and solve for the unknown.

$\therefore x - 2 = 4$

$\therefore x = 6$

Example 2.3

Solve for y: $5^y + 3.5^{y+1} = 400$.

Solution:

$5^y + 3.5^{y+1} = 400$

We have two terms with unknowns in the exponent that we need to combine. To do this, we need to recognise that $5^{y+1} = 5^y.5^1$. Now we can simplify the second term on the left-hand side.

$\therefore 5^y + 3.5.5^y = 400$

$\therefore 5^y + 15.5^y = 400$

On the left-hand side we have two terms that we can add. If you need to, you can substitute 5^y for A if this makes the step easier.

$\therefore 16.5^y = 400$ or

$$\begin{aligned}
\therefore 5^y + 15.5^y &= 400 \quad \text{Let } 5^y \text{ be A} \\
\therefore \text{A} + 15\text{A} &= 400 \\
\therefore 16\text{A} &= 400 \quad \text{Substitute back for } 5^y \\
\therefore 16.5^y &= 400
\end{aligned}$$

Now we can isolate the term with the unknown by dividing both sides by 16.

$\therefore 5^y = 25$

From here on, it is easy to solve for y.

$\therefore 5^y = 5^2$
$\therefore y = 2$

Note

$3^x + 3^{2x} \neq 2.3^x$ and $3^x + 3^{2x} \neq 2.3^{2x}$
$3^x \times 3^{2x} = 3^{x+2x} = 3^{3x}$

Example 2.4

Solve for t: $7^t = 50 - 7^{t+2}$

Solution:

$$\begin{aligned}
7^t &= 50 - 7^{t+2} \\
\therefore 7^t + 7^{t+2} &= 50 \quad \text{Get all terms with unknowns in the exponent on one side} \\
\therefore 7^t + 7^t.7^2 &= 50 \quad \text{Split the two term exponent into its components} \\
\therefore 7^t + 49.7^t &= 50 \quad \text{Collect the like terms} \\
\therefore 50.7^t &= 50 \\
\therefore 7^t &= 1 \quad \text{Remember that } 1 = a^0 \\
\therefore 7^t &= 7^0 \\
\therefore t &= 0
\end{aligned}$$

Note

If $a^x = a^y$ then $x = y$ if $a > 0$ and $a \neq 1$.

Exercise 2.1

Solve for the unknown in the following:

1. $2^{a+5} = 32$
2. $16^{2b+5} = 64^{b+3}$
3. $81^{x+3} = 27^{x-4}$
4. $81^y + 9^{2y+1} = 270$
5. $2^x + 2^{x+2} = 40$

The full solutions are at the end of the unit.

Solve literal equations

Very often in the real world, equations will have more than one variable. Take the equation $E = mc^2$. Although the implications of this equation are hugely significant, the equation itself is very simple. It has the two variables and one constant. The constant is c, which is the speed of light in a vacuum, while the variables are E (energy) and m (mass).

We can use this equation as it is to calculate the total amount of energy contained in a given mass. Alternatively, we can rearrange it to calculate the amount of mass needed to produce a given energy.

$$m = \frac{E}{c^2}$$

We call equations like this literal equations. Other common examples are the area of a circle ($A = \pi r^2$) or the volume of a rectangular prism ($V = l \times b \times h$).

When we 'solve' literal equations, all we are really doing is isolating one of the variables on one side of the equation.

Example 2.5

The area of a triangle is $A = \frac{1}{2} \times b \times h$. What is the base of the triangle in terms of its area and height?

Solution:

$$A = \frac{1}{2} \times b \times h$$

We need to isolate the variable for the base (b) on one side of the equation. To do this, we can start by multiplying both sides of the equation by 2.

$$\therefore 2A = b \times h$$

Now we can divide both sides of the equation by h.

$$\therefore \frac{2A}{h} = b$$

We normally write the equation with the isolated variable on the left-hand side.

$$\therefore b = \frac{2A}{h}$$

Example 2.6

Given the formula $b = G \times \dfrac{q}{G + r^2}$

1. Make G the subject of the formula.
2. Express r in terms of b, G and q.

Solutions:

1. Make G the subject of the formula is just another way of saying isolate the variable G on one side of the equation.

$$\begin{aligned} b &= G \times \frac{q}{G + r^2} \\ \therefore b &= \frac{Gq}{G + r^2} \quad \text{Multiply both sides by } G + r^2 \\ \therefore b(G + r^2) &= Gq \quad \text{Divide both sides by } b \text{ and by } G \\ \therefore \frac{G + r^2}{G} &= \frac{q}{b} \quad \text{Write the fraction on the left-hand side as two fractions} \\ \therefore \frac{G}{G} + \frac{r^2}{G} &= \frac{q}{b} \\ \therefore 1 + \frac{r^2}{G} &= \frac{q}{b} \\ \therefore \frac{r^2}{G} &= \frac{q}{b} - 1 \quad \text{Divide both sides by } r^2 \\ \therefore \frac{1}{G} &= \frac{\frac{q}{b} - 1}{r^2} \end{aligned}$$

We need to have G on the left-hand side. **Remember that you can do anything to an equation so long as you do the same thing to both sides of the equation.** Take the inverse of both sides of the equation which will flip both sides around; the numerator will become the denominator and the denominator will become the numerator.

$$\therefore G = \frac{r^2}{\frac{q}{b} - 1}$$

2. Being asked to express one variable in terms of others is another way of asking you to isolate that variable on one side of the equation, or to solve the equation for that variable. In Question 1 above, we arrived at the following:

$$\begin{aligned} \frac{r^2}{G} &= \frac{q}{b} - 1 \quad \text{Multipy both side by } G \\ \therefore r^2 &= G\left(\frac{q}{b} - 1\right) \end{aligned}$$

Now we have r^2 on the left-hand side. To get just r, we need to take the square root of both sides. Remember when you take the square root, the answer could be positive or negative.

$$\therefore r = \pm\sqrt{G\left(\frac{q}{b} - 1\right)} \quad \text{Remember to take the square root of the whole right-hand side}$$

Exercise 2.2

1. Solve for y in the following: $2x + 3y = 9$.
2. Make a the subject of the formula: $s = ut + \frac{1}{2}at^2$.
3. Solve for r: $A = \pi R^2 - \pi r^2$.
4. If $r = \sqrt{a^2 + b^2}$, solve for b.
5. $F_R = \frac{1}{2}\rho A C_D v^2$. If $F_R = 24$, $A = 0.5$, $C_D = 0.01$ and $\rho = 1.2$, solve for v.

The full solutions are at the end of the unit.

Note

When solving literal equations remember the following:

- To isolate the required variable, perform the operation that will 'undo' the operations currently being performed on the variable.
- If the unknown variable is in two or more terms, collect these terms on one side of the equation and try and take it out as a common factor.
- If we have to take the square root of both sides, remember that there will be a positive and a negative answer.
- If the variable that needs to be isolated is in the denominator, start by multiplying both sides by the lowest common denominator.

Summary

In this unit you have learnt the following:

- How to solve equations where the unknown is in the exponent (exponential equations) by getting the bases on both sides of the equation to be the same using the exponent laws.
- How to rearrange equations with more than one unknown in terms of any of the unknowns.

Unit 2: Assessment

Suggested time to complete: 30 minutes

1. Solve for the unknown in the following:
   a. $12^{2x-3} = 1$
   b. $25^{1-2x} - 5^4 = 0$

c. $-\frac{1}{2}.6^{\frac{m}{2}+3} = -18$

2. When measuring the diameter of a large pipe, the formula is $D = L + \frac{W^2}{2L}$. Solve for W.
3. The formula for the volume of a geometric shape is given as $V = \frac{1}{3}\pi r^2 h$
   a. Make r the subject of the formula.
   b. Determine the value of r to two decimal places if $V = 250\ \text{mm}^3$ and $h = 25\ \text{mm}$.

The full solutions are at the end of the unit.

Unit 2: Solutions

Exercise 2.1

1.
$$\begin{aligned} 2^{a+5} &= 32 \\ \therefore 2^{a+5} &= 2^5 \\ \therefore a+5 &= 5 \\ \therefore a &= 0 \end{aligned}$$

2.
$$\begin{aligned} 16^{2b+5} &= 64^{b+3} \\ \therefore \left(4^2\right)^{2b+5} &= \left(4^3\right)^{b+3} \quad \text{Remember that } (a^m)^n = a^{mn} \\ \therefore 4^{2(2b+5)} &= 4^{3(b+3)} \\ \therefore 4^{4b+10} &= 4^{3b+9} \\ \therefore 4b+10 &= 3b+9 \\ \therefore b &= -1 \end{aligned}$$
You could also have changed the bases to 2.

3.
$$\begin{aligned} 81^{x+3} &= 27^{x-4} \\ \therefore \left(3^4\right)^{x+3} &= \left(3^3\right)^{x-4} \\ \therefore 3^{4x+12} &= 3^{3x-12} \\ \therefore 4x+12 &= 3x-12 \\ \therefore x &= -24 \end{aligned}$$

4.
$$\begin{aligned} 81^y + 9^{2y+1} &= 270 \\ \therefore \left(9^2\right)^y + 9^{2y+1} &= 270 \\ \therefore 9^{2y} + 9^{2y}.9^1 &= 270 \\ \therefore 9^{2y} + 9.9^{2y} &= 270 \\ \therefore 10.9^{2y} &= 270 \\ \therefore 9^{2y} &= 27 \\ \therefore \left(3^2\right)^{2y} &= 3^3 \\ \therefore 3^{4y} &= 3^3 \\ \therefore 4y &= 3 \\ \therefore y &= \frac{3}{4} \end{aligned}$$

5.

$$\begin{aligned} 2^x + 2^{x+2} &= 40 \\ \therefore 2^x + 2^x.2^2 &= 40 \\ \therefore 2^x + 4.2^x &= 40 \\ \therefore 5.2^x &= 40 \\ \therefore 2^x &= 8 \\ \therefore 2^x &= 2^3 \\ \therefore x &= 3 \end{aligned}$$

Back to Exercise 2.1

Exercise 2.2

1.

$$\begin{aligned} &2x + 3y = 9 \\ &\quad \therefore 3y = 9 - 2x \\ &\quad\quad \therefore y = \frac{9 - 2x}{3} \end{aligned}$$

2.

$$\begin{aligned} s &= ut + \frac{1}{2}at^2 \\ \therefore s - ut &= \frac{1}{2}at^2 \\ \therefore 2(s - ut) &= at^2 \\ \therefore \frac{2s - 2ut}{t^2} &= a \\ \therefore a &= \frac{2s - 2ut}{t^2} \end{aligned}$$

3.

$$\begin{aligned} A &= \pi R^2 - \pi r^2 \\ \therefore \pi r^2 &= \pi R^2 - A \\ \therefore r^2 &= \frac{\pi R^2 - A}{\pi} \\ \therefore r &= \pm\sqrt{\frac{\pi R^2 - A}{\pi}} \end{aligned}$$

4.

$$\begin{aligned} r &= \sqrt{a^2 + b^2} \\ \therefore r^2 &= \left(\sqrt{a^2 + b^2}\right)^2 \\ \therefore r^2 &= a^2 + b^2 \\ \therefore b^2 &= r^2 - a^2 \\ \therefore b &= \pm\sqrt{r^2 - a^2} \end{aligned}$$

5.

$$\begin{aligned} F_R &= \frac{1}{2}\rho A C_D v^2 \\ \therefore 2F_R &= \rho A C_D v^2 \\ \therefore v^2 &= \frac{2F_R}{\rho A C_D} \\ \therefore v &= \pm\sqrt{\frac{2F_R}{\rho A C_D}} \end{aligned}$$

$$\text{But } F_D = 24,\ A = 0.5,\ C_D = 0.01,\ \rho = 1.2$$

$$\begin{aligned} \therefore v &= \pm\sqrt{\frac{2 \times 24}{1.2 \times 0.5 \times 0.01}} \\ \therefore v &= \pm\sqrt{8\ 000} \\ \therefore v &= \pm 89.44 \end{aligned}$$

Back to Exercise 2.2

Unit 2: Assessment

1. Solve for the unknown in the following:

a.

$$\begin{aligned} 12^{2x-3} &= 1 \\ \therefore 12^{2x-3} &= 12^0 \\ \therefore 2x - 3 &= 0 \\ \therefore 2x &= 3 \\ \therefore x &= \frac{3}{2} \end{aligned}$$

b.

$$\begin{aligned} 25^{1-2x} - 5^4 &= 0 \\ \therefore \left(5^2\right)^{1-2x} - 5^4 &= 0 \\ \therefore 5^{2-4x} - 5^4 &= 0 \\ \therefore 5^{2-4x} &= 5^4 \\ \therefore 2 - 4x &= 4 \\ \therefore 4x &= -2 \\ \therefore x &= -\frac{1}{2} \end{aligned}$$

c.

$$\begin{aligned} -\frac{1}{2}.6^{\frac{m}{2}+3} &= -18 \\ \therefore 6^{\frac{m}{2}+3} &= 36 \\ \therefore 6^{\frac{m}{2}+3} &= 6^2 \\ \therefore \frac{m}{2} + 3 &= 2 \\ \therefore \frac{m}{2} &= -1 \\ \therefore m &= -2 \end{aligned}$$

2.

$$D = L + \frac{W^2}{2L}$$
$$\therefore D - L = \frac{W^2}{2L}$$
$$\therefore W^2 = 2L(D - L)$$
$$\therefore W = \pm\sqrt{2L(D - L)}$$

3.

a.

$$V = \frac{1}{3}\pi r^2 h$$
$$\therefore 3V = \pi r^2 h$$
$$\therefore r^2 = \frac{3V}{\pi h}$$
$$\therefore r = \pm\sqrt{\frac{3V}{\pi h}}$$

b.

$$r = \pm\sqrt{\frac{3V}{\pi h}}$$
$$= \pm\sqrt{\frac{3 \times 250 \text{ mm}^3}{25\pi \text{ mm}}}$$
$$= \pm 3.09 \text{ mm}$$

However, as this is for the volume of a physical object, a negative answer does not make sense. Therefore $r = 3.09 \text{ mm}$.

Back to Unit 1: Assessment

Unit 3: Solve algebraic inequalities

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Solve linear inequalities with a single unknown or variable.
- Represent the solution to linear inequalities:
    - In set builder notation
    - In interval notation
    - On the number line.

What you should know

Before you start this unit, make sure you can:

- Solve linear equations; if you need to revise solving linear equations, see Unit 1 in this subject outcome.

Introduction

Think about the following situation. A shopkeeper rents a shop for $\mathrm{R}1\ 200$ a month. He sells socks for $\mathrm{R}120$ a pair. If each pair of socks costs him $\mathrm{R}70$, how many pairs of socks must he sell in order to cover his rent. Try to work out the answer on your own?

The first thing we need to do is work out the profit he makes on each pair of socks. This is $\mathrm{R}120-\mathrm{R}70=\mathrm{R}50$. In order to make $\mathrm{R}1\ 200$, he will need to sell $\dfrac{\mathrm{R}1\ 200}{\mathrm{R}50}=24$ pairs of socks.

However, can you see that this is the least number of pairs of socks he needs to sell? He needs to sell **at least** 24 pairs of socks to cover his rent. Another way of saying this is that he needs to sell 24 or more pairs of socks.

We can write this mathematically using an inequality sign like this: $\text{pairs of socks to sell}\geq 24$.

Not all calculations have one specific answer. Sometimes the answer lies in a range. We use **inequalities** to work out what these ranges are.

Linear inequalities are similar to linear equations and the methods used to solve them are very similar. The only difference occurs when we multiply both sides of an inequality sign by a negative number as we will see below.

Solve simple linear inequalities

We are going to solve for x in the following linear inequality: $3x-4\leq 8$. But before we do, let's look at how to solve the similar equation $3x-4=8$.

$3x - 4 = 8$
$\therefore 3x = 12$
$\therefore x = 4$

We can draw this answer on the number line as shown in Figure 1.

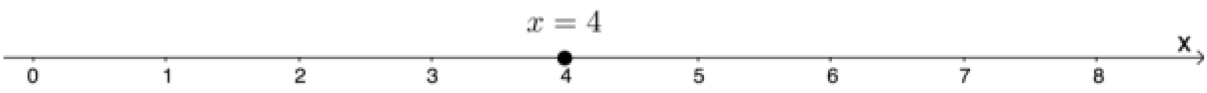


Figure 1: Solution to $3x - 4 = 8$

Solving the inequality $3x - 4 \leq 8$ is very similar.
$3x - 4 \leq 8$
$\therefore 3x \leq 12$
$\therefore x \leq 4$

Now if we sketch this solution on the number line, it will look like Figure 2. We use a **solid dot** to indicate that x can be equal to 4. The line shows that x can be any real number less than 4.

Figure 2: Solution to $3x - 4 \leq 8$

We can see that x does not have a single specific value. x can be any real number from 4 or less. We can check this by substituting values into the original inequality to see if the inequality is true.

- $x = 4$: $3(4) - 4 = 12 - 4 = 8$ which is **less than or equal to** 8.
- $x = 3$: $3(3) - 4 = 9 - 4 = 5$ which is **less than or equal to** 8.
- $x = -4$: $3(-4) - 4 = -12 - 4 = -16$ which is **less than or equal to** 8.

But what happens if we try a value for x that is greater than 4?

- $x = 5$: $3(5) - 4 = 15 - 4 = 11$ which is **not** less than or equal to 8.

Note

$<$	Less than	$x < 4$
$>$	Greater than	$x > 3$
$\leq$	Less than or equal to	$x \leq 5$
$\geq$	Greater than or equal to	$x \geq -2$

Example 3.1

Solve for x in $5x - 4 > 7$ and represent the answer on a number line.

Solution:

In order to solve this inequality, we treat it just like we would an ordinary linear equation.

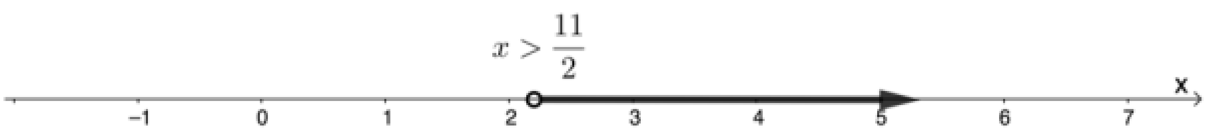
$$\begin{aligned} 5x - 4 &> 7 \\ \therefore 5x &> 11 \quad \text{Add 4 to both sides of the inequality} \\ \therefore x &> \frac{11}{5} \quad \text{Divide both sides of the inequality by 5} \end{aligned}$$

We can also check to see if our solution is correct by substituting any value greater than $\frac{11}{5}$ (or 2.2) into the original inequality to see if it is true. Let's substitute $x = 3$.

LHS: $5(3) - 4 = 15 - 4 = 11$
RHS: 7

The LHS is greater than the RHS. Therefore, the inequality is true, and our solution is correct.

In this case x must be **greater than** but not **equal** to $\frac{11}{5}$. So we use an **open dot** on the value of $\frac{11}{5}$ to indicate that x cannot have this value, and a line to indicate that x can be any real number greater than $\frac{11}{2}$.



Interval and set builder notation

Representing the solution to an inequality on a number line is effective but it can take quite a long time to draw. Mathematicians, therefore, use other ways to represent the sets of numbers represented by these solutions.

The first notation is called **interval notation**. Interval notation uses round and square brackets to indicate if the end point of the range of numbers is included or excluded from the set. Here are some examples to illustrate.

- $(5, 11)$: The round brackets indicate that the numbers themselves are **not** included. This interval includes all real numbers greater than but not equal to 5 and less than but not equal to 11.
- $[-2, 6]$: Square brackets indicate that the numbers **are** included. This interval includes all real numbers greater than or equal to -2 and less than or equal to 6.
- $[-7, \infty)$: Round brackets are always used for positive and negative infinity. This interval includes all real numbers greater than or equal to -7.

Note

Interval notation can only be used to represent an interval of **real numbers**. If the interval or set is only a sub-set of the real numbers, such as integers or natural numbers, we need to use set builder notation.

Set builder notation is slightly more involved than interval notation but it does have the advantage of being able to represent any set of numbers we can think of. The structure of all sets in set builder notation follows the same set of rules. Here is an example:

$\{x|x \in \mathbb{R},\ 2 < x \leq 17\}$

This means the set of all x such that x is a real number (x is an element of the real numbers or $x \in \mathbb{R}$) and x is greater than 2 and less than or equal to 17.

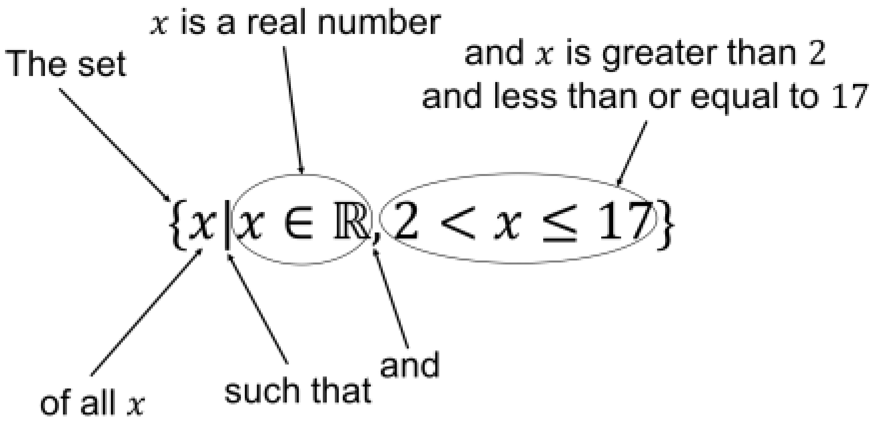


Figure 3: Set builder notation

We can represent all the natural numbers between 4 and 20 (including 4 but not including 20) in set builder notation like this: .$\{x|x \in \mathbb{N},\ 4 \leq x < 20\}$. The set of all x such that x is an element of the natural numbers and x is greater than or equal to 4 and less than 20.

Note

Unless otherwise indicated, assume that all sets have x (or whatever number) as an element of the real numbers $\mathbb{R}$.

Example 3.2

Solve for y in $5y + 5 \geq 3(2y - 3)$ and represent the answer on a number line, in interval notation and in set builder notation.

Solution:

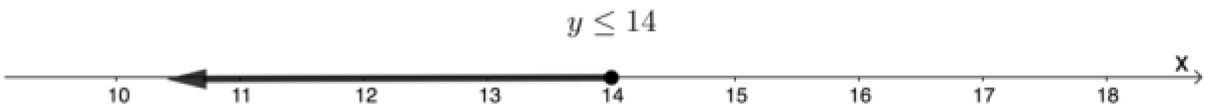
$$\begin{aligned} 5y + 5 &\geq 3(2y - 3) \\ \therefore 5y + 5 &\geq 6y - 9 \\ \therefore -y &\geq -14 \end{aligned}$$

We now have the situation where we need to multiply both sides of the inequality by -1. When we do so, we need to flip the inequality sign. The reason for this is simple. Consider $3 < 5$. This is true. Now if we multiply (or divide) both sides by -1, we get $-3 < -5$ which is **NOT** true. The only way to make the inequality true again, is to flip the inequality sign to get $-3 > -5$.

So, in our case, when we multiply (or divide) both sides by -1, we need to also flip the inequality sign.

$$\therefore y \leq 14$$

Number line:



Interval notation: $(-\infty, 14]$. Remember, we cannot include $-\infty$ in the interval so we use a round bracket.

Set builder notation: $\{y|y \in \mathbb{R},\ y \leq 14\}$

Exercise 3.1

Solve for x in the following inequalities and represent the solutions on a number line, in interval notation and in set builder notation.

1. $6x + 5 > 3x - 7$
2. $5(2x - 3) \leq -4(x + 2)$
3. $\dfrac{3x + 4}{4} - \dfrac{3x - 3}{3} \geq 1$

The full solutions are at the end of the unit.

Solve compound inequalities

In the examples we have seen so far, the range of values that the unknown could take to satisfy the original inequality had only one bound; an upper bound or a lower bound.

This means that the possible values for the unknown are either all less than or equal to a certain number (the upper bound) or greater than or equal to a certain number (a lower bound).

But we can also create inequalities that limit the permissible values of the unknown to a specific range between an upper and lower bound. These are called compound inequalities. Here is an example:

$-4 < x \leq 7$. In this case, x can take any real value between -4 (but not including -4) and 7 (including 7). This is represented in interval notation as $(-4, 7]$ and on a number line as shown in Figure 4.

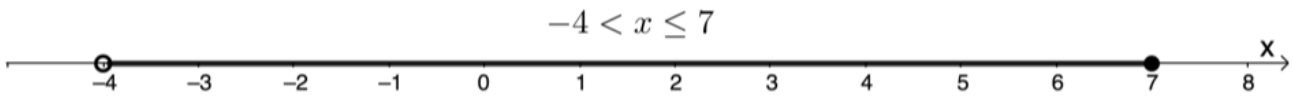


Figure 4: Number line for $-4 < x \leq 7$

Example 3.3

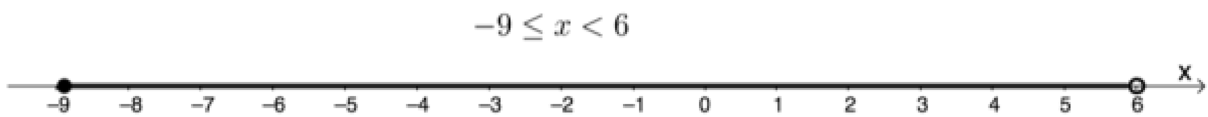
Solve for x in the following: $-5 \leq x + 4 < 10$ and represent your answer on a number line and using interval notation.

Solution:

We solve compound inequalities the same way as simple inequalities, except that we need to do the same thing to all three parts of the inequality.

$$\begin{aligned} & -5 \leq x + 4 < 10 \\ \therefore\ & -5 - 4 \leq x + 4 - 4 < 10 - 4 \quad \text{Subtract 4 from all three parts} \\ \therefore\ & -9 \leq x < 6 \end{aligned}$$

We represent this solution using interval notation as $[-9, 6)$ and on a number line as follows.



Example 3.4

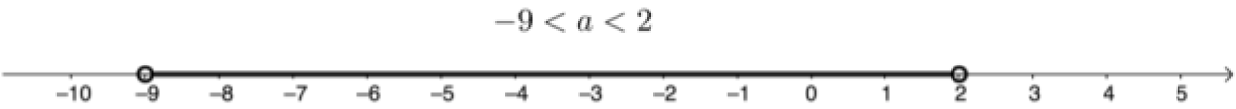
Solve for a in $5 < 7 - a < 16$ and represent your answer on a number line and using interval notation.

Solution:

$$\begin{aligned} & 5 < 7 - a < 16 \\ \therefore\ & -2 < -a < 9 \quad \text{Subtract 7 from all three parts} \\ \therefore\ & 2 > a > -9 \quad \text{Divide through by } -1 \text{ and change direction of inequalities} \end{aligned}$$

Interval notation: $(-9, 2)$

Number line:



Example 3.5

Solve for x in $4x + 7 < -3$ or $3x + 3 > 5$ and represent your answer on a number line and using interval notation.

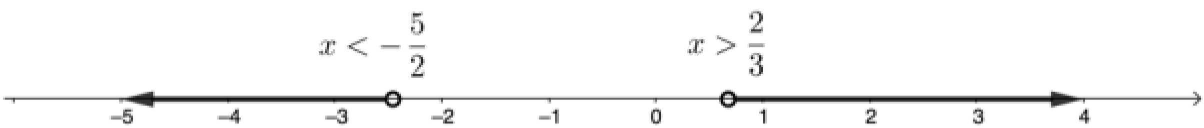
Solution:

This is a slightly different kind of compound inequality where the two parts are separated. To solve for the unknown, we have to solve each part of the inequality separately and then combine the results.

$$\begin{aligned} 4x + 7 < -3 &\quad \text{or} \quad 3x + 3 > 5 \\ \therefore 4x < -10 &\quad \text{or} \quad 3x > 2 \\ \therefore x < -\frac{10}{4} &\quad \text{or} \quad x > \frac{2}{3} \\ \therefore x < -\frac{5}{2} &\quad \text{or} \quad x > \frac{2}{3} \end{aligned}$$

We represent this using interval notation as $(-\infty, -\frac{5}{2})$ or $(\frac{2}{3}, \infty)$.

The number line representation is as follows:



Example 3.6

Solve for x in $-8 \leq 4 - \frac{2x}{3} \leq 0,\ x \in \mathbb{N}$ and represent your answer on a number line.

Solution:

$$\begin{aligned} -8 \leq 4 - \frac{2x}{3} \leq 0 & \\ \therefore -24 \leq 12 - 2x \leq 0 &\quad \text{Multiply through by 3} \\ \therefore -36 \leq -2x \leq -12 &\quad \text{Subtract 12 from all three parts} \\ \therefore 18 \geq x \geq 6 &\quad \text{Divide through by } -2 \end{aligned}$$

When we come to represent the solution on a number line, we need to remember that $x \in \mathbb{N}$. The number line representation is as follows:

$$6 \leq x \leq 18$$

In this case, we cannot use a line like we do to represent all the real numbers. We have to use separate closed dots to represent only the natural numbers between and including 6 and 18.

Note

Unless told otherwise, you can assume that the unknown can take any real value.

Exercise 3.2

1. Solve for the unknown in each of the following, representing your answers using interval notation and on number lines.
   a. $4 > -6x - 6 > -2$
   b. $-5 \leq \dfrac{2a - 4}{3} < 1$
   c. $\dfrac{-3x + 6}{4} \geq 6 \quad \text{or} \quad 2x > \dfrac{7}{2}$
   d. $4x + 3 \leq 3\left(4x + \dfrac{1}{3}\right)$
2. Solve for x and represent your answer on a number line and using set builder notation.

$$\frac{3}{2}(2x - 1) \leq 2(x + 4), \;\; x \in \mathbb{N}$$

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to solve simple linear inequalities.
- How to solve compound inequalities.
- How to represent the solution to an inequality on a number line.
- How to represent the solution to an inequality using interval notation.

Unit 3: Assessment

Suggested time to complete: 30 minutes

1. Solve for x in the following inequalities, representing your answer on number lines.
   a. $-2 \leq x+6 < 9$
   b. $2-3x \leq 3-5x$
   c. $-5 < 1+3x \leq 10,\ x \in \mathbb{Z}$
2. Solve for the unknown in each case, representing your answers using interval notation and set builder notation.
   a. $\dfrac{4y-2}{6} > 2y+1$
   b. $\dfrac{1-s}{3} + \dfrac{1-s}{2} \leq 5$
   c. $-\dfrac{2x+6}{3} > 4 \quad \text{or} \quad \dfrac{3x-3}{4} > 0$
   d. $\dfrac{7x+3}{4} < 4(2x-7)$

The full solutions are at the end of the unit.

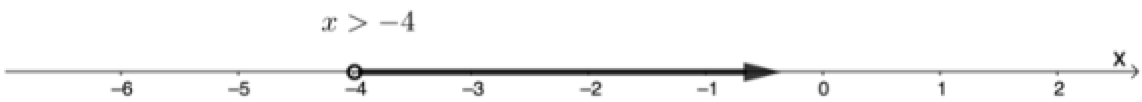
Unit 3: Solutions

Exercise 3.1

1.

$$\begin{aligned} 6x+5 &> 3x-7 \\ \therefore 3x &> -12 \\ \therefore x &> -4 \end{aligned}$$

Number line:

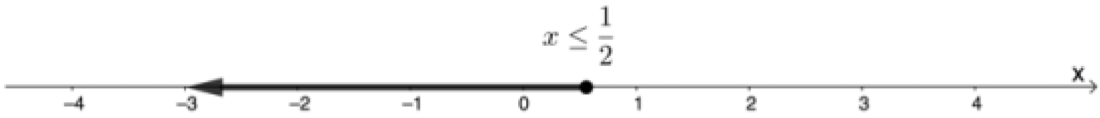


Interval notation: $(-4, \infty)$
Set builder notation: $\{x|x \in \mathbb{R},\ x > -4\}$

2.

$$\begin{aligned} 5(2x-3) &\leq -4(x+2) \\ \therefore 10x-15 &\leq -4x-8 \\ \therefore 14x &\leq 7 \\ \therefore x &\leq \frac{1}{2} \end{aligned}$$

Number line:



Interval notation: $(-\infty, \frac{1}{2}]$

Set builder notation: $\{x | x \in \mathbb{R},\ x \leq \frac{1}{2}\}$

3.

$$\begin{aligned} \frac{3x+4}{4} - \frac{3x-3}{3} &\geq 1 \\ \therefore 3(3x+4) - 4(3x-3) &\geq 12 \\ \therefore 9x + 12 - 12x + 12 &\geq 12 \\ \therefore -3x &\geq -12 \\ \therefore x &\leq 4 \end{aligned}$$

Number line:

Interval notation: $(-\infty, 4]$
Set builder notation: $\{x | x \in \mathbb{R},\ x \leq 4\}$

Back to Exercise 3.1

Exercise 3.2

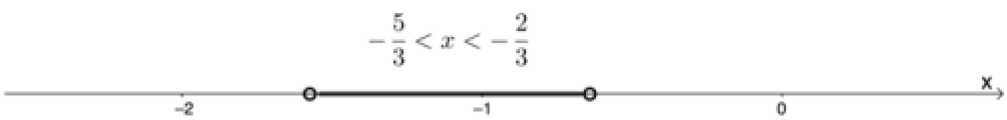
1.

a.

$$\begin{aligned} 4 > -6x - 6 &> -2 \\ \therefore 10 > -6x &> 4 \\ \therefore -\frac{10}{6} < x &< -\frac{4}{6} \\ \therefore -\frac{5}{3} < x &< -\frac{2}{3} \end{aligned}$$

Interval notation: $(-\frac{5}{3}, -\frac{2}{3})$

Number line:

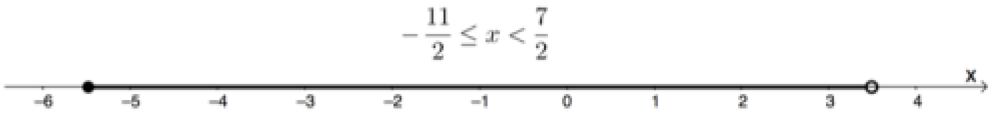


b.

$$\begin{aligned} -5 \leq \frac{2a-4}{3} &< 1 \\ \therefore -15 \leq 2a - 4 &< 3 \\ \therefore -11 \leq 2a &< 7 \\ \therefore -\frac{11}{2} \leq a &< \frac{7}{2} \end{aligned}$$

Interval notation: $[-\frac{11}{2}, \frac{7}{2})$

Number line:

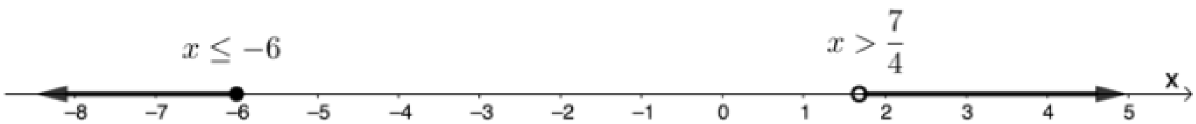


c.

$$\begin{aligned} \frac{-3x+6}{4} &\geq 6 \quad \text{or} \quad 2x > \frac{7}{2} \\ \therefore -3x+6 &\geq 24 \quad \text{or} \quad x > \frac{7}{4} \\ \therefore -3x &\geq 18 \quad \text{or} \quad x > \frac{7}{4} \\ \therefore x &\leq -6 \quad \text{or} \quad x > \frac{7}{4} \end{aligned}$$

Interval notation: $(-\infty, -6]$ or $(\frac{7}{4}, \infty)$

Number line:



d.

$$\begin{aligned} 4x+3 &\leq 3\left(4x+\frac{1}{3}\right) \\ \therefore 4x+3 &\leq 12x+1 \quad \text{Subtract } 12x \text{ from both sides of the inequality} \\ \therefore -8x &\leq -2 \\ \therefore x &\geq \frac{1}{4} \end{aligned}$$

Interval notation: $[\frac{1}{4}, \infty)$

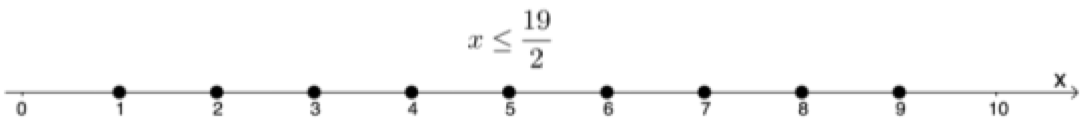
Number line:

2.

$$\begin{aligned} \frac{3}{2}(2x-1) &\leq 2(x+4),\ x \in \mathbb{N} \\ \therefore 3(2x-1) &\leq 4(x+4),\ x \in \mathbb{N} \\ \therefore 6x-3 &\leq 4x+16,\ x \in \mathbb{N} \\ \therefore 2x &\leq 19,\ x \in \mathbb{N} \\ \therefore x &\leq \frac{19}{2},\ x \in \mathbb{N} \end{aligned}$$

x can only be a natural number. Therefore, the first permitted solution less than $\frac{19}{2}$ (or 9.5) is 9. Even though there is no specific lower bound, a lower bound of 1 is enforced by the fact that $x \in \mathbb{N}$.

Number line:



Set builder notation: Set builder notation: $\{x|x \in \mathbb{N}, x \leq \frac{19}{2}\}$

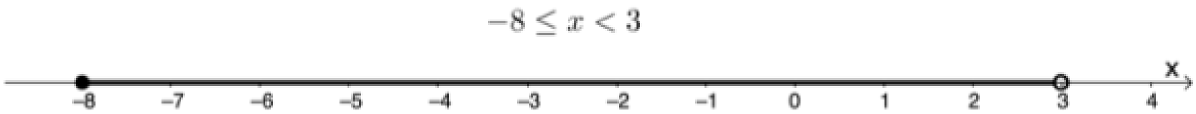
Back to Exercise 3.2

Unit 3: Assessment

1.

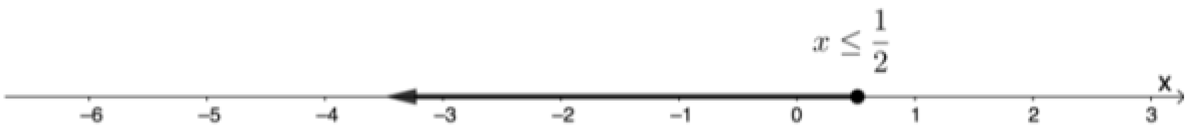
a.

$$-2 \leq x + 6 < 9$$
$$\therefore -8 \leq x < 3$$



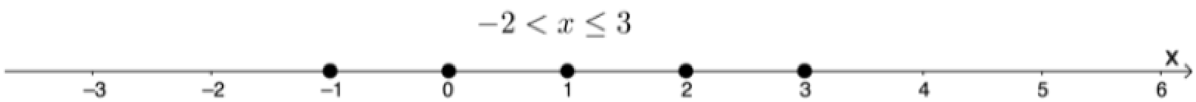
b.

$$2 - 3x \leq 3 - 5x$$
$$\therefore -3x \leq 1 - 5x \quad \text{Subtract 2 from both sides of the inequality}$$
$$\therefore 2x \leq 1 \quad \text{Add } 5x \text{ to both sides of the inequality}$$
$$\therefore x \leq \frac{1}{2}$$



c.

$$-5 < 1 + 3x \leq 10, x \in \mathbb{Z}$$
$$\therefore -6 < 3x \leq 9, x \in \mathbb{Z}$$
$$\therefore -2 < x \leq 3, x \in \mathbb{Z}$$



2.

a.

$$\frac{4y - 2}{6} > 2y + 1$$
$$\therefore 4y - 2 > 12y + 6$$
$$\therefore -8y > 8$$
$$\therefore y < -1$$

Interval notation: $(-\infty, -1)$

Set builder notation: $\{y|y \in \mathbb{R}, y < -1\}$

b.

$$\frac{1-s}{3}+\frac{1-s}{2}\leq 5$$
$$\therefore 2(1-s)+3(1-s)\leq 30$$
$$\therefore 2-2s+3-3s\leq 30$$
$$\therefore -5s\leq 25$$
$$\therefore s\geq -5$$

Interval notation: $[-5,\infty)$

Set builder notation: $\{s|s\in\mathbb{R},s\geq -5\}$

c.

$$-\frac{2x+6}{3}>4 \quad \text{or} \quad \frac{3x-3}{4}>0$$
$$\therefore -(2x+6)>12 \quad \text{or} \quad 3x-3>0$$
$$\therefore -2x-6>12 \quad \text{or} \quad 3x-3$$
$$\therefore -2x>18 \quad \text{or} \quad x>1$$
$$\therefore x<-9 \quad \text{or} \quad x>1$$

Interval notation: $(-\infty,-9)$ or $(1,\infty)$

Set builder notation: $\{x|x\in\mathbb{R},x<-9,x>1\}$

d.

$$\frac{7x+3}{4}<4(2x-7)$$
$$\therefore 7x+3<16(2x-7)$$
$$\therefore 7x+3<32x-112$$
$$\therefore -25x<-115$$
$$\therefore x>\frac{115}{25}$$
$$\therefore x>\frac{23}{5}$$

Interval notation: $(\frac{23}{5},\infty)$

Set builder notation: $\{x|x\in\mathbb{R},x>\frac{23}{5}\}$

Back to Unit 3: Assessment

Media Attributions

Unit 4: Solve simultaneous equations

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Solve systems of simultaneous equations where both equations are linear equations algebraically by means of:
  - Elimination
  - Substitution.
- Solve systems of simultaneous equations where both equations are linear equations graphically by finding the point of intersection of the functions.

What you should know

Before you start this unit, make sure you can:

- Solve linear equations; if you need to revise solving linear equations, see Unit 1 in this subject outcome.
- Sketch linear functions on the Cartesian plane; if you need help with this, refer to Subject outcome 2.1, Unit 1: Linear functions.

Introduction

In Unit 1 of this subject outcome, we learnt how to solve linear equations and quadratic equations. We saw that the highest power on the unknown in linear equations is **1** and this corresponds to **one solution**. The highest power on the unknown in quadratic equations is **2** and this corresponds to **two solutions**.

In both these cases however, there is still only **one unknown** and we only need one equation to solve for that unknown. But what if there are two unknowns like in the following case?

A hardware store sells drill bit sets. The BIG set contains 27 bits while the SMALL set contains 12 bits. In total, a college bought 459 bits for its workshops. How many of each set did the college buy?

In this case, we have **two unknowns**; the number of BIG sets bought, and the number of SMALL sets bought. If we let the number of BIG sets be x and the number of SMALL sets be y, we can set up an equation as follows.

$27x + 12y = 459$

You probably recognise this as a linear function or a straight-line graph. We can rewrite the equation as $y = \frac{27}{12}x + \frac{459}{12}$ ($y = mx + c$ or $y = ax + q$) where the gradient is $\frac{27}{12}$ and the y-intercept is $\frac{459}{12}$. If we draw a sketch of the function, we get the graph in Figure 1.

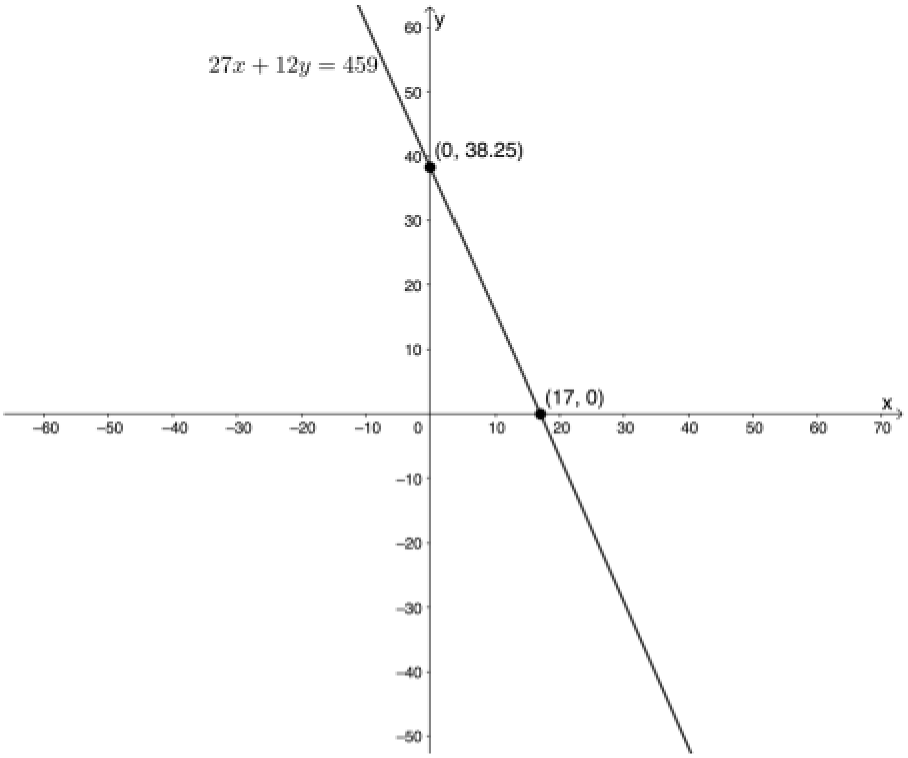


Figure 1: Graph of $27x + 12y = 459$

We can see from the graph that there are infinitely many possible solutions. All the points that lie on the graph are possible solutions for x and y. Which one do we choose? Clearly, we need more information.

Now, what if we know that each BIG set costs $\text{R}68$ and each SMALL set costs $\text{R}42$ and that the college paid a total of $\text{R}1\ 262$ for all the sets? We could set up another equation.

$68x + 42y = 1262$

This is also a linear function. What if we drew the graph of this function on the same set of axes as before (see Figure 2)?

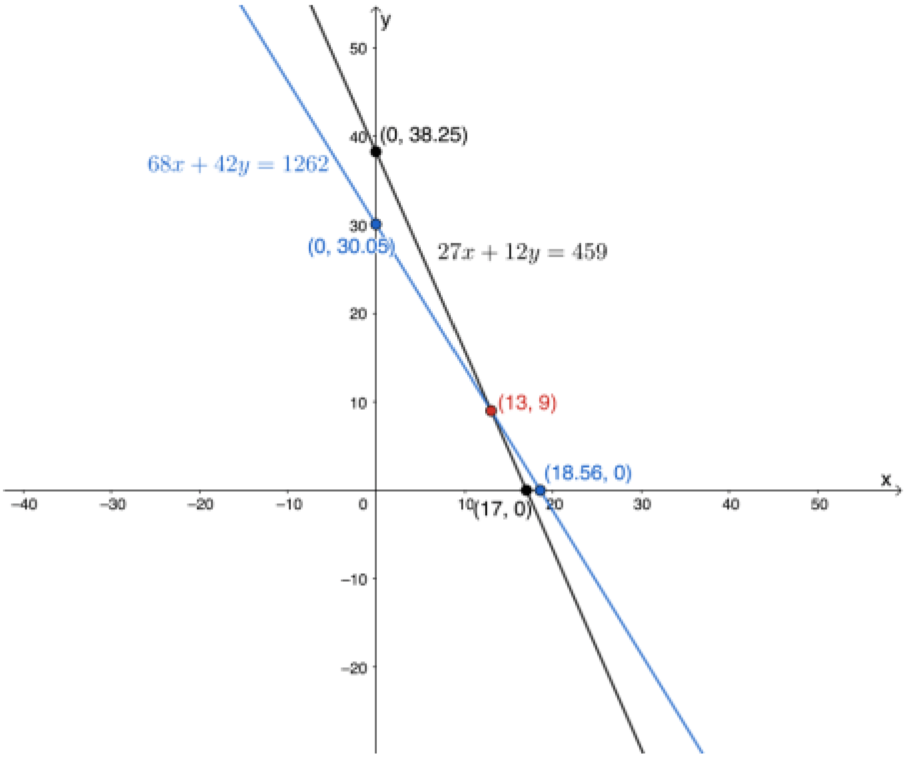


Figure 2: Graphs of $27x + 12y = 459$ *and* $68x + 42y = 1262$

Now we have a single point that lies on both graphs or a single set of values for x and y that satisfies both equations: $x = 13$ and $y = 9$. Therefore, 13 BIG sets and 9 SMALL sets were bought.

We can see that in order to solve for **two unknowns**, we need **two equations**. If we want to solve for three unknowns, we need three equations. If we want to solve for n unknowns, we need n equations.

We call sets of equations like this **simultaneous equations**. We will only solve two simultaneous equations at a time, and these will always be linear equations. To do so we have three options: substitution, elimination or graphically (as above).

Solve simultaneous equations using substitution

The process of solving simultaneous equations by substitution is quite simple.

1. Take one of the equations (usually the simplest) and rearrange it to express one of the variables in terms of the other.
2. Substitute this expression into the other equation. Now we only have one equation with one variable.
3. Solve the equation for this variable.
4. Substitute the solution for this variable back into the originally rearranged equation to solve for the other variable.
5. Write your final answer.

To help you understand this, work through the following example.

Example 4.1

Solve for x and y:

$x + y = 1$ and $3 = y + 2x$

Solution:

It is a very good idea to start by numbering each equation. This makes them easier to refer to in your working.

$x + y = 1$ (1)

$3 = y + 2x$ (2)

Step 1: Take one of the equations (usually the simplest) and rearrange it to express one of the variables in terms of the other.

From (1):

$$x + y = 1$$
$$\therefore y = 1 - x \qquad (3)$$

Note that we could have expressed x in terms of y.

Step 2: Substitute this expression into the other equation. Now we only have one equation with one variable. It is always a good idea to substitute using brackets to make sure you don't make any sign errors.

Substitute (3) into (2):

$3 = (1 - x) + 2x$

Step 3: Solve the equation.

$$\begin{aligned} 3 &= (1 - x) + 2x \\ \therefore 3 &= 1 - x + 2x \\ \therefore 2 &= x \end{aligned}$$

Step 4: Substitute the solution for this variable back into the originally rearranged equation to solve for the other variable. You can substitute this value into any of the previous equations but usually it is easiest to substitute into the equation you rearranged. Again, it is a good idea to substitute using brackets to avoid sign errors.

Substitute $x = 2$ into (3):

$$\begin{aligned} y &= 1 - (2) \\ \therefore y &= -1 \end{aligned}$$

Step 5: Write your final answer.

$$\begin{aligned} x &= 2 \\ y &= -1 \end{aligned}$$

Example 4.2

Solve for a and b:

$4a - 3b = 80$ and $4a - 18b = 125$

Solution:

$4a - 3b = 80$ (1)

$4a - 18b = 125$ (2)

From (1):
$4a = 80 + 3b$ (3)

Because the other equation also has a $4a$ term it is not necessary to express the equation as $a = \ldots$. This is also the reason it is more convenient to aim to substitute for a rather than b.

Substitute (3) into (2):

$$(80 + 3b) - 18b = 125$$
$$\therefore 80 + 3b - 18b = 125$$
$$\therefore -15b = 45$$
$$\therefore b = -3$$

Substitute $b = -3$ into (3):

$$4a = 80 + 3(-3)$$
$$\therefore 4a = 80 - 9$$
$$\therefore a = \frac{71}{4}$$

$a = \dfrac{71}{4}$
$b = -3$

Exercise 4.1

Solve for the unknowns in each case by substitution.

1. $3x = y + 27$ and $2y = 9 - 3x$
2. $s + t = 8$ and $3t + 2s = 21$
3. $\dfrac{a}{2} + b = 4$ and $\dfrac{a}{4} - \dfrac{b}{4} = 1$

The full solutions are at the end of the unit.

Note

If you would like more practise solving simultaneous equations using the substitution method, visit this link.

Solve simultaneous equations using elimination

Solving simultaneous equations by elimination is another technique you can use to solve these equations algebraically (i.e. without the use of graphs). Whenever you are asked to solve simultaneous equations algebraically, you are free to use either the substitution or elimination method.

The best way to learn how to solve simultaneous equations by elimination is to work through examples.

Example 4.3

Solve for x and y:

$3x + y = 2$ and $3x - y = 24$

Solution:

Once again, it is a very good idea to start by numbering each equation. This makes them easier to refer to in your working.

$3x + y = 2$ (1)
$3x - y = 24$ (2)

Notice how both equations have a $3x$ term. If we subtracted equation (2) from equation (1), we would eliminate this term.
(1) – (2):

$$\begin{array}{r} 3x + y = 2 \\ -(3x - y = 24) \\ \hline 0 + 2y = -22 \end{array}$$

Note: When subtracting equations, be careful with the signs. $y - (-y) = y + y = 2y$. Now we can solve for y.

$$2y = -22$$
$$\therefore y = -11$$

Substitute $y = -11$ into (1):

$$3x + (-11) = 2$$
$$\therefore 3x = 13$$
$$\therefore x = \frac{13}{3}$$

$$x = \frac{13}{3}$$
$$y = -11$$

Notice that if we had added equations (1) and (2) in the beginning, instead of subtracting equation (2) from (1), the $y\text{-term}$ would have been eliminated. Try this alternative method for yourself, and check that you get the same answers.

Example 4.4

Solve for a and b:

$2a - 3b = 5$ and $3a - 2b = 20$

Solution:

$2a - 3b = 5$ (1)
$3a - 2b = 20$ (2)

There are no obvious terms to eliminate. However, if we multiplied equation (1) by 2 and multiplied equation (2) by 3, we would create a $6b$ term in each equation that could be eliminated. Note, however, that we could also create a $6a$ term in each equation by multiplying equation (1) by 3 and equation (2) by 2. Either strategy will work.

(1) multiplied by 2: $4a - 6b = 10$ (3)
(2) multiplied by 3: $9a - 6b = 60$ (4)

Subtract (4) from (3):

$\therefore a = 10$

Substitute $a = 10$ into (1):

$$\begin{aligned} 2(10) - 3b &= 5 \\ \therefore 20 - 3b &= 5 \\ \therefore -3b &= -15 \\ \therefore b &= 5 \end{aligned}$$

$$\begin{aligned} a &= 10 \\ b &= 5 \end{aligned}$$

Exercise 4.2

Solve for the unknowns in each case by elimination.

1. $3x - y = 2$ and $6x + y = 25$
2. $4y + 3x = 100$ and $4y - 19x = 12$
3. $2c + d = 1$ and $\frac{c}{2} + \frac{d}{3} = 1$
4. $4s - 3t = 19$ and $8s - 2t = 2$

The full solutions are at the end of the unit.

Note

If you would like more practise solving simultaneous equations using the elimination method, visit this link.

Solve simultaneous equations graphically

We started this unit by solving a set of simultaneous equation graphically. We drew the graphs of the two functions and the point where these graphs intersected gave us the solution (see Figure 2).

This is possibly the most intuitive way to solve simultaneous equations and it can be used effectively when the equations involved are not both linear. For example, we can solve the following simultaneous equations graphically by drawing both graphs and reading off where they intersect (see Figure 3).

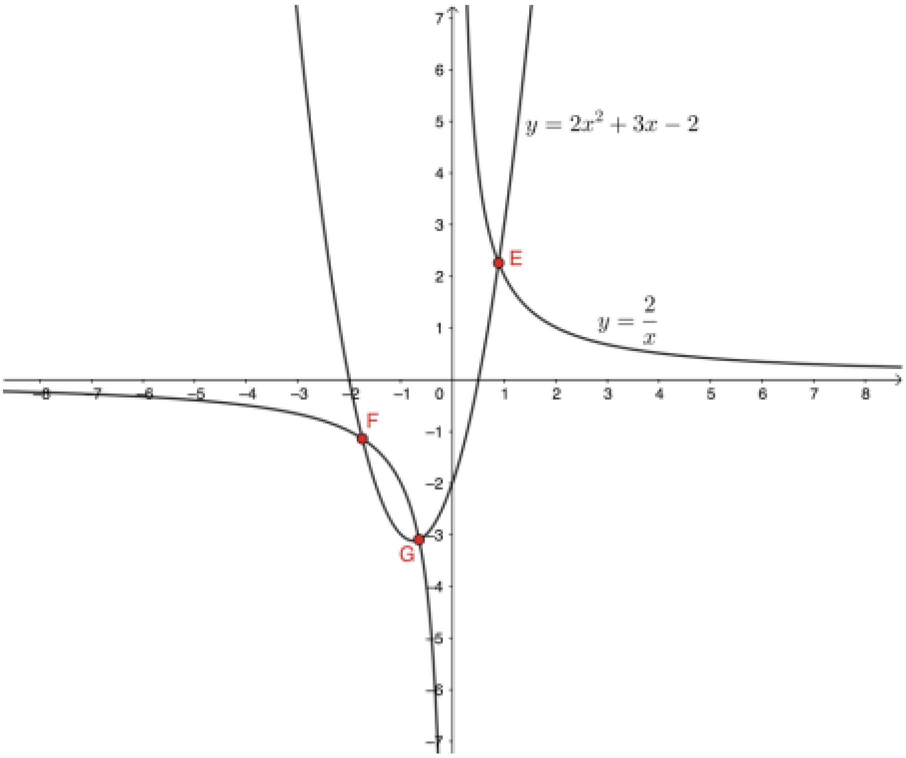


Figure 3: Graphical solution to $y = 2x^2 + 3x - 2$ *and* $y = \frac{2}{x}$

Figure 3 shows that there are three solutions represented by the coordinates of points E, F and G.

For now, we will stick to solving simultaneous equations graphically where both equations are linear.

Example 4.5

Solve the following system of simultaneous equations graphically:

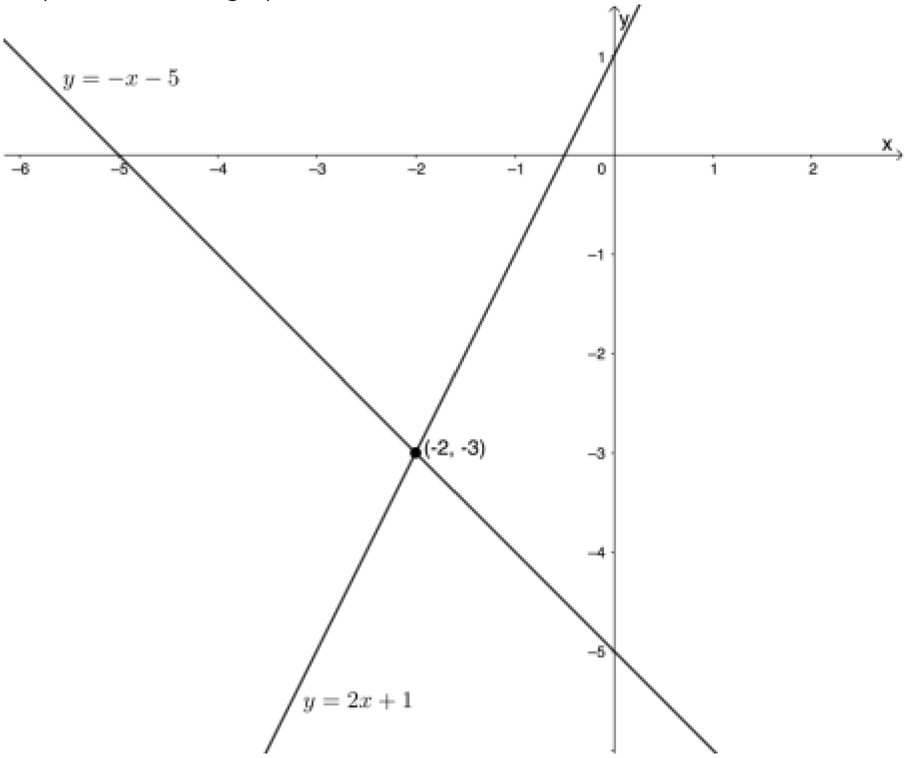
$10x - 5y = -5$ and $13y + 13x + 65 = 0$

Solution:

Step 1: Start by numbering the equations so they are easy to refer to.
$10x - 5y = -5$ (1)
$13y + 13x + 65 = 0$ (2)

Step 2: Write both equations in the form of $y = ax + q$ (or $y = mx + c$).
From (1): $y = 2x + 1$ (3)
From (2): $y = -x - 5$ (4)

Step 3: Sketch the graphs on the same set of axes.



Step 4: Find the point of intersection and read off the coordinates of the point.
The graphs intersect at $(-2,-3)$.

Step 5: Write your final answer.
$x=-2$
$y=-3$

Note

There are several online graphing tools that you can use to you sketch these graphs so that you can check your work. Two very good ones are:

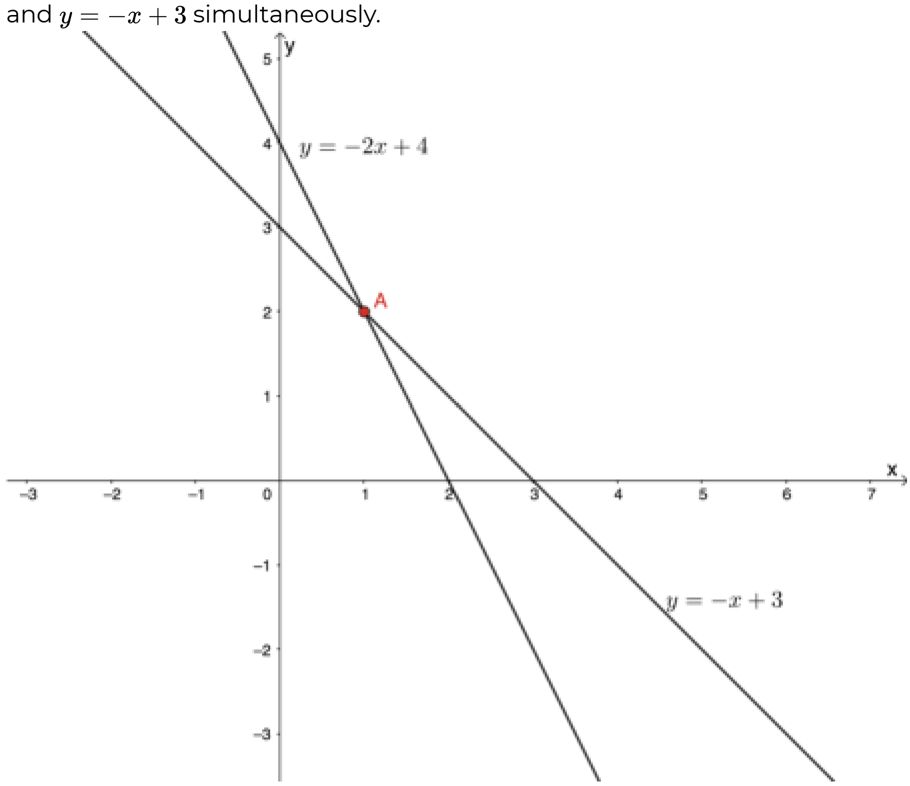
- Graph Sketch
- Desmos

Both are free to use, and work well on all platforms (PC, laptop, tablet and mobile).

If you need help sketching the graphs that represent linear equations, review Subject outcome 2.1, Unit 1: Linear functions.

Exercise 4.3

1. The graph below shows sketches of $y = -2x + 4$ and $y = -x + 3$. Solve the equations $y = -2x + 4$ and $y = -x + 3$ simultaneously.



2. Solve the following simultaneous equations graphically:
   $3x - 2y = 4$ and $2x + y = 5$
3. Consider the equations $y = 4x - 2$ and $2y - 8x = \frac{1}{2}$. Graphically or otherwise, state whether there is a solution and, if so, what this solution is.

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to solve systems of simultaneous linear equations using substitution.
- How to solve systems of simultaneous linear equations using elimination.
- How to solve systems of simultaneous linear equations graphically by finding the point of intersection of their graphs.

Unit 4: Assessment

Suggested time to complete: 60 minutes

1. Solve for the unknowns graphically and algebraically.
   $2x - y = -3$ and $2y + 6x - 1 = 0$

2. Solve the following systems of simultaneous equations algebraically.
    a. $5a + 3b + 11 = 0$ and $4b = -2a - 10$
    b. $3p + 31 = 5q$ and $2p + q = 1$
    c. $\dfrac{x+1}{y} = 7$ and $\dfrac{x}{y+1} = 6$

The following questions are taken from Engineering Maths First Aid Kit: Simultaneous Equations released under a Creative Commons BY-NC-ND licence.

3. Solve algebraically for the unknowns in the following simultaneous equations.
    a. $3x + 2y = 36$ and $5x + 4y = 64$
    b. $2x + 13y = 36$ and $13x + 2y = 69$

The full solutions are at the end of the unit.

Unit 4: Solutions

Exercise 4.1

1. $3x = y + 27$ (1)
   $2y = 9 - 3x$ (2)

   Substitute (1) into (2):
   $$2y = 9 - (y + 27)$$
   $$\therefore 2y = 9 - y - 27$$
   $$\therefore 3y = -18$$
   $$\therefore y = -6$$

   Substitute $y = -6$ into (1):
   $$3x = (-6) + 27$$
   $$\therefore 3x = 21$$
   $$\therefore x = 7$$

   $$x = 7$$
   $$y = -6$$

2. $s + t = 8$ (1)
   $3t + 2s = 21$ (2)

   From (1):
   $$s + t = 8$$
   $$\therefore s = 8 - t \quad (3)$$

   Substitute (3) into (2):
   $$3t + 2(8 - t) = 21$$
   $$\therefore 3t + 16 - 2t = 21$$
   $$\therefore t = 5$$

   Substitute $t = 5$ into (3):
   $$s = 8 - (5)$$
   $$\therefore s = 3$$

$s = 3$
$t = 5$

3. $\frac{a}{2} + b = 4$ (1)
$\frac{a}{4} - \frac{b}{4} = 1$ (2)

From (1):
$$\frac{a}{2} + b = 4$$
$$\therefore a + 2b = 8$$
$$\therefore a = 8 - 2b \qquad (3)$$

Substitute (3) into (2):
$$\frac{(8 - 2b)}{4} - \frac{b}{4} = 1$$
$$\therefore 8 - 2b - b = 4$$
$$\therefore -3b = -4$$
$$\therefore b = \frac{4}{3}$$

Substitute $b = \frac{4}{3}$ into (3):
$$a = 8 - 2\left(\frac{4}{3}\right)$$
$$\therefore a = 8 - \frac{8}{3}$$
$$= \frac{24 - 8}{3}$$
$$= \frac{16}{3}$$

$$a = \frac{16}{3}$$
$$b = \frac{4}{3}$$

Back to Exercise 4.1

Exercise 4.2

1. $3x - y = 2$ (1)
$6x + y = 25$ (2)

Add (2) to (1):
$$\begin{array}{rl} 3x - y &= 2 \\ +(6x + y &= 25) \\ \hline 9x \quad &= 27 \end{array}$$
$$\therefore x = 3$$

Substitute $x = 3$ into (1):
$$3(3) - y = 2$$
$$\therefore 9 - y = 2$$
$$\therefore y = 7$$

$x = 3$
$y = 7$

2. $4y + 3x = 100$ (1)
$4y - 19x = 12$ (2)

Subtract (2) from (1):

$$\begin{array}{r} 4y + 3x = 100 \\ -\underline{(4y - 19x = 12)} \\ 22x = 88 \end{array}$$

$\therefore x = 4$

Substitute $x = 4$ into (1):

$$\begin{aligned} 4y + 3(4) &= 100 \\ \therefore 4y + 12 &= 100 \\ \therefore 4y &= 88 \\ \therefore y &= 22 \end{aligned}$$

$x = 4$
$y = 22$

3. $2c + d = 1$ (1)
$\dfrac{c}{2} + \dfrac{d}{3} = 1$ (2)

Multiply (2) by 3: $\dfrac{3c}{2} + d = 3$ (3)

Subtract (3) from (1):

$$\begin{array}{r} 2c + d = 1 \\ -\underline{\left(\dfrac{3c}{2} + d = 3\right)} \\ \dfrac{c}{2} \qquad = -2 \end{array}$$

$\therefore c = -4$

Substitute $c = -4$ into (1):

$$\begin{aligned} 2(-4) + d &= 1 \\ \therefore -8 + d &= 1 \\ \therefore d &= 9 \end{aligned}$$

$c = -4$
$d = 9$

4. $4s - 3t = 19$ (1)
$8s - 2t = 2$ (2)

Multiply (1) by 2: $8s - 6t = 38$ (3)

Subtract (3) from (2):

$$\begin{array}{r} 8s - 2t = 2 \\ -\underline{(8s - 6t = 38)} \\ 4t = -36 \end{array}$$

$\therefore t = -9$

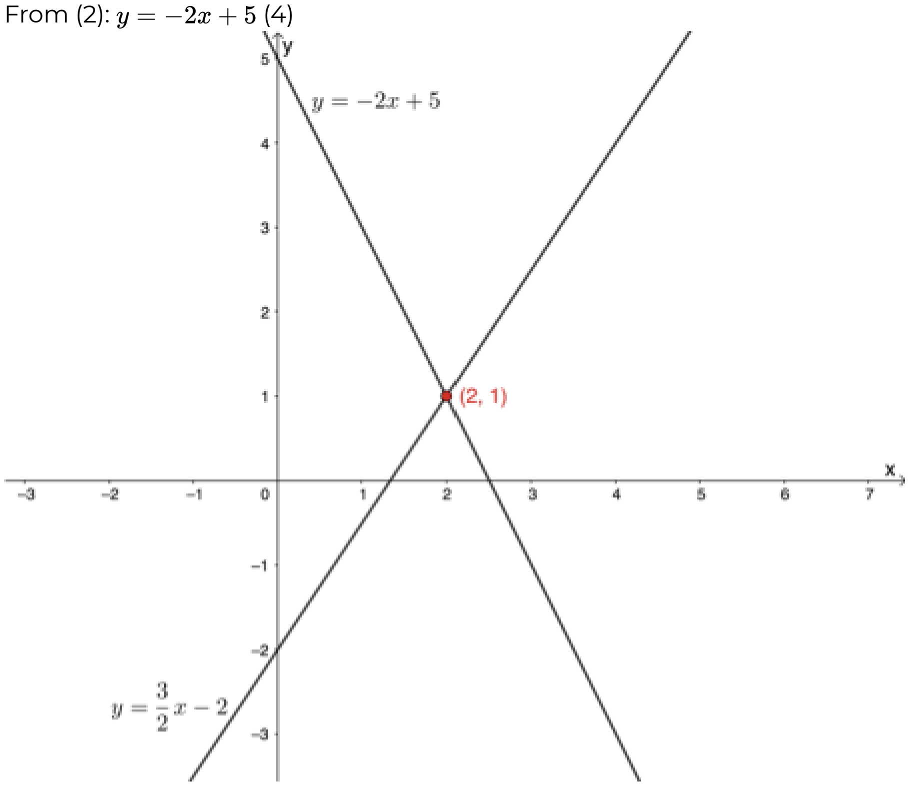
Substitute $t = -9$ into (2):

$8s - 2(-9) = 2$
$\therefore 8s + 18 = 2$
$\therefore 8s = -16$
$\therefore s = -2$

$s = -2$
$t = -9$

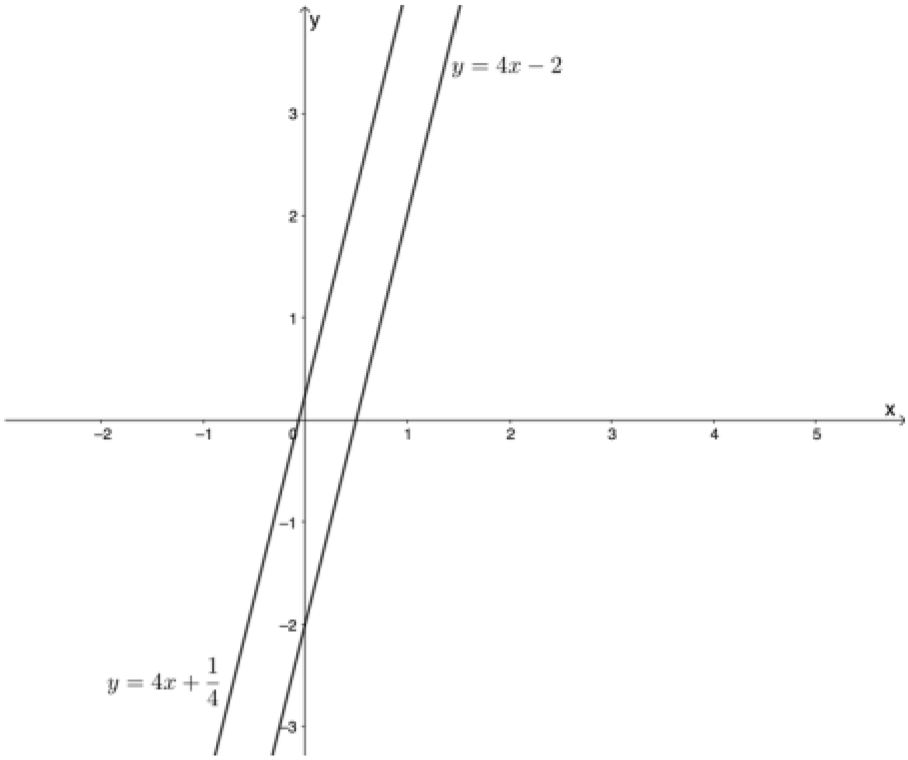
Back to Exercise 4.2

Exercise 4.3

1. From the graph, we can see that the point of intersection is the point $(1, 2)$. Therefore, the solution is:
   $x = 1$
   $y = 2$
2. $3x - 2y = 4$ (1)
   $2x + y = 5$ (2)
   From (1): $y = \frac{3}{2}x - 2$ (3)
   From (2): $y = -2x + 5$ (4)
   

   $x = 2$
   $y = 1$
3. $y = 4x - 2$ (1)
   $2y - 8x = \frac{1}{2}$ (2)
   From (2): $y = 4x + \frac{1}{4}$ (3)

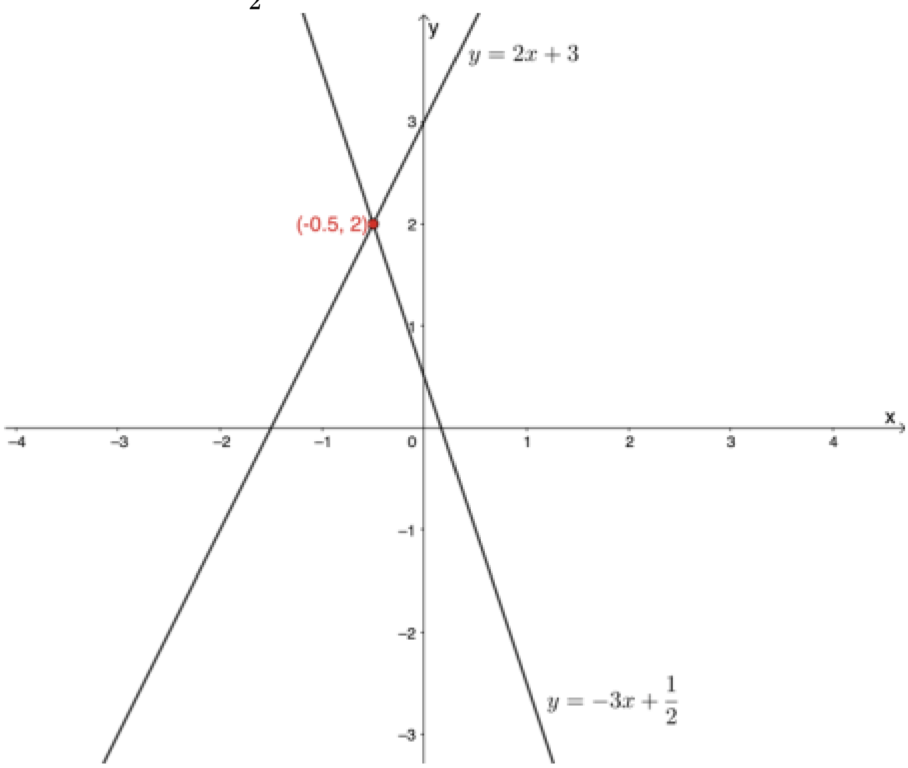
   The graphs of equations (1) and (3) have the same gradient ($m = 4/a = 4$). Therefore, these graphs are parallel and will never intersect. Therefore, there is no solution to these simultaneous equations.



Back to Exercise 4.3

Unit 4: Assessment

1. $2x - y = -3$ (1)
   $2y + 6x - 1 = 0$ (2)
   From (1): $y = 2x + 3$ (3)
   From (2): $y = -3x + \frac{1}{2}$ (4)

   Substitute (3) into (4):

$$2x + 3 = -3x + \frac{1}{2}$$
$$\therefore 5x = -\frac{5}{2}$$
$$\therefore x = -\frac{1}{2}$$

Substitute $x = -\frac{1}{2}$ into (1):

$$2\left(-\frac{1}{2}\right) - y = -3$$
$$\therefore -1 - y = -3$$
$$\therefore y = 2$$

$$x = -\frac{1}{2}$$
$$y = 2$$

2. **Note:** the substitution method has been used in the solutions. You are permitted to use the elimination method if you prefer.

   a. $5a + 3b + 11 = 0$ (1)
   $4b = -2a - 10$ (2)

   From (2):
   $$4b = -2a - 10$$
   $$\therefore 2a = -4b - 10$$
   $$\therefore a = -2b - 5 \qquad (3)$$

   Substitute (3) into (1):
   $$5(-2b - 5) + 3b + 11 = 0$$
   $$\therefore -10b - 25 + 3b + 11 = 0$$
   $$\therefore -7b = 14$$
   $$\therefore b = -2$$

   Substitute $b = -2$ into (3):
   $$a = -2(-2) - 5$$
   $$\therefore a = -1$$

   $$a = -1$$
   $$b = -2$$

   b. $3p + 31 = 5q$ (1)
   $2p + q = 1$ (2)

   From (2): $q = -2p + 1$ (3)

   Substitute (3) into (1):
   $$3p + 31 = 5(-2p + 1)$$
   $$\therefore 3p + 31 = -10p + 5$$
   $$\therefore 13p = -26$$
   $$\therefore p = -2$$

   Substitute $p = -2$ into (3):
   $$q = -2(-2) + 1$$
   $$\therefore q = 5$$

   $$p = -2$$
   $$q = 5$$

c. $\dfrac{x+1}{y} = 7$ (1)

$\dfrac{x}{y+1} = 6$ (2)

From (1):

$x+1 = 7y$

$\therefore x = 7y - 1 \quad (3)$

From (2):

$x = 6y + 6$ (4)

Substitute (3) into (4):

$7y - 1 = 6y + 6$

$\therefore y = 7$

Substitute $y = 7$ into (3):

$x = 7(7) - 1$

$\therefore x = 48$

$x = 48$

$y = 7$

3. **Note:** the elimination method has been used in the solutions. You are permitted to use the substitution method if you prefer.

a. $3x + 2y = 36$ (1)

$5x + 4y = 64$ (2)

Multiply (1) by 2: $6x + 4y = 72$ (3)

Subtract (3) from (2):

$$\begin{array}{r} 5x + 4y = 64 \\ -\underline{(6x + 4y = 72)} \\ -x \qquad\quad = -8 \end{array}$$

$\therefore x = 8$

Substitute $x = 8$ into (1):

$3(8) + 2y = 36$

$\therefore 24 + 2y = 36$

$\therefore 2y = 12$

$\therefore y = 6$

$x = 8$

$y = 6$

b. $2x + 13y = 36$ (1)

$13x + 2y = 69$ (2)

Multiply (1) by 13: $26x + 169y = 468$ (3)
Multiply (2) by 2: $26x + 4y = 138$ (4)

Subtract (4) from (3):

$$\begin{array}{r} 26x + 169y = 468 \\ -\underline{(26x + 4y = 138)} \\ 165y = 330 \end{array}$$

$\therefore y = 2$

Substitute $y = 2$ into (1):

$$2x + 13(2) = 36$$
$$\therefore 2x + 26 = 36$$
$$\therefore x = 5$$

$$x = 5$$
$$y = 2$$

Back to Unit 4: Assessment

Media Attributions

SUBJECT OUTCOME V

FUNCTIONS AND ALGEBRA: SKETCH AND INTERPRET FUNCTIONS AND GRAPHS

Subject outcome

Subject outcome 2.1: Use a variety of techniques to sketch and interpret information from graphs of algebraic and transcendental functions.

Learning outcomes

- Generate graphs by means of point-by-point plotting using/supported by available technology.
- Use the generated graphs to make and test conjectures.
- Investigate and generalise the impact of a and q on the functions:
  - $y = ax + q$
  - $y = ax^2 + q$
  - $y = \frac{a}{x} + q$
  - $y = a \cdot b^x + q$ where $b > 0$
  - $y = \sin x + q$
  - $y = \cos x + q$
  - $y = \tan x + q$
- Define functions.
- Identify the following characteristics of functions:
  - Domain and range
  - Intercepts with axes
  - Turning points, minima and maxima
  - Asymptotes
  - Shape and symmetry
  - Periodicity and amplitude
  - Functions or non-functions
  - Continuous or discontinuous
- Sketch graphs and find equations of graphs for the following:
  - $y = ax + q$
  - $y = ax^2 + q$
  - $y = \frac{a}{x} + q$
  - $y = a \cdot b^x + q$ where $b > 0$
  - $y = \sin x + q$
  - $y = \cos x + q$
  - $y = \tan x + q$

Unit 1 outcomes

By the end of this unit you will be able to:

- Define a function.
- Identify if a relationship is a function or not.
- Sketch and find the equation of a linear function ($y = ax + q$ or $y = mx + c$).
- Explain the effects on the shape of the graphs of linear functions of a and q or m and c.
- Find the equation of a linear function from its graph or other details.
- State the domain and range of a linear function.

Unit 2 outcomes

By the end of this unit you will be able to:

- Identify the following characteristics of quadratic functions:
  - turning points
  - minima and maxima
  - shape and symmetry.
- Sketch and find the equation of the graph $y = ax^2 + q$.
- Investigate and generalise the impact of a and q on $y = ax^2 + q$.

Unit 3 outcomes

By the end of this unit you will be able to:

- Identify the following characteristics of functions:
  - continuous or discontinuous
  - asymptotes.
- Sketch and find the equation of the graph $y = \frac{a}{x} + q$.
- Investigate and generalise the impact of a and q on $y = \frac{a}{x} + q$.

Unit 4 outcomes

By the end of this unit you will be able to:

- Sketch and find the equation of the graph $y = a.b^x + q, b > 0$.
- Investigate and generalise the impact of a and q on $y = a.b^x + q, b > 0$.

Unit 5 outcomes

By the end of this unit you will be able to:

- Identify the following characteristics of trigonometric functions:
  - periodicity
  - amplitude.
- Sketch the graph $y = a\sin x + q$, $y = a\cos x + q$ and $y = a\tan x + q$.
- Identify the asymptotes of $y = a\tan x + q$.
- Investigate and generalise the impact of a and q on $y = asinx + q$, $y = a\cos x + q$ and $y = a\tan x + q$.

Unit 1: Linear functions

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Define a function.
- Identify if a relationship is a function or not.
- Sketch and find the equation of a linear function ($y = ax + q$ or $y = mx + c$).
- Explain the effects on the shape of the graphs of linear functions of a and q or m and c.
- Find the equation of a linear function from its graph or other details.
- State the domain and range of a linear function.

What you should know

Before you start this unit, make sure you can:

- Manipulate and simplify algebraic expressions. Go over Subject outcome 2.2, Unit 1: Simplifying algebraic expressions if you need more help with the basics.
- Solve linear equations. Go over Subject outcome 2.3, Unit 1: Solve linear and quadratic equations if you need more help with the basics.
- Plot points on the Cartesian plane.

Here is a short self-assessment to make sure you have the skills you need to proceed with this unit.

Solve for x in each case.

1. $5x = 2x + 45$
2. $-7x = 8(1 - x)$
3. $\frac{1}{5}(x - 1) = \frac{1}{3}(x - 2) + 3$

Solutions

1. $$\begin{aligned} 5x &= 2x + 45 \\ \therefore 3x &= 45 \\ \therefore x &= 15 \end{aligned}$$
2. $$\begin{aligned} -7x &= 8(1 - x) \\ \therefore -7x &= 8 - 8x \\ \therefore x &= 8 \end{aligned}$$
3.

$$\begin{aligned}\frac{1}{5}(x-1) &= \frac{1}{3}(x-2)+3\\ \therefore 3(x-1) &= 5(x-2)+45\\ \therefore 3x-3 &= 5x-10+45\\ \therefore -2x &= 38\\ \therefore x &= -19\end{aligned}$$

Introduction

Functions are one of the primary reasons mathematics is so important and useful. Functions let us describe and explore the relationships between different quantities. This in turn helps us design and build real things like buildings, planes, computers, and cell phones. Once we can explore the relationship between things, we can use this information to predict changes in populations and the economy, and even fight diseases like cancer and HIV.

Functions

What is a function?

Before we can start working with linear functions, we need to know what functions are. Let's start with an activity.

Activity 1.1: What is a function?

Time required: 45 minutes

What you need:

- a pen or pencil
- four bottle tops and jar lids
- a ruler
- string
- a calculator
- blank paper or a notebook

What to do:

1. Collect four bottle tops and jar lids of different sizes (so circles of different sizes). Now, carefully use your ruler to measure the diameter of each top or lid (that is the straight-line distance from one side to the other through the centre). Use a piece of string to measure the circumference (the distance around the outside edge of each top or lid) by tightly wrapping the string around the lid and then measuring the length of string needed to go once all the way around.

   NOTE: If you cannot get real lids, then just draw a few circles of different sizes on a piece of paper using a compass.

   Record your measurements for each top or lid in a table like this one:

	Lid 1	Lid 2	Lid 3	Lid 4
Diameter				
Circumference				
$\frac{\text{circumference}}{\text{diameter}}$				

In the last row, calculate the circumference divided by the diameter for each lid to 2 decimal places.

Now answer these questions:

a. What do you notice about your answers?
b. Do you recognise this number? What number is it?
c. If you know the diameter of a lid, how could you predict its circumference? Why?
d. Write an equation that shows the relationship between the circumference and diameter.
e. If the diameter of a certain lid is 12 cm, what is its circumference?
f. If the circumference of a certain lid is 15.24 cm, what is its diameter?
g. If you know the diameter is there any chance that there is more than one corresponding circumference?

2. Now look at the partly completed table of values in Table 1. The values have been generated using the equation $y = \pm\sqrt{25 - x^2}$. This just so happens to be the equation of a circle with a radius of five units and its centre on the origin of the Cartesian plane.

Table 1: Table of values of x and y related by the equation $y = \pm\sqrt{25 - x^2}$

Values of x	-5	-4	-3	-2	-1	0	1	2	3	4	5
Values of y	0		± 4	± 4.58						± 3	

a. Complete the table (rounding your answers off to two decimal places where necessary).
b. Does every value of x correspond with only a single value of y?
c. Draw your own circle with a radius of five units and its centre on the origin of a Cartesian plane. Why do you think each value of x is associated with more than one value of y?

What did you find?

In Question 1, you measured the diameter and circumference of various differently-sized lids. You should have noticed that all the answers in the bottom row of your table were about 3.14. In other words, in each case, the circumference was about 3.14 times greater than the diameter. Table 2 shows the values someone measured.

Table 2: Completed table of the measurement of the diameters and circumferences of various lids

	Lid 1	Lid 2	Lid 3	Lid 4
Diameter	2 cm	2.4 cm	5.2 cm	10.7 cm
Circumference	6.3 cm	7.5 cm	16.4 cm	33.9 cm
$\frac{\text{circumference}}{\text{diameter}}$	3.15	3.13	3.15	3.14

We can see that all the answers are more or less the same and that they are all about 3.14. You might know that this number is an estimate of pi (π).

Because the ratio of the length of the circumference to the diameter of a circle is a constant number, we can use this to predict the circumference of any lid if we know its diameter.

We can say that $\frac{\text{circumference}}{\text{diameter}} \approx 3.14$. Therefore, $\text{circumference} \approx \text{diameter} \times 3.14$. So, if the diameter is 12 cm, the circumference will be $\text{circumference} \approx 12 \times 3.14 = 37.68\text{cm}$.

If we know that $\frac{\text{circumference}}{\text{diameter}} \approx 3.14$, we can also say that $\frac{\text{circumference}}{3.14} \approx \text{diameter}$. So, if the circumference is 15.24 cm, the diameter will be diameter $\approx \frac{\text{circumference}}{3.14} = \frac{15.24\text{cm}}{3.14} = 4.85\text{cm}$.

For every diameter (or circumference), there is only one size of circle that can exist so there is only one corresponding circumference (or diameter). We can see from the equation $\frac{\text{circumference}}{\text{diameter}} \approx 3.14$ that no matter what diameter we put into the equation, we will only ever get one answer for the circumference.

In Question 2, you used the equation for a circle of radius five units and centre on the origin of the Cartesian plane ($y = \pm\sqrt{25 - x^2}$) to complete a table of values.

You should have seen that for every value of x we feed into the equation, we get two values of y out of the equation (except for when $x = -5$ or $x = 5$).

If we draw a circle of radius five units with its centre on the origin of the Cartesian plane, we get Figure 1.

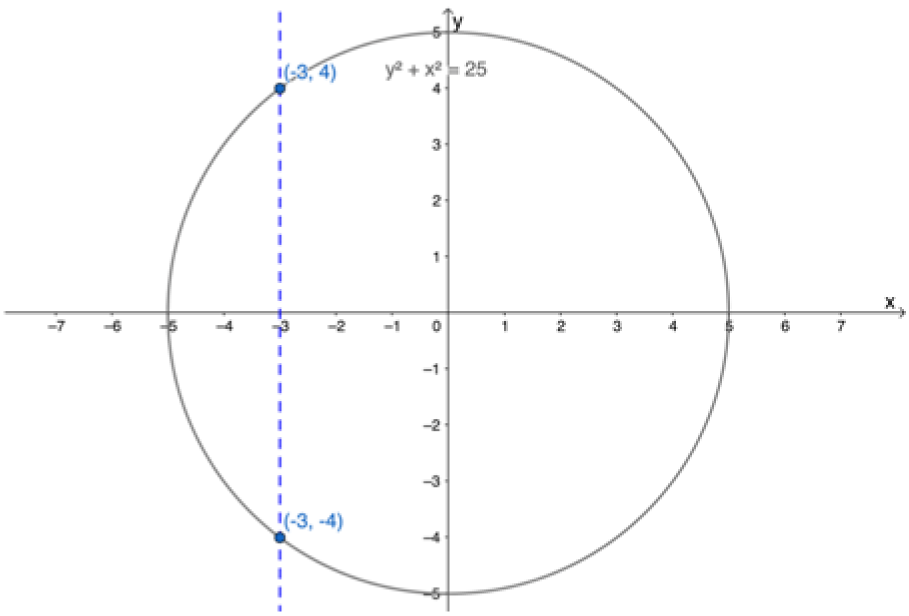


Figure 1: Graph of the circle given by $y = \pm\sqrt{25 - x^2}$

Looking at Figure 1, we can see why each value of x that we put into the equation gives us two values of y out. For example, putting $x = -3$ into the equation gives us $y = 4$ and $y = -4$.

In Activity 1.1 we saw two different kinds of relationships. The relationship between diameter and circumference was a one-to-one relationship. In other words, each diameter corresponded to **one and only one** circumference. We could use the relationship between diameter and circumference to determine exactly the circumference for a given diameter (or the diameter for a given circumference).

However, the relationship between x and y given by the equation $y = \pm\sqrt{25 - x^2}$ was different. It was an example of a one-to-many relationship. Each value of x corresponded to more than one value of y. We could not use this relationship to determine an exact y value for a given x value (except for $x = \pm 5$). We got more than one answer with no way to know which one to choose.

We call both of these types of relationships between values **relations**. But the relationship between diameter and circumference is a special kind of relation called a **function**. It is special because we can use it to determine an exact circumference for any given diameter. We only got one answer when we put a diameter into the equation.

Before we move on, have a look at this next example.

Example 1.1

In South Africa, electricity costs $170\ \text{c}$ per kWh (kilowatt hour) with a basic delivery charge of $\text{R}\ 485$ per month.

1. How much will a household pay if they use $50\ \text{kWh}$ in a month?
2. How much will a household pay if they use $75\ \text{kWh}$ in a month?
3. How much will a household pay if they don't use any electricity in a month?
4. Write an equation that describes the relationship between the amount of electricity a household uses and the amount they have to pay.
5. How much electricity can a household use in a month if they can only spend $\text{R}600$ on electricity?
6. Is this relation between how much electricity a household uses and how much they have to pay a function or not?

Solution

1. We know that for every kWh, households will have to pay $\text{R}1.70$. Therefore, they will have to pay $\text{R}1.70/\text{kWh} \times 50\ \text{kWh} = \text{R}85.00$. But there is also a basic delivery charge of $\text{R}485$ per month. So their total bill will be $\text{R}85 + \text{R}485 = \text{R}570$ for the month.
2. If they use $75\ \text{kWh}$ they will have to pay $\text{R}1.70/\text{kWh} \times 75\ \text{kWh} = \text{R}127.50$. But there is also a basic delivery charge of $\text{R}485$ per month. So their total bill will be $\text{R}127.50 + \text{R}485 = \text{R}612.50$ for the month.
3. If they don't use any electricity, they will still have to pay the basic service charge of $\text{R}485$ per month.
4. If we let the amount of electricity used be U and the total cost (in Rands) for a month be C then the equation will be $C = 1.70U + 485$.
5. If they can only spend $\text{R}600$ on electricity, then $C = 600$ and we need to solve for U.

$$\begin{aligned} C &= 1.70U + 485 \\ \therefore 1.70U &= C - 485 \\ \therefore U &= \frac{C - 485}{1.70} \\ \text{But } C &= 600 \\ \therefore U &= \frac{600 - 485}{1.70} = 67.6\ \text{kWh} \end{aligned}$$

6. Because we only ever get one answer from the equation $C = 1.70U + 485$ for each value of U, this relation is a function.

Variables and function machines

Think back to Activity 1.1. Here we saw that there was a relationship between diameter and circumference. We call the values for diameter and circumference **variables** as their values were not fixed but could change or vary.

In our case, the circumference depended on the diameter. The bigger the diameter, the greater the circumference. We call circumference the **dependent** variable and diameter the **independent** variable.

In Example 1.1, the amount you pay for electricity **depends** on the amount of electricity you use. The cost was the **dependent** variable and the number of kilowatt hours was the **independent** variable.

In both cases, we can think of these relations as special number machines that take **inputs** (the independent variables) and give back **outputs** (the matching dependent variables). The number machine for circumference and diameter takes the input of diameter, multiplies this by π and returns the circumference (figure 2). The number machine for cost of electricity and kilowatt hours, takes the input of kilowatt hours, multiples this by 1.70, adds 485, and returns the corresponding output of cost (figure 3).

It is because each machine only ever gives a single output for each input that we know they are the special types of relations called functions.

We say that **circumference is a function of diameter** or that **cost is a function of kilowatt hours**.

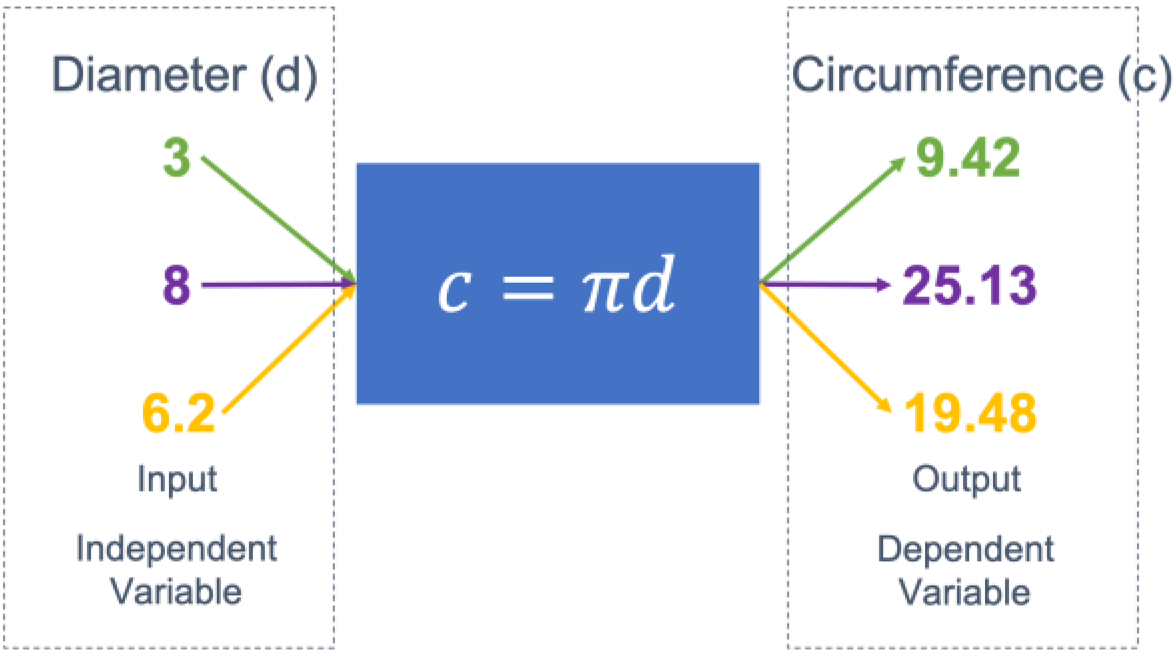


Figure 2: Diameter to circumference mapping

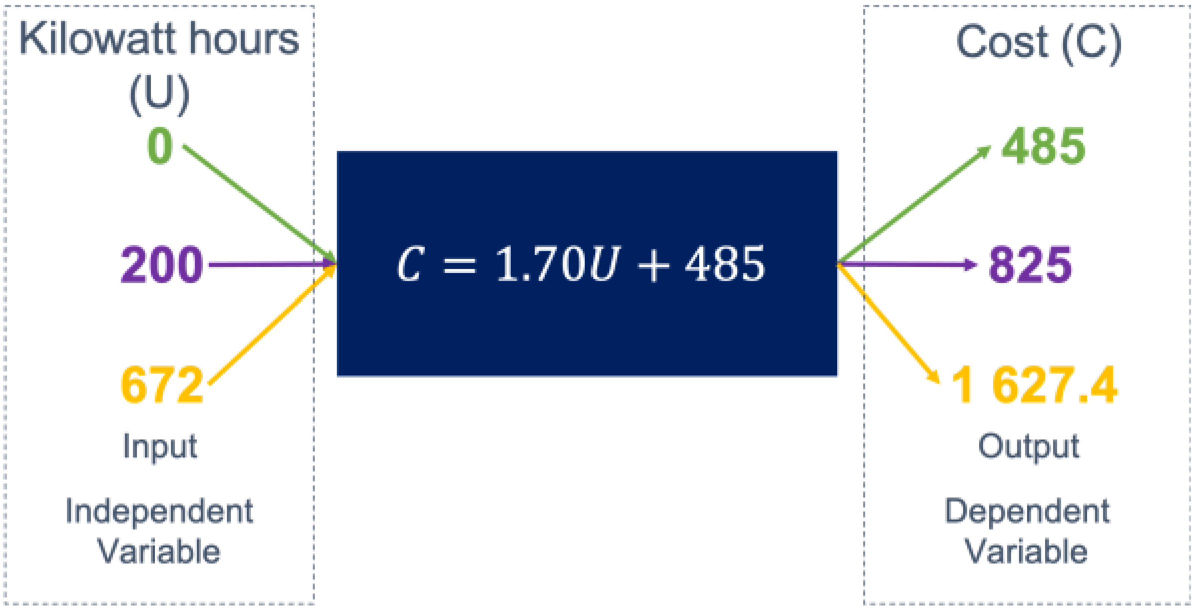


Figure 3: kW to cost mapping mapping

These functions map each an input value to a single output value. We can represent the relationship between these values using **mapping diagrams** as shown in Figures 4 and 5. We call the set of input values the **domain** and the set of corresponding output values the **range**.

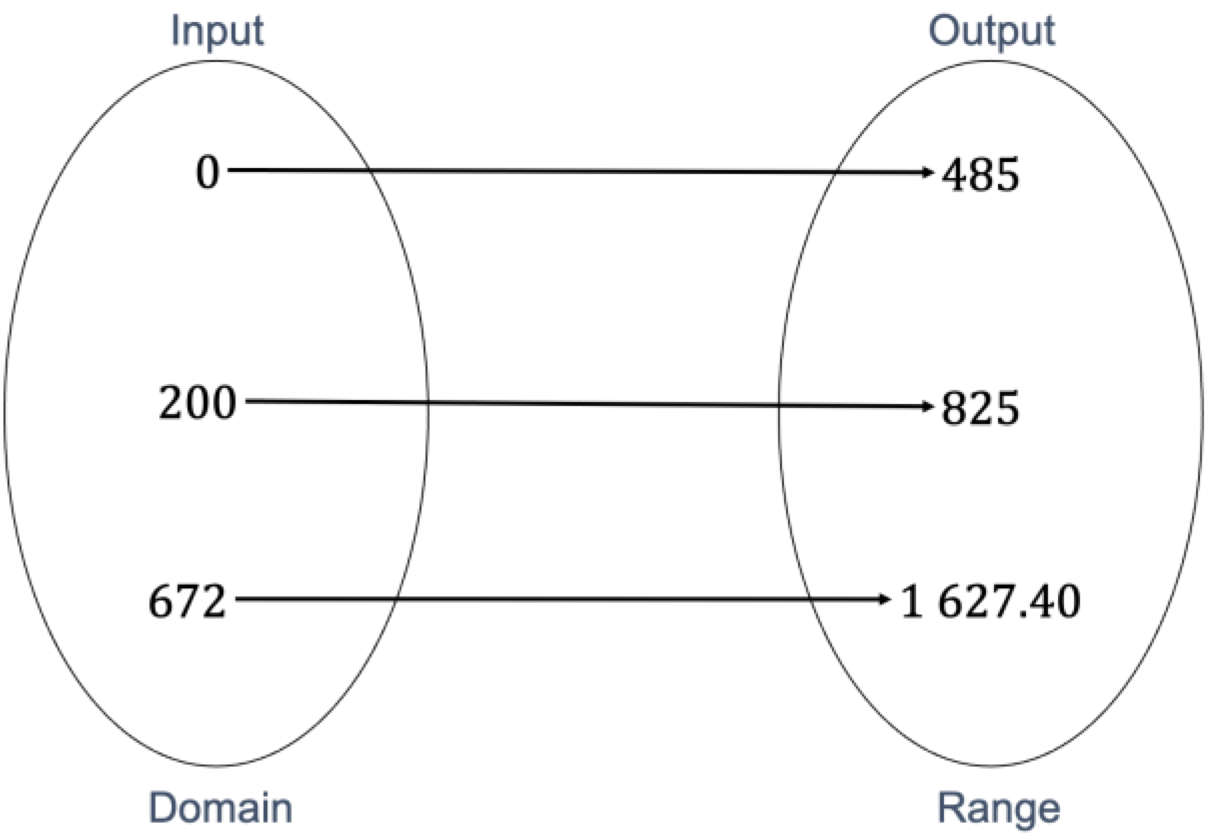


Figure 4: kW to cost function mapping

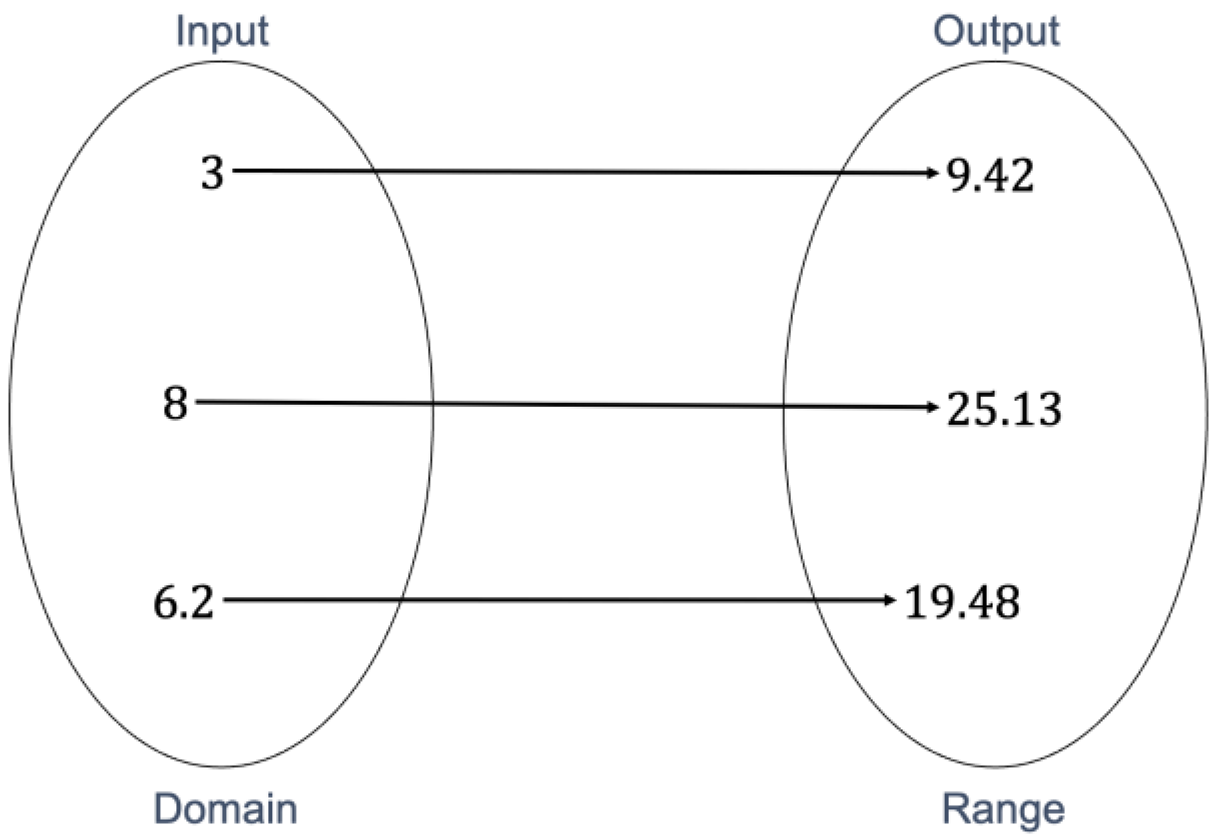
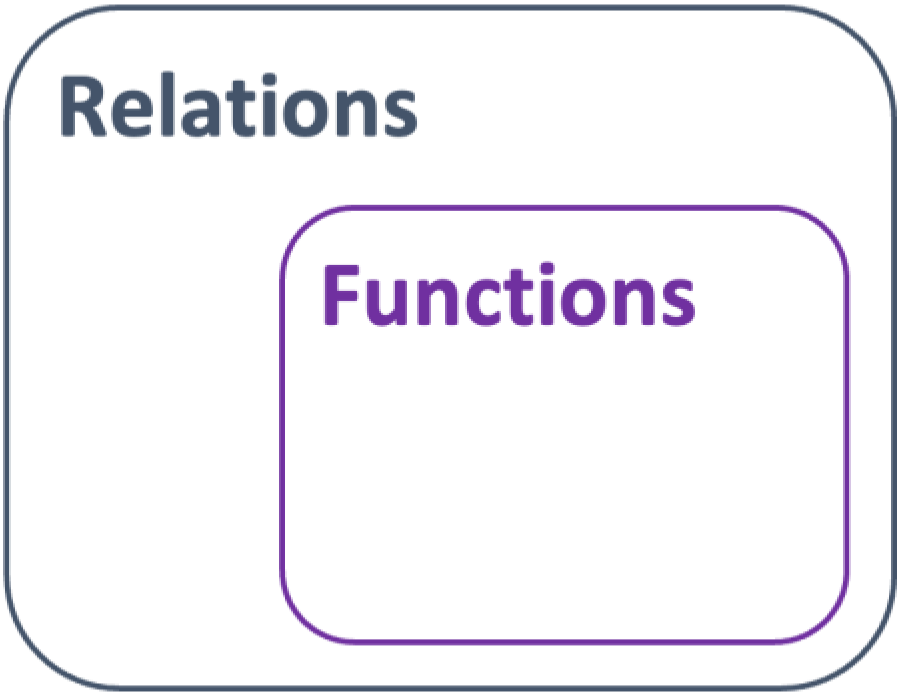


Figure 5: Diameter to circumference function mapping

Now we can properly define functions.

A function is a mathematical relation between two variables that maps each element of the **domain** (the set of input values) to **one, and only one,** element in the **range** (the set of output values).

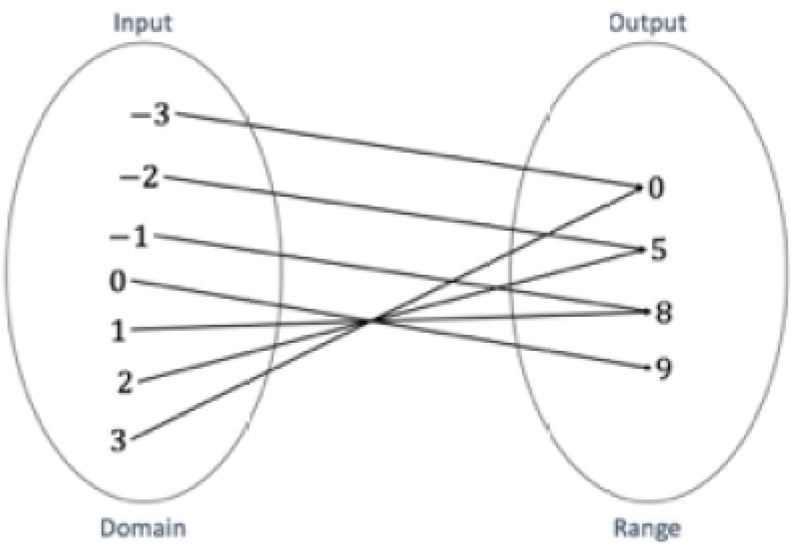
This means that every function is a relation but not every relation is a function.



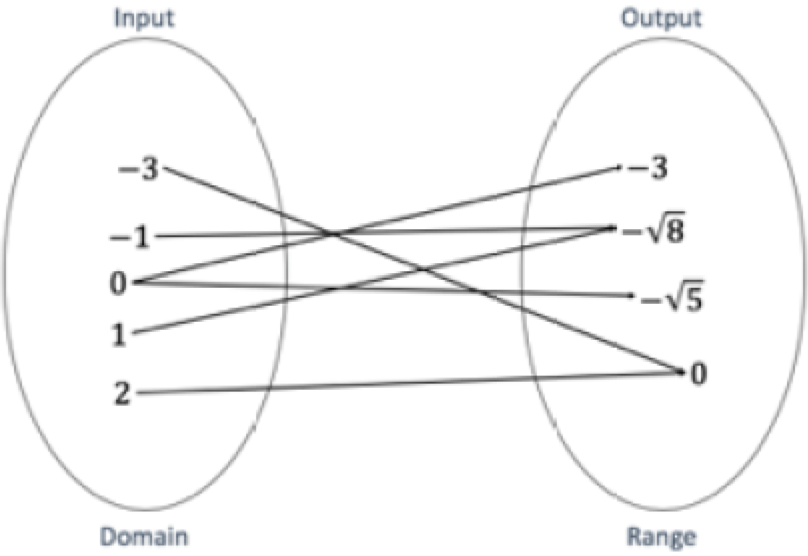
Example 1.2

Are the following mappings of functions or not?

1.



2.



Solution

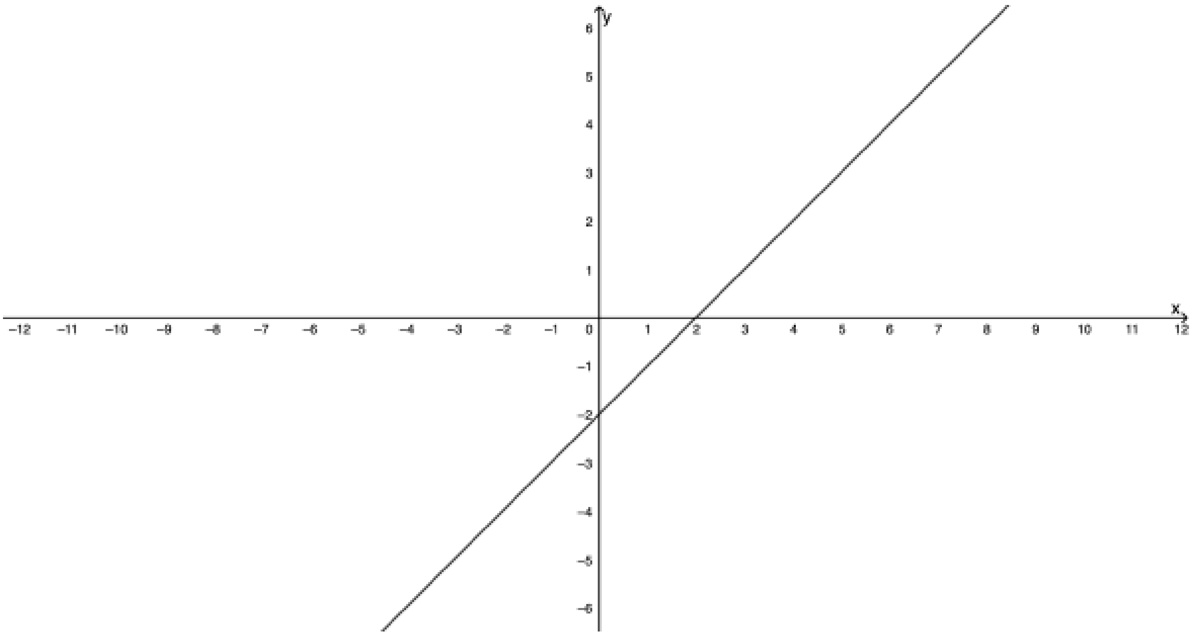
1. This mapping is of a function. In every case, each input maps to one, and only one, output. It does not matter that more than one input maps to the same output because there is still only one output for each input we put into the machine. This shows us that functions can be one-to-one or many-to-one (where you can get the same output from more than one input but there is still only one output for each input).
2. This mapping is not of a function. In the case of 0, the input maps to **more than one** output. This is an example of a one-to-many relationship. One-to-many relations cannot be functions because an input can give **more than one** output.

We have a very simple test to check if a relation is a function or not. It is called the **vertical line test** and we apply it to the picture or graph of the relation. In the test, move a ruler, or any other vertical line, across the graph from left to right. If, at any point, the vertical line cuts the graph more than once, it means that for at least one input (x value), the relation gives more than one output (y value) and it is, therefore, not a function.

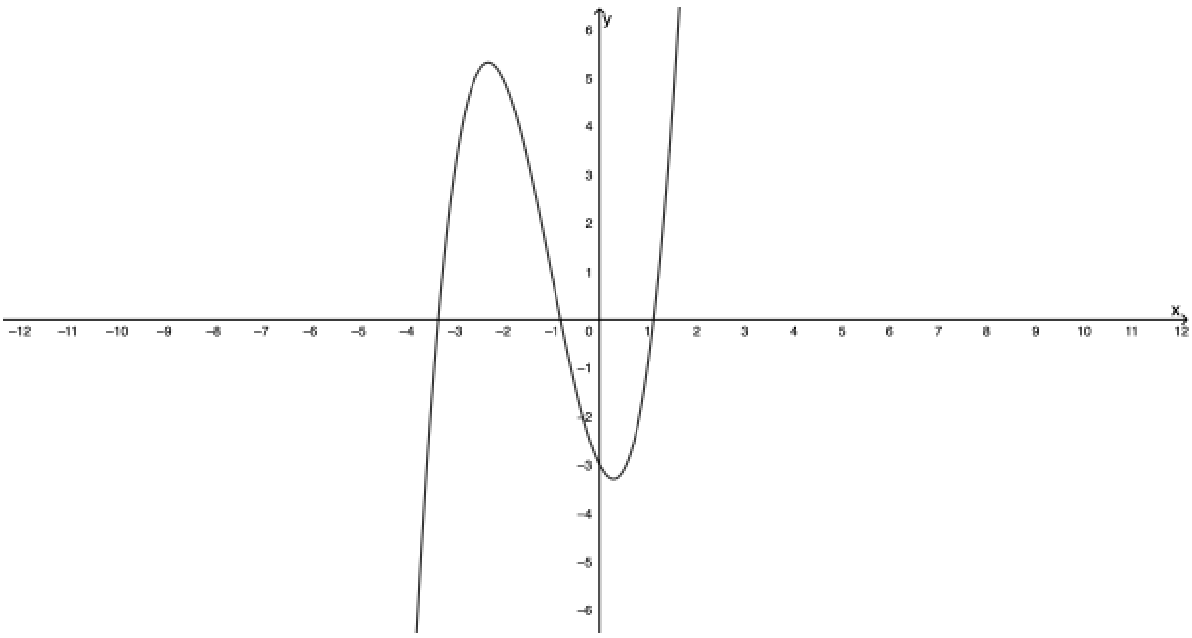
Example 1.3

Do the following graphs represent functions or not?

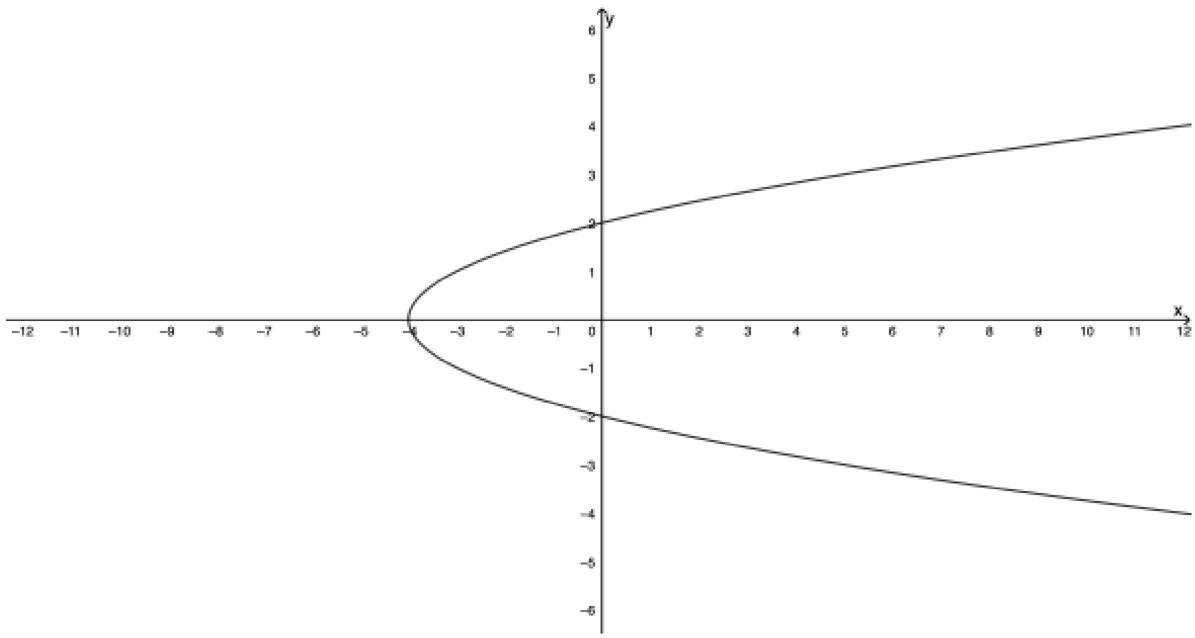
1.


y
x
6
5
4
3
2
1
−12
−11
−10
−9
−8
−7
−6
−5
−4
−3
−2
−1
0
1
2
3
4
5
6
7
8
9
10
11
12
−1
−2
−3
−4
−5
−6

2.


y
x
6
5
4
3
2
1
−12
−11
−10
−9
−8
−7
−6
−5
−4
−3
−2
−1
0
1
2
3
4
5
6
7
8
9
10
11
12
−1
−2
−3
−4
−5
−6

3.



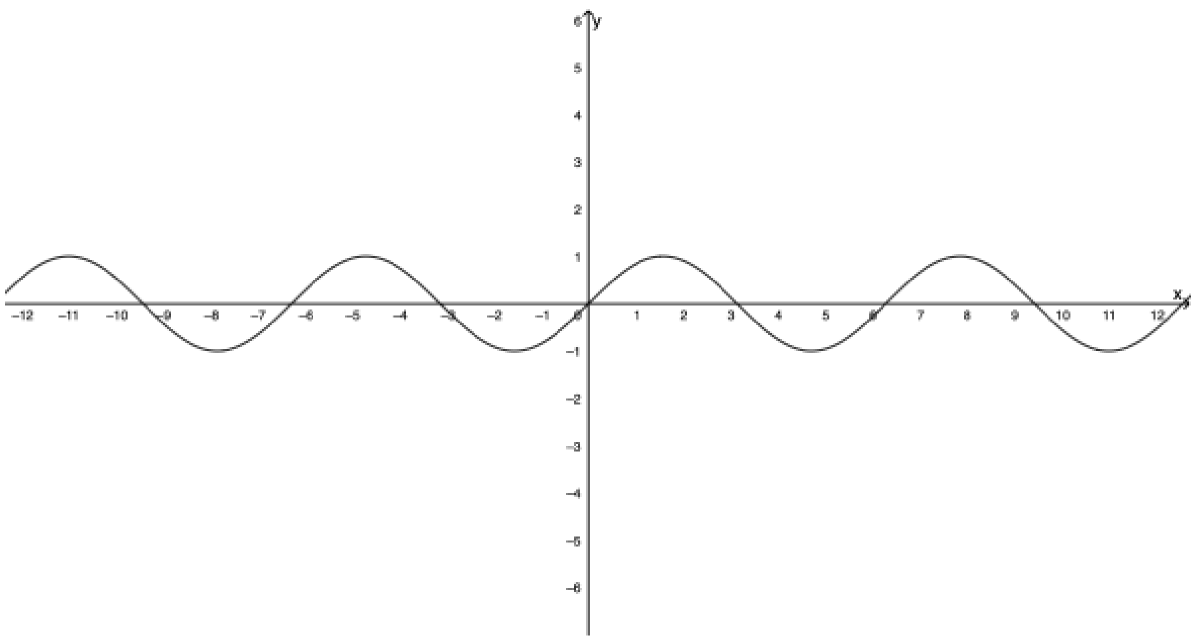
Solution

1. This graph does represent a function. At no point does a vertical line cut the graph more than once. Therefore, no input (x) value ever gives more than one output (y) value.
2. This graph does represent a function. At no point does a vertical line cut the graph more than once. Therefore, no input (x) value ever gives more than one output (y) value.
3. This graph does not represent a function. For all the input values greater than -4 a vertical line cuts the graph more than once. Therefore, there is more than one output (y) value for these input (x) values.

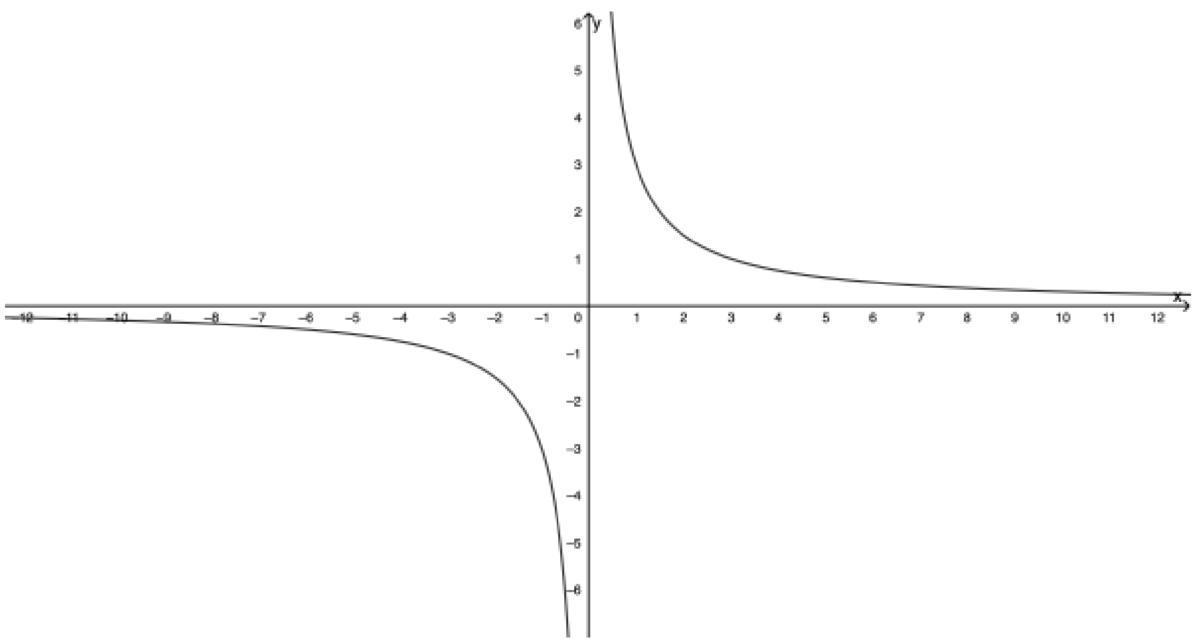
Exercise 1.1

Do the following graphs represent functions or not?

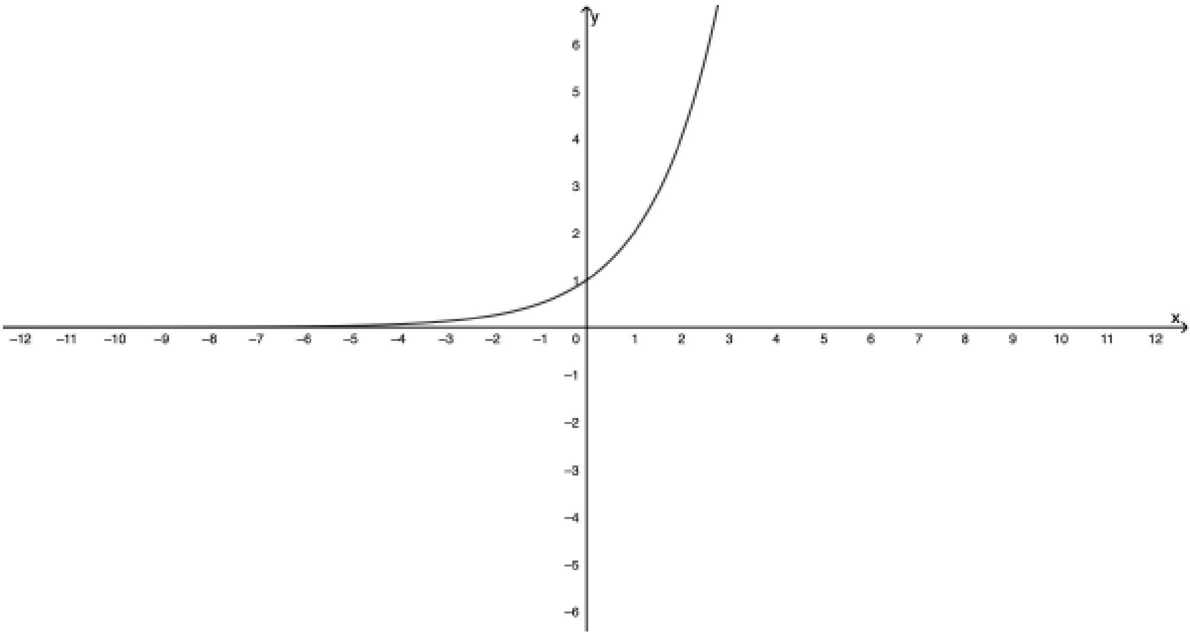
1.



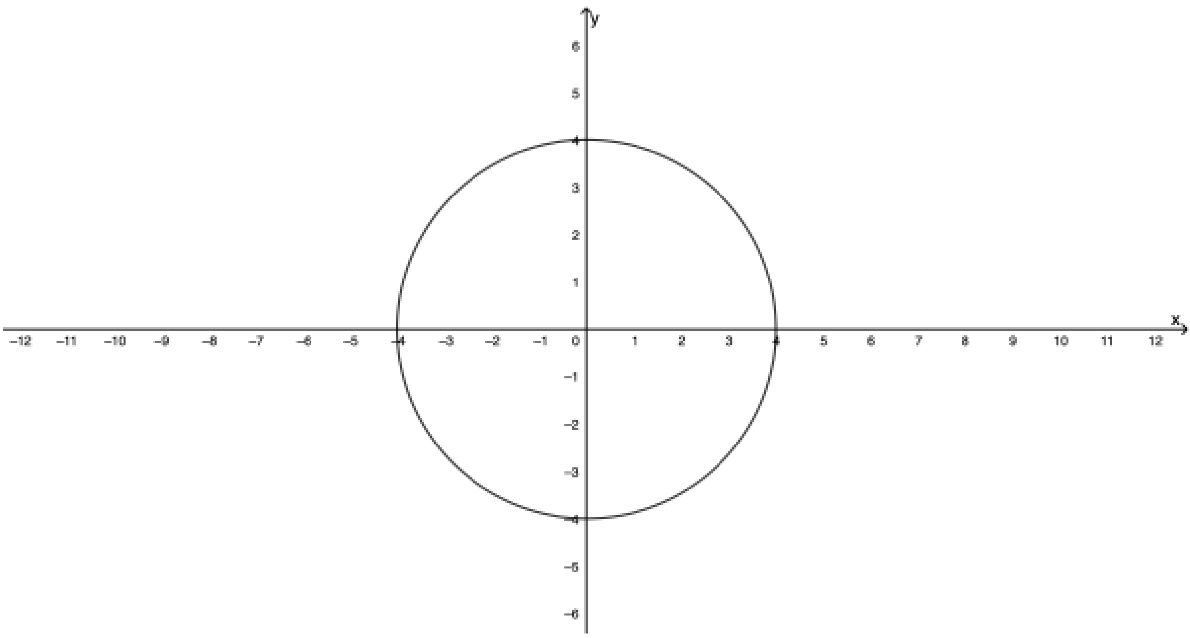
2.



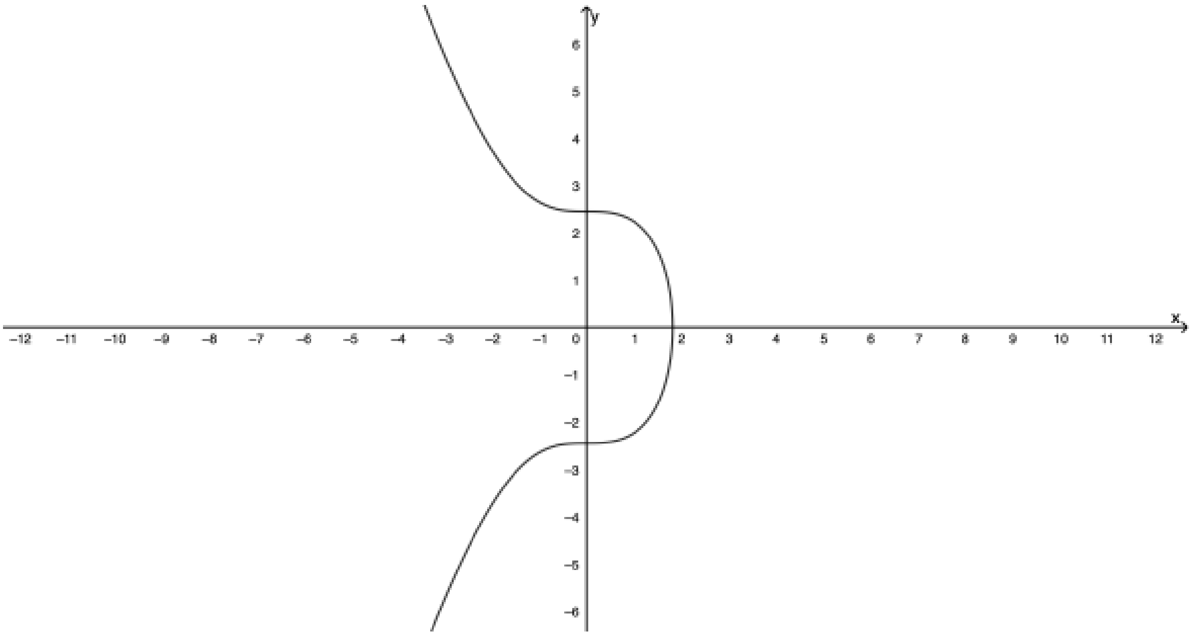
3.



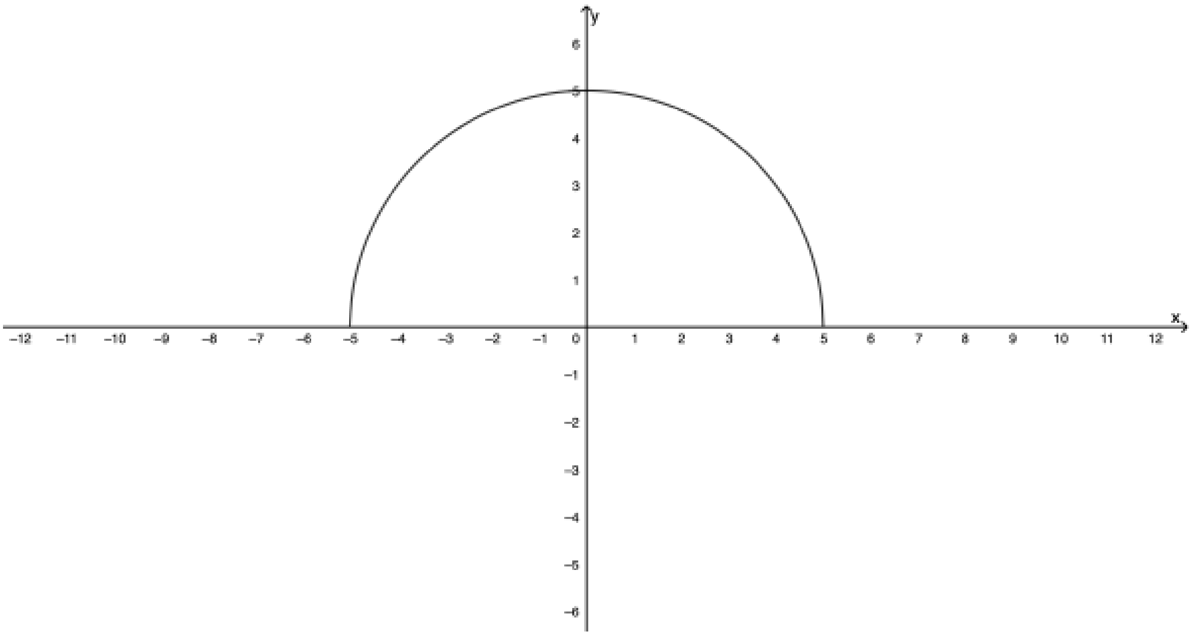
4.



5.



6.



The full solutions are at the end of the unit.

Note

If you have an internet connection, you can practise using an online vertical line tester. Here you will find a variety of different graphs. Choose one and then press the Start button to move the vertical line. Which of the 6 different graphs are not functions?

Representing functions

Now that we have a definition of a function and a way to test to see if the relationship between variables is a function or not, let's see how we can represent functions in different ways.

Activity 1.2 will help you to understand the different ways we have of representing functions.

Activity 1.2: Representing functions

Time required: 15 minutes

What you need:

- a pen or pencil
- blank paper or a notebook

What to do:

Have a look at this relation: $y = x - 5$.

1. Is this relation a function?
2. Complete the table.

Input variable x	-3		5
Output variable y		-5	

3. Draw a mapping diagram using the values from the completed table.
4. Write each related input and output as an ordered pair, then plot these ordered pairs on a Cartesian plane and join the points with a line.
5. What kind of graph is this?
6. What kind of equation is $y = x - 5$? Why do you think it is called this?

What did you find?

Each input will only ever produce a single output. Therefore, the relation is a function.

Here is the completed table.

Input variable x	-3	0	5
Output variable y	-8	-5	0

Figure 6 shows the corresponding mapping diagram.

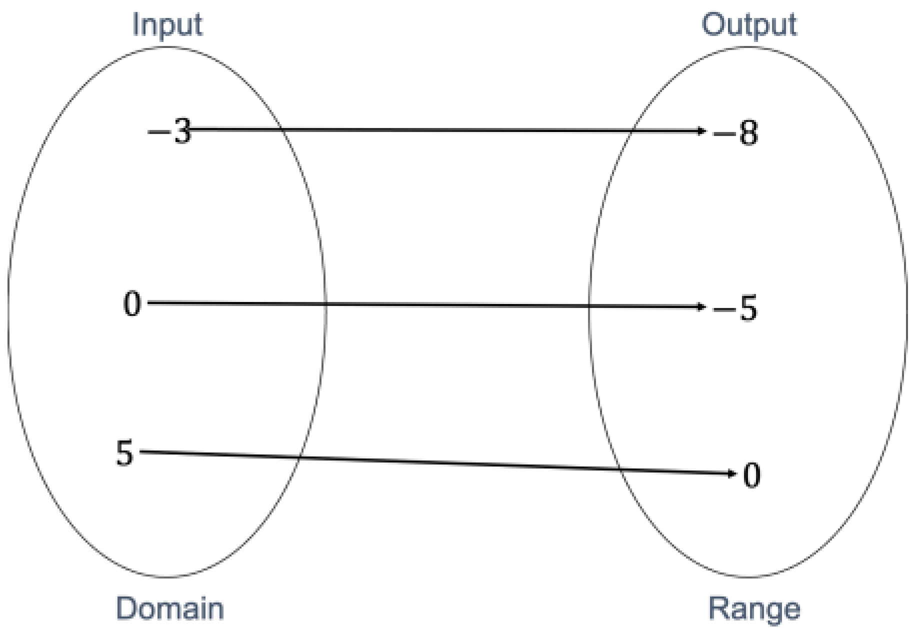


Figure 6: Mapping diagram of $y = x - 5$

The three ordered pairs are $(-3, -8), (0, -5), (5, 0)$.

Figure 7 shows a Cartesian plane with these three points plotted and joined with a line.

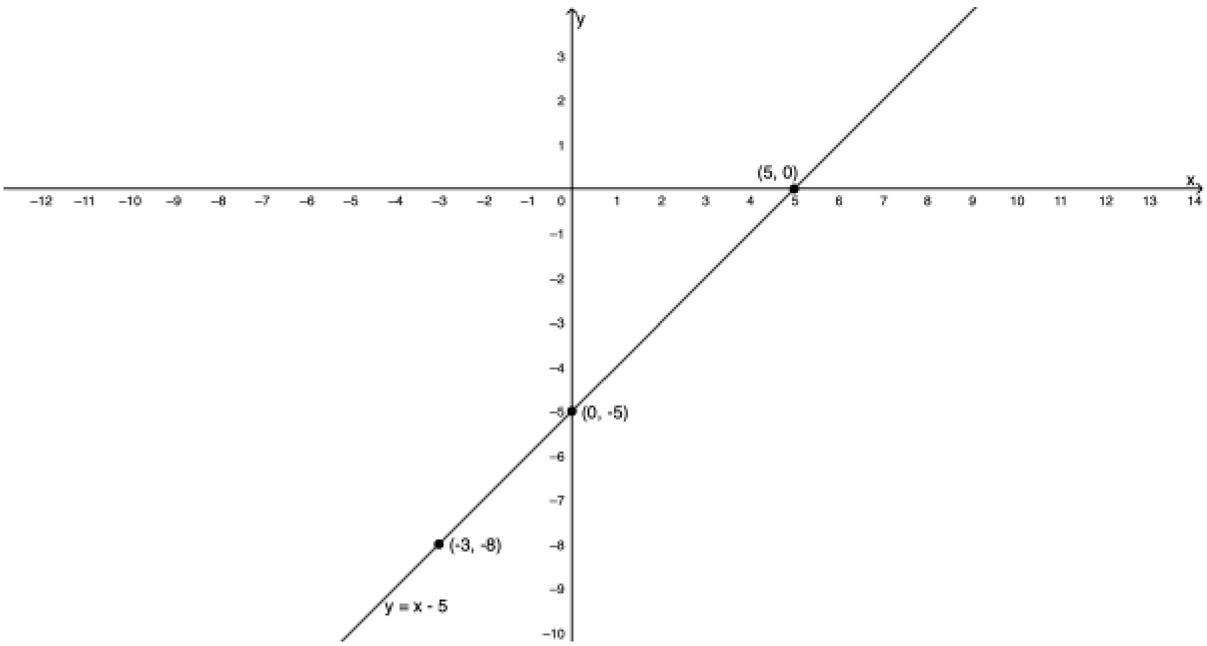


Figure 7: Graph of $y = x - 5$

The graph is a straight line, so we know that $y = x - 5$ is a linear equation as it produces a straight-line graph. Linear is another word for straight line.

All the representations of $y = x - 5$ in Activity 1.2 are possible ways to represent a function. We can represent it as an equation, a table of values, a mapping diagram and as a picture or graph.

However, there is one more way we have of representing functions. It is called **function notation**. We can represent $y = x - 5$ as $f(x) = x - 5$. We read this 'f of x is equal to x minus 5'.

Function notation is really useful because it allows us to name different functions. For example, we could have $f(x) = x - 5$, $g(x) = x^2 - 3x + 2$ and $h(x) = \frac{x}{4} + 4$ all plotted on the same Cartesian plane and we would still be able to tell which graph is which by labelling the graphs f, g and h.

Now, when we say 'the function g' or 'the function h of x' we know exactly which function we are referring to. This is much better than having all these functions written as $y = \ldots$ and not being able to tell them apart. We are like parents giving their children names!

Function notation is also great for representing different function values. Take the function f above. We can write $f(3)$ ('f of three'). All this means is 'what is the value of the function when $x = 3$?'. Have a look at Example 1.4 to see how to calculate this.

Example 1.4

1. If $f(x) = x - 5$, calculate the value of:
   a. $f(3)$
   b. $f(-2)$
   c. $f(a)$.
2. If $f(x) = x^2 + 2x + 1$ and $g(x) = 7 + x^2$, evaluate:
   a. $f(4)$
   b. $g(-2)$
   c. $f(g(1))$
   d. for what value(s) of x is $f(x) = g(x)$.
3. If $h(x) = 3x + 7$, for what value of x is $h(x) = 13$?

Solution

1.
   a.
   $$f(x) = x - 5$$
   $$\therefore f(3) = (3) - 5 = -2$$
   b.
   $$f(x) = x - 5$$
   $$\therefore f(-2) = (-2) - 5 = -7$$
   c.
   $$f(x) = x - 5$$
   $$\therefore f(a) = (a) - 5 = a - 7$$
2.
   a.

$$\begin{aligned} f(x) &= x^2 + 2x + 1 \\ \therefore f(4) &= (4)^2 + 2(4) + 1 \\ &= 16 + 8 + 1 \\ &= 25 \end{aligned}$$

b.

$$\begin{aligned} g(x) &= 7 + x^2 \\ \therefore g(-2) &= 7 + (-2)^2 \\ &= 7 + 4 \\ &= 11 \end{aligned}$$

c.

We first need to evaluate $g(1)$.

$$\begin{aligned} g(x) &= 7 + x^2 \\ \therefore g(1) &= 7 + (1)^2 \\ &= 7 + 1 \\ &= 8 \\ \text{Now } f(g(1)) = f(8) &= (8)^2 + 2(8) + 1 \\ &= 64 + 16 + 1 \\ &= 81 \end{aligned}$$

d.

$$\begin{aligned} f(x) &= g(x) \\ \therefore x^2 + 2x + 1 &= 7 + x^2 \qquad \text{The } x^2 \text{ terms cancel out} \\ \therefore 2x + 1 &= 7 \\ \therefore 2x &= 6 \\ \therefore x &= 3 \end{aligned}$$

3.

$h(x) = 3x + 7$ and $h(x) = 13$. Therefore, we can state that

$$\begin{aligned} 3x + 7 &= 13 \\ \therefore 3x &= 6 \\ \therefore x &= 2 \end{aligned}$$

Exercise 1.2

1. If $f(x) = 2x - 3$, represent $f(x)$ as a:
   a. table of values using the following values for x: $-2, -1, 0, 1, 2$
   b. mapping diagram (using the values from your table of values)
   c. graph (plotting all the points from your table of values).
2. If $f(x) = x^2 - 2$, $g(x) = x - 3$ and $k(x) = 3$, find the value of the following:
   a. $f(-1)$
   b. $f(-6)$
   c. $k(2)$

d. $k(-1) + f(3) - g(7)$

e. $g(f(4))$

3. The cost of petrol and diesel are described by $P = 15.61L$ and $D = 13.47L$ where P is the cost of petrol in Rands, D is the cost of diesel in Rands and L is the amount in litres.

   a. What is the cost of $30\ \ell$ of petrol?

   b. Evaluate $D(35)$.

   c. How many litres of petrol can you buy for $\text{R}300$?

   d. How many litres of diesel can you buy for $\text{R}450$?

   e. How much more expensive is petrol than diesel? Show your answer as a function.

The full solutions are at the end of the unit.

Note

If you have an internet connection, watch the video called “What is a function”, which gives a great summary of everything we have learnt about functions so far. While you watch, you should make your own summary.

What is a function (Duration 4.30)

Linear functions

We already know what linear equations are. Remember, these are equations where the highest power on the unknown is one. Linear functions are simply the representation of the relationship between input and output variables described by a linear equation. They have the general form of $y = ax + q$ or $y = mx + c$ where a and q or m or c are constants.

We have also seen that linear functions result in graphs which are straight lines. The best way to learn more about linear functions is to play with them. This next activity will help you learn how to sketch linear functions.

Sketching linear functions

Activity 1.3: Sketching linear functions

Time required: 25 minutes

What you need:

- a pen or pencil
- blank paper or a notebook
- an internet connection (optional)

What to do:

1. Have a look at the following linear functions:
   $f(x) = 2x + 1$
   $g(x) = -2x + 1$
   $h(x) = x + 1$
   a. Complete the following table of values for each function.

x	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
$f(x)$					
$g(x)$					
$h(x)$					

   b. What is the least number of points you need in order to plot a linear function?
   c. Which do you think are the easiest and most convenient points from the table of values with which to plot each function?
   d. On the same set of axes, plot all three functions. You can choose any points from the table to plot each function.
   e. What is the same about each function and what is the same about each graph? Remember that the general form of the linear function is $y = mx + c$.
   f. What is different about each function and what is different about each graph?
2. If you have an internet connection, visit this linear function interactive simulation to explore the linear function some more.
   Here you will find a linear function in its general form with sliders that let you change the values of m and c.
   a. Change the value of c. What effect does this have on the graph?
   b. What can you say about the value of c in $y = mx + c$?

c. Change the value of m. What effect does this have on the graph?

d. What is the difference between a large value of m like 5 and a small value of m like $\frac{1}{5}$?

e. What is the difference between positive and negative values of m?

f. What does a value of $m = 0$ mean? Why is this?

What did you find?

1.

a. Here is the completed table.

x	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
$f(x)$	-1	0	1	2	3
$g(x)$	3	2	1	0	-1
$h(x)$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2

b. To plot a straight line, you need at least two different points.

c. The easiest points to use are those where either the x or x value of the point is zero. These points lie on an axis and are where the graph cuts or **intercepts** the axis. For example, for $f(x)$, the point $(-\frac{1}{2}; 0)$ is the point where the graph intercepts the x-axis.

d. Figure 8 shows all three functions plotted on the same set of axes. The black crosses indicate the points where the graphs cut or intercept each axis.

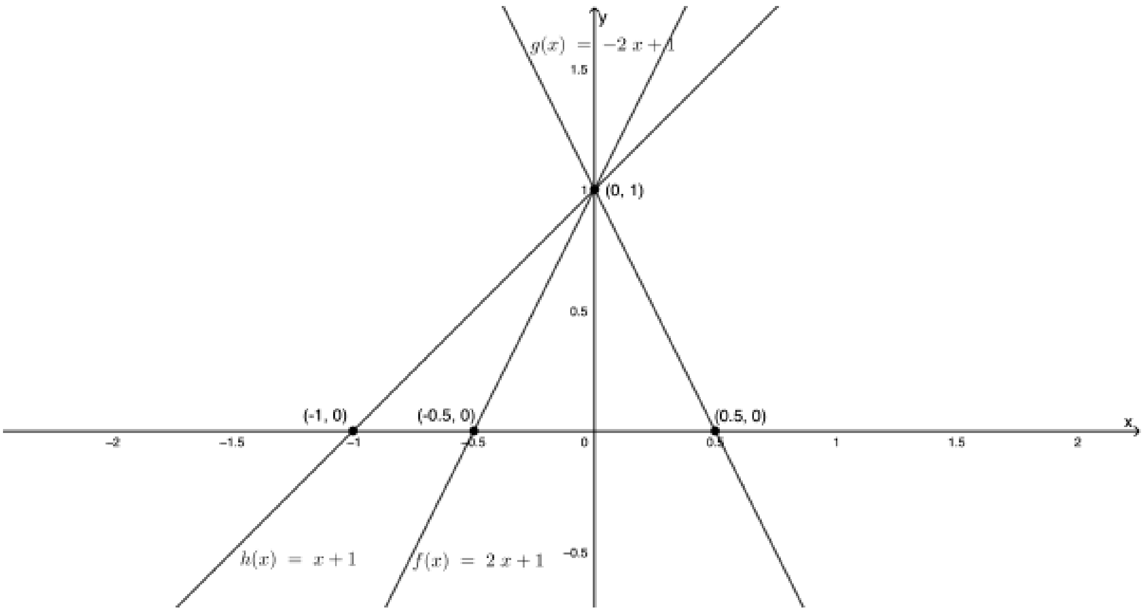


Figure 8: Sketches of $f(x)$, $g(x)$ and $h(x)$

   e. Each function has the same value of c which is 1. Each graph intercepts the y-axis at this same value.
   f. Each function has a different value of m. Each graph has a different slope or gradient.

2. Changing the variables in a function and seeing what affect they have on the shape and position of the graph of the function is a great way of getting to know a function better. This is what we were able to do by playing with the linear function interactive simulation.

   a. As we change the value of c, the graph moves up and down. When we **increase** the value of c, we move the graph **up**. When we decrease the value of c, we move the graph **down**.
   b. The value of c in $y = mx + c$ is the point where the graph cuts or intercepts the y-axis. We call this point the **y-intercept**.
   c. As we change the value of m, we change the slope or **gradient** of the graph.
   d. A large value of m gives a graph with a very steep slope. A small value of m gives a graph with a very shallow slope.
   e. When m is **positive**, the graph slopes **up** from left to right. When m is **negative** the graph slopes **down** from left to right.
   f. When $m = 0$, the graph is a flat horizontal line. When $m = 0$, the function becomes $y = c$. In other words, it makes no difference what input value we choose, the output value will always be c. Thus, all the points on the graph will have a y-coordinate of c.

We learnt a lot from that last activity. Let's look at two different ways to sketch a linear function.

Any two points

One way we have of sketching a linear function is to find and plot any two points. We can do this by choosing any two input values and then finding out what the corresponding function or output values are. For example, if we had $q(x) = 2x - 1$, we might choose $x = 1$ and $x = 2$. Then $q(1) = 2(1) - 1 = 1$ and $q(2) = 2(2) - 1 = 3$. We would then plot the points $(1, 1)$ and $(2, 3)$ and join them with a straight line as shown in Figure 9.

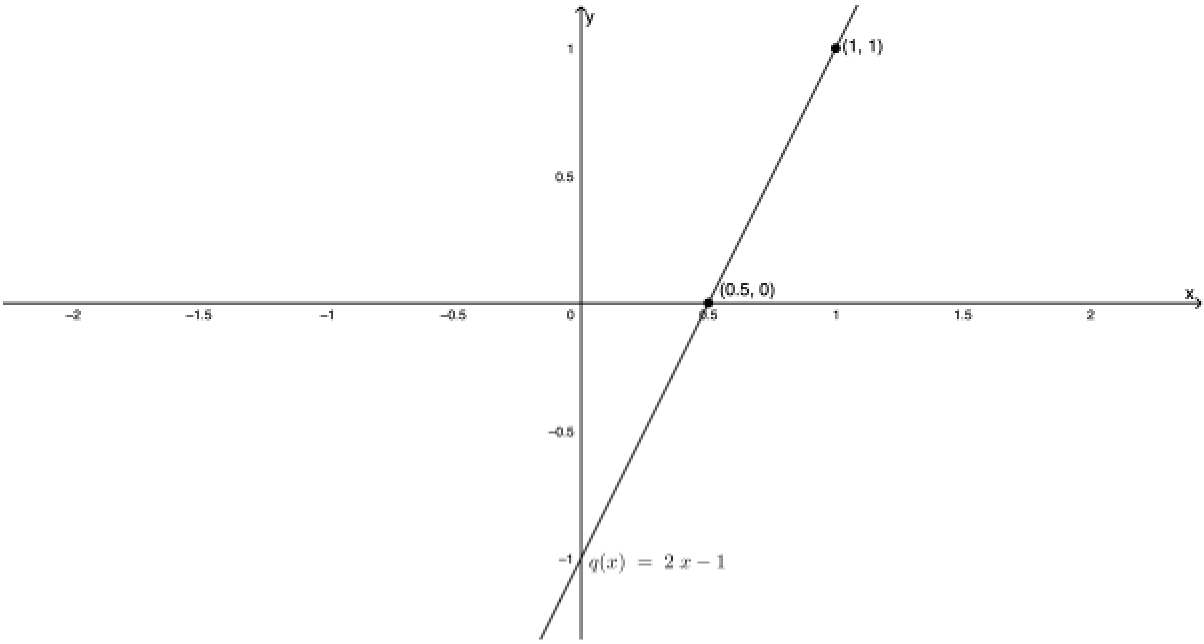


Figure 9: Graph of $q(x)$ by any two points method

Dual-intercept method

However, we can simplify things if we plot the points where the graph intercepts with the axes. Where the graph crosses the x-axis, this is the **x-intercept** and the y-coordinate of the point is zero. Where the graph crosses the y-axis, this is the **y-intercept** and the x-coordinate of the point is zero.

So, plotting $q(x) = 2x - 1$:

- x-intercept (make $y = 0$): $0 = 2x - 1 \therefore 2x = 1 \therefore x = \frac{1}{2}$. Therefore, the point to plot is $(\frac{1}{2}, 0)$.
- y-intercept (make $x = 0$): $q(0) = 2(0) - 1 = -1$. Therefore, the point is $(0, -1)$.

If we plot these points and join them with a straight line, we get the graph in Figure 10 which is identical to the graph in Figure 9 because we plotted the same function.

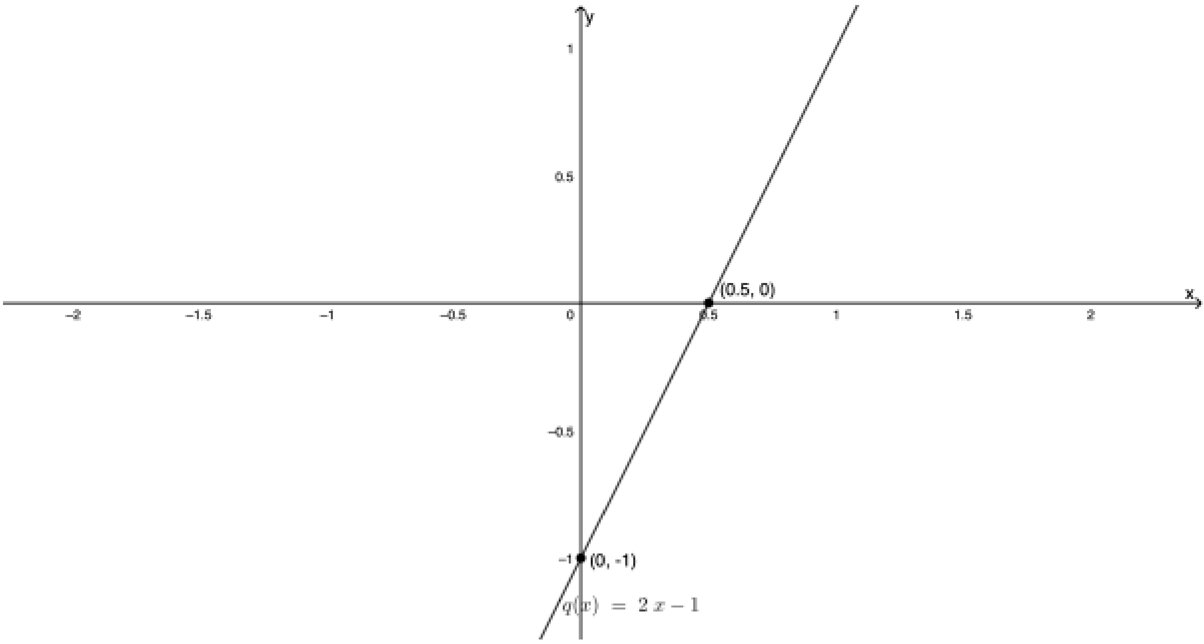


Figure 10 : graph of $q(x)$ by dual-intercept method

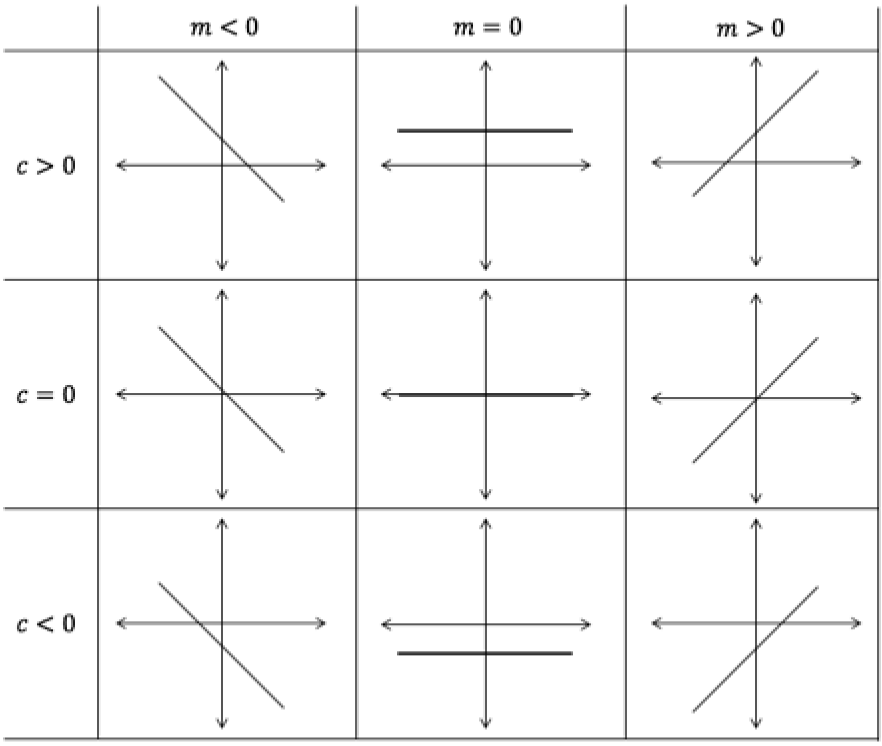
Can you see why we can only look at the value of c as a short-cut for finding the y-intercept? Can you also see why it is called the dual-intercept method for sketching a linear function? We find both intercepts.

The gradient of a straight line graph

We have already seen that when the linear function is in the form $y = mx + c$, the value of c immediately tells us what the y-intercept of the graph is. The value of m tells us what the slope or gradient of the graph is. The bigger the value of m, the steeper the graph.

Here is a summary of what we know so far.

Take note!



Remember, that we sometimes write the standard form of the linear function as $y = ax + q$. In this case q tells us what the y-intercept is and the value of a gives us the gradient of the graph. Let's take a closer look at the gradient.

This activity will help you understand what the gradient of a straight line is and how to measure it.

Activity 1.4: The gradient of a straight line graph

Time required: 30 minutes

What you need:

- a pen or pencil
- blank paper or a notebook
- an internet connection (optional)

What to do:

1. Figure 11 is a graph of the function $y = 2x + 1$. Look at Figure 11 and answer the following questions.

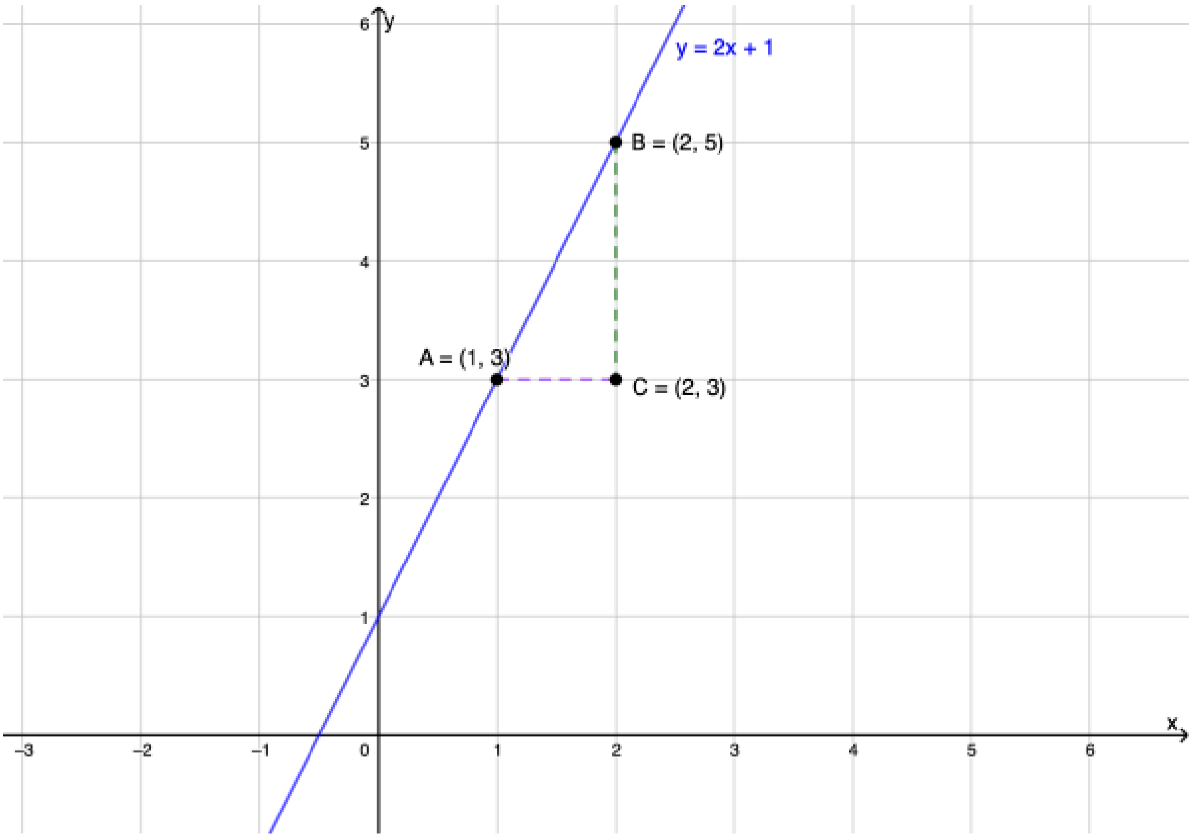


Figure 11: Graph of $y = 2x + 1$.

a. What is the length of the horizontal purple line? How does this length relate to the difference in the x-coordinates of points A and B?

b. What is the length of the vertical green line? How does this length relate to the difference in the x-coordinates of points A and B?

c. Calculate the value of $\frac{\text{length of green line}}{\text{length of purple line}}$. What characteristic of the graph does this value measure?

d. Pick any two other points on the graph and again measure the difference in their x- and y-coordinates in order to work out the same ratio of $\frac{\text{length of green line}}{\text{length of purple line}}$.

e. What characteristic of the graph is this value of $\frac{\text{length of green line}}{\text{length of purple line}}$ equal to?

2. If you have an internet connection, open the gradient of a straight line graph interactive simulation.

Here you will find a linear function of the form $y = mx + c$ with sliders so that you can change the values of m and c. Set the graph to represent the function $y = \frac{1}{2}x - 1$. Drag the purple point to $(2; 0)$ and the green point to $(4; 1)$.

By looking at the equation of the function, what do you expect the value of $\frac{\text{length of green line}}{\text{length of purple line}}$ to be? Measure the length of the lines and calculate the value. Is it what you expected? Now move either or both points to other places on the graph and recalculate the value of $\frac{\text{length of green line}}{\text{length of purple line}}$. Do you get the same answer? Why do you think this is?

Now, set the graph to represent $y = -2x + 1$ and move the purple point to $(-1; 3)$ and the green point to $(1; -1)$. What do you expect the value of $\frac{\text{length of green line}}{\text{length of purple line}}$ to be? Measure the length of the lines and calculate the value. Is it what you expected?

What changes to your calculations did you have to make to arrive at the answer of -2 for $\frac{\text{length of green line}}{\text{length of purple line}}$? Move the green and purple points to any other parts of the graph and recalculate the value of $\frac{\text{length of green line}}{\text{length of purple line}}$. Do you still get the same answer?

Change the value of m to any other value you like and verify that you are able to use the same method as above to measure or calculate the gradient.

Write a general expression that will allow you to measure the gradient of any straight-line graph between any two points on the line.

What did you find?

In Question 1, we looked at Figure 11 showing the graph of $y = 2x + 1$. The length of the purple line was two units. This was the difference between the x-coordinates of points A and B ($(\text{x-coordinate of B}) - (\text{x-coordinate of A}) = 2 - 1 = 1)$. The length of the green line was two units and was the difference between the y-coordinates of the points A and B ($(\text{y-coordinate of B}) - (\text{y-coordinate of A}) = 5 - 3 = 2)$.

Therefore $\frac{\text{length of green line}}{\text{length of purple line}} = \frac{2}{1} = 2$. This is a measure of how steep the graph is; in other words, its gradient.

No matter which other points we choose, the value of $\frac{\text{length of green line}}{\text{length of purple line}}$ stays 2. For example, if we chose the points $(0; 1)$ and $(3; 7)$, the value of $\frac{\text{length of green line}}{\text{length of purple line}}$ would be:

$$\frac{\text{length of green line}}{\text{length of purple line}} = \frac{7 - 1}{3 - 0} \frac{6}{3} = 2$$

The value of $\frac{\text{length of green line}}{\text{length of purple line}}$ is the same as the value of m in the equation of the function. We know that m describes the gradient of the graph. Therefore $\frac{\text{length of green line}}{\text{length of purple line}}$ is a measure of the gradient of the graph as well.

If you were able to practise using the gradient of a straight line graph interactive simulation of the function $y = \frac{1}{2}x - 1$, you would have found that no matter where you dragged the green and purple points to, the value of $\frac{\text{length of green line}}{\text{length of purple line}}$ was always equal to the value of m in the equation of the function which was $\frac{1}{2}$.

For the function $y = -2x + 1$, the same thing was true. To find the value of $\frac{\text{length of green line}}{\text{length of purple line}}$ between the points $(-1; 3)$ and $(1; -1)$ our calculations were as follows:

$$\frac{\text{length of green line}}{\text{length of purple line}} = \frac{3 - (-1)}{-1 - 1} = \frac{4}{-2} = -2$$

Or

$$\frac{\text{length of green line}}{\text{length of purple line}} = \frac{-1 - 3)}{1 - (-1)} = \frac{-4}{2} = -2$$

It does not matter in which order we subtracted the coordinates so long as we always used the same order.

If we have any two points on a straight line, for example $(x_1; y_1)$ and $(x_2; y_2)$, then we can calculate the gradient of the straight line as follows:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

It does not matter which is point 1 and which is point 2, so long as you **subtract the coordinates in the same order**.

The expression for the gradient of a straight line covered above is important and you will need to use it often in many different scenarios. Other ways you may see the gradient of a straight line expressed are:

$\frac{\text{rise}}{\text{run}}$ or $\frac{\text{change in } y}{\text{change in } x}$ or $\frac{\text{vertical change}}{\text{horizontal change}}$

All these mean the same thing. **When moving from left to right between any two points**, how much the graph goes up (or down) from the one point to the next, divided by how far the graph moves to the right to achieve this up or down change.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x}$$

Now that we know what gradient is and how to measure it, we can look at another way to sketch linear functions.

Example 1.5

1. Sketch the function $t(x) = 1 - 2x$ by finding the y-intercept and then measuring out the gradient to find another point on the line.

2. Sketch the linear function $r(x)$ that has a y-intercept at $(0, 3)$ and a gradient of $\frac{3}{2}$.
3. A straight line is **parallel** to the line $3y + 4x - 7 = 0$ and passes through the point $(0, -2)$. What is the equation of the line? Sketch the line. Note: Parallel lines have the same gradient.

Solution

1.

First, we need to get our function into standard form.$t(x) = -2x + 1$

Now we can see that the y-intercept is the point $(0, 1)$ and the gradient is -2 or $\frac{-2}{1}$. This means that for every one unit we move to the **right**, we have to move two units **down**. Starting at the y-intercept, we measure out the gradient – one unit **right** and then two units **down**. We mark this point and then join our two points with a straight line (see Figure 12).

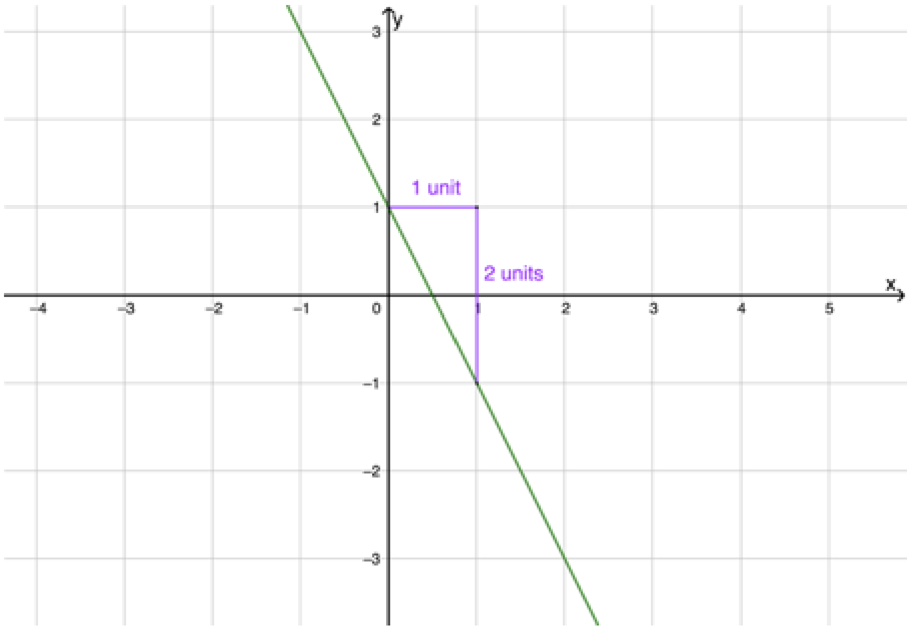


Figure 12: Graph of $t(x)$

Note: It is important that when measuring a gradient, you always move from left to right (i.e. in the positive horizontal direction) and then up or down depending on whether the gradient is positive or negative. In the case above, if we moved from right to left (the negative horizontal direction), we would have to go up. Going down (in the negative vertical direction) would have measured a gradient of $\frac{-2}{-1} = \frac{2}{1}$ not $\frac{-2}{1}$.

2.

We know that the gradient of the line is $\frac{3}{2}$. In other words, if we move two units to the right, we have to move three up. The y-intercept is the point $(0, 3)$, so this is a good place from which to start measuring the gradient. Figure 13 shows the graph.

Figure 13: Graph of $r(x)$

3.

We are told that our graph is parallel to $3y + 4x - 7 = 0$. Therefore, it has the same gradient. To find the gradient, we have to get the parallel line into standard form.

$$\begin{aligned} 3y + 4x - 7 &= 0 \\ \therefore 3y &= -4x + 7 \\ \therefore y &= \frac{-4}{3}x + \frac{7}{3} \end{aligned}$$

The gradient of the graph is $\frac{-4}{3}$. We are also told that it passes through the point $(0; -2)$. So, starting at the point $(0; -2)$ (which is the y-intercept but could really be any point on the line), we measure out the gradient – three units to the right and then 4 units down. The graph is shown in Figure 14.

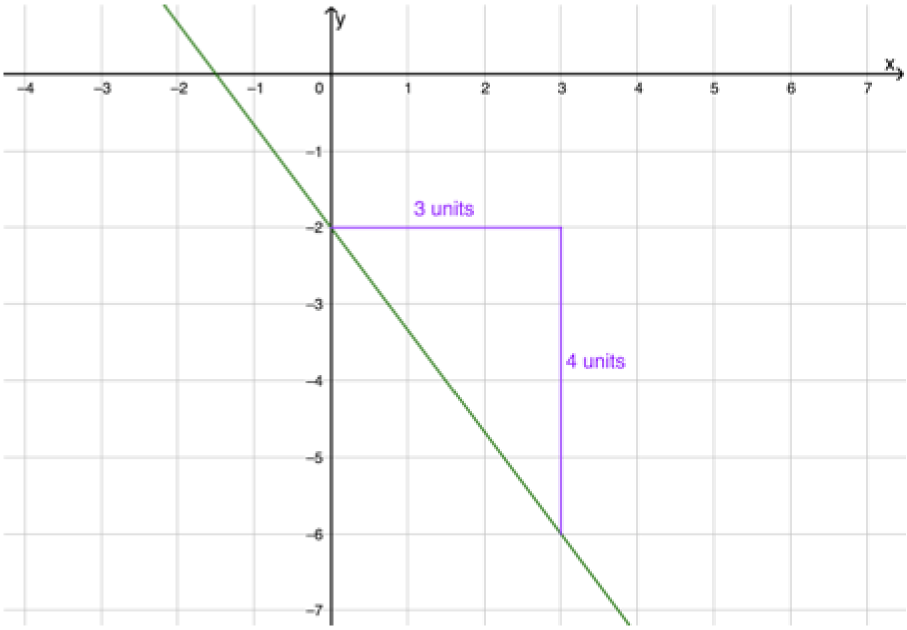


Figure 14: Graph of the function with gradient $\frac{-4}{3}$ *passing through the point* $(0;-2)$

Exercise 1.3

Use any method to plot the following linear functions on the same set of axes.

1. $2y = 3x - 1$
2. $y = x - 3$
3. $2x - 3y = 12$
4. $y + 3x - 3 = 1$

The full solutions are at the end of the unit.

Domain and range

Have you wondered yet why we sketch straight line graphs by just finding any two points and then join them with a solid line? Think about the function $f(x) = 3x - 2$. Is there any value of x that we cannot put into the equation? Is there any value of y that the function cannot be equal to?

The answer to both questions is 'no'. We say that the **domain** (remember, this is just a fancy name for the set of all input values) is all real numbers. Similarly, the **range** (the set of all output values) is also all real numbers. If we were to plot all the possible input and output pairs for the function on the same set of axes, there would be so many points that they would blur together and form a solid line.

But we can restrict what input values a function is allowed to have and, in so doing, restrict the output values that a function can produce. Sometimes the nature of the function itself automatically restricts what input values are allowed. Look at this example:

$$t(x) = \frac{x+3}{x-1}$$

We know that we are never allowed to divide by zero which means the denominator in this function may never be zero. This means that $x \neq 1$. In this case, the function is allowed to take any real number as an x value **except** 1. We can write this domain formally using **set notation** like this:

Domain: $\{x : x \in \mathbb{R}, x \neq 1\}$

Here is another domain: $\{x : x \in \mathbb{R}, -2 \leq x < 5\}$. This one says that the domain is all the real numbers between -2 and 5, including -2 but not including 5.

We could also write this slightly less formally using **interval notation** as $[-2; 5)$. The square bracket tells us that the domain **includes** -2 and the round bracket tells us that it does not include 5. When using interval notation, we assume that the interval contains all real numbers.

Generally, unless there is a specific restriction imposed, linear functions can take any real number as an input and it is possible to generate any real number as an output. Therefore, the Domain and Range of linear functions is almost always:

- Domain: $\{x : x \in \mathbb{R}\}$ or $(-\infty; \infty)$ – the round brackets indicate that the domain does not include $-\infty$ or ∞.
- Range: $\{x : x \in \mathbb{R}\}$ or $(-\infty; \infty)$ – the round brackets indicate that the range does not include $-\infty$ or ∞.

Exercise 1.4

1. Write the following in set notation:
   a. $(-\infty; 6)$
   b. $[-12; 13)$
   c. $(-\sqrt{3}; \infty)$
2. Write the following in interval notation:
   a. $\{p : p \in \mathbb{R}\}$
   b. $\{k : k \in \mathbb{R}, k > \frac{2}{5}\}$
   c. $\{r : r \in \mathbb{R}, 13 < r \leq 42\}$

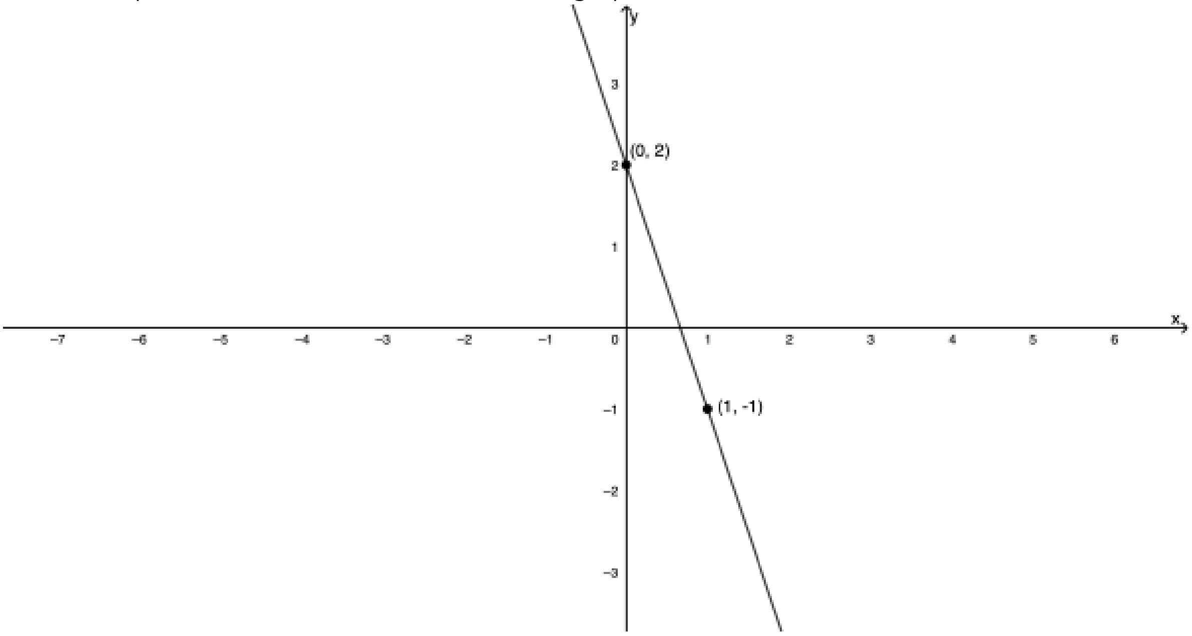
The full solutions are at the end of the unit.

Finding the equation of a linear function

We know how to draw a graph for a linear function if we know its equation. But can we find its equation if, all we have is its graph?

Example 1.6

1. Find the equation of the function shown in the graph below.



2. A linear function, $s(x)$, passes through the points $(-6, 1)$ and $(2, -3)$. What is its equation?
3. $g(x)$ is parallel to $3y - 5x + 6 = 0$ and passes through $(-2, -\frac{3}{2})$. What is the equation of $g(x)$?

Solution

1.
The graph passes through the points $(0, 2)$ and $(1, -1)$. We can see that $(0, 2)$ is the y-intercept, therefore we know that $c = 2$. We can use these two points to find the gradient.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - (-1)}{0 - 1} = \frac{3}{-1} = -3$$

Therefore, the equation of the function is $y = -3x + 2$.

2.
We are told that the function passes through points $(-6, 1)$ and $(2, -3)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3 - 1}{2 - (-6)} = \frac{-4}{8} = \frac{-1}{2}$$

Therefore, $s(x) = -\frac{1}{2}x + q$. To find the value of q, we substitute either point into the equation of $s(x)$. For example, we know that $s(2) = -3$.

$$s(2) = 2(-\frac{1}{2}) + q = -3$$
$$\therefore -1 + q = -3$$
$$\therefore -1 + 3 = -q$$
$$\therefore q = -2$$

So $s(x) = -\frac{1}{2}x - 2$

3.

$g(x)$ is parallel to $3y - 5x + 6 = 0$. Writing $3y - 5x + 6 = 0$ in standard form, we get

$$\begin{aligned} 3y - 5x + 6 &= 0 \\ \therefore 3y &= 5x - 6 \\ \therefore y &= \frac{5}{3}x - 2 \end{aligned}$$

Therefore $g(x) = \frac{5}{3}x + q$. We are also told that $g(x)$ passes through $(-2; -\frac{3}{2})$. Therefore, $g(-2) = \frac{3}{2}$

$$\begin{aligned} g(-2) = \frac{5}{3}(-2) + q &= \frac{3}{2} \\ \therefore \frac{-10}{3} + q &= \frac{3}{2} \\ \therefore -20 + 6q &= 9 \\ \therefore 6q &= 29 \\ \therefore q &= \frac{29}{6} \end{aligned}$$

So finally we know that $g(x) = \frac{5}{3}x + \frac{29}{6}$.

Note

Watch these two videos if you need more help finding the equation of a linear function.

Finding The Equation Of A Line Through 2 Points PART 1 (Duration: 2.15)

Finding The Equation Of A Line Through 2 Points PART 2 (Duration: 2.39)

Summary

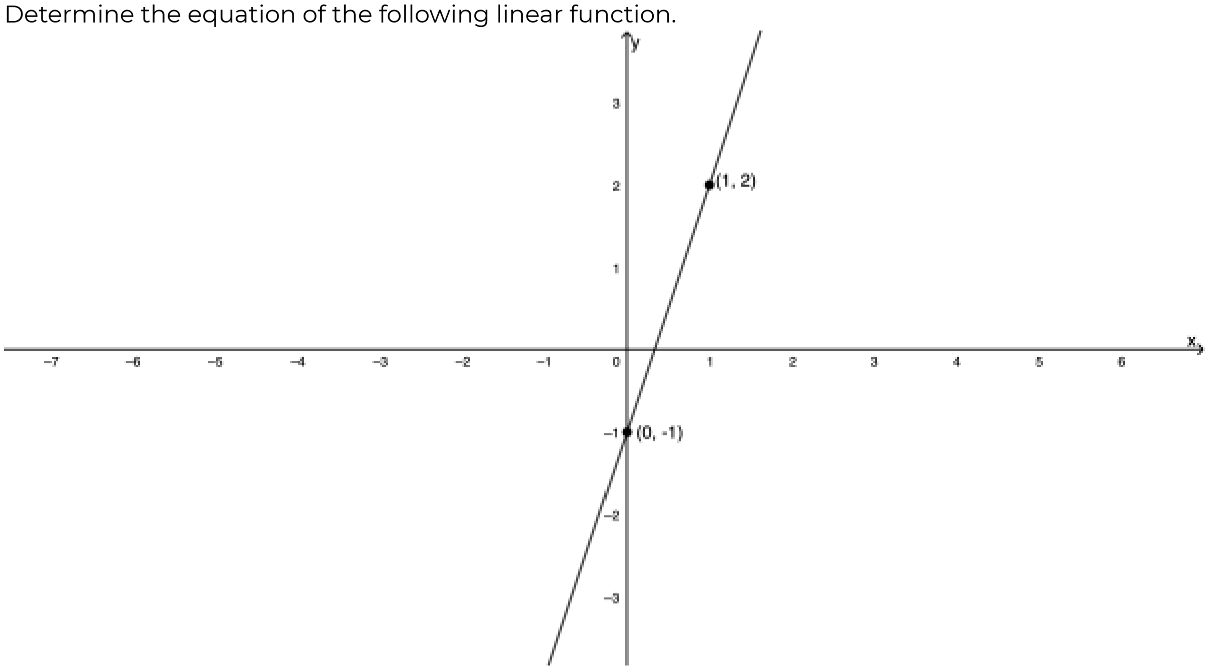
In this unit you have learnt the following:

- That a function is a mathematical relationship between inputs and outputs that maps each input to one, and only one, output.
- The difference between a function and a relation.
- The standard form of a linear function is $y = mc + c$ (or $y = ax + q$).
- The effect of m and c (or a and q) on the graph of a linear function.
- What we mean by the domain and range of a linear function and how to represent these in set and interval notation.
- How to plot the graph of a linear function using any two points, the dual-intercept method and the gradient-intercept method.
- How to find the equation of a linear function.

Unit 1: Assessment

Suggested time to complete: 40 minutes

1. Write the following linear functions in standard form.
   a. $2y + 3x = 1$
   b. $y + 2x - 3 = 1$
2. Determine the x- and y-intercepts of $x + 2y - 2 = 0$.
3. Determine the equation of the following linear function.



4. Are the following statements true or false?
   a. The gradient of $2y = 3x - 1$ is 3.
   b. The gradient of $2 - y = 2x - 1$ is -2.
   c. The y-intercept of $2y + 3x = 6$ is 6.
5. $f(x) = 3 - 3x$ and $g(x) - 1 = \frac{1}{3}x$.

a. Sketch both graphs on the same set of axes using any method. Make sure to label your axes and functions.

b. Find the point where the graphs cross or intersect.

The full solutions are at the end of the unit.

Unit 1: Solutions

Exercise 1.1

1. Moving a vertical line across the graph from left to right, at no point does the line cut the graph more than once. Therefore, the graph represents a function.
2. Moving a vertical line across the graph from left to right, at no point does the line cut the graph more than once. Therefore, the graph represents a function.
3. Moving a vertical line across the graph from left to right, at no point does the line cut the graph more than once. Therefore, the graph represents a function.
4. For input values between -4 and 4, a vertical line will cut the graph more than once. Therefore, the graph does not represent a function.
5. For input values less than about 1.9 a vertical line will cut the graph more than once. Therefore, the graph does not represent a function.
6. Moving a vertical line across the graph from left to right, at no point does the line cut the graph more than once. Therefore, the graph represents a function.

Back to Exercise 1.1

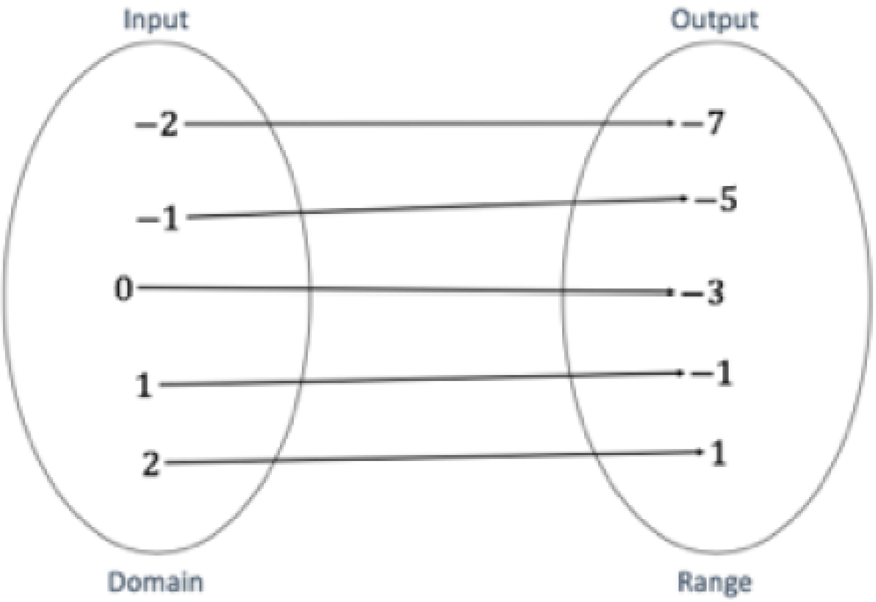
Exercise 1.2

1. $f(x) = 2x - 3$

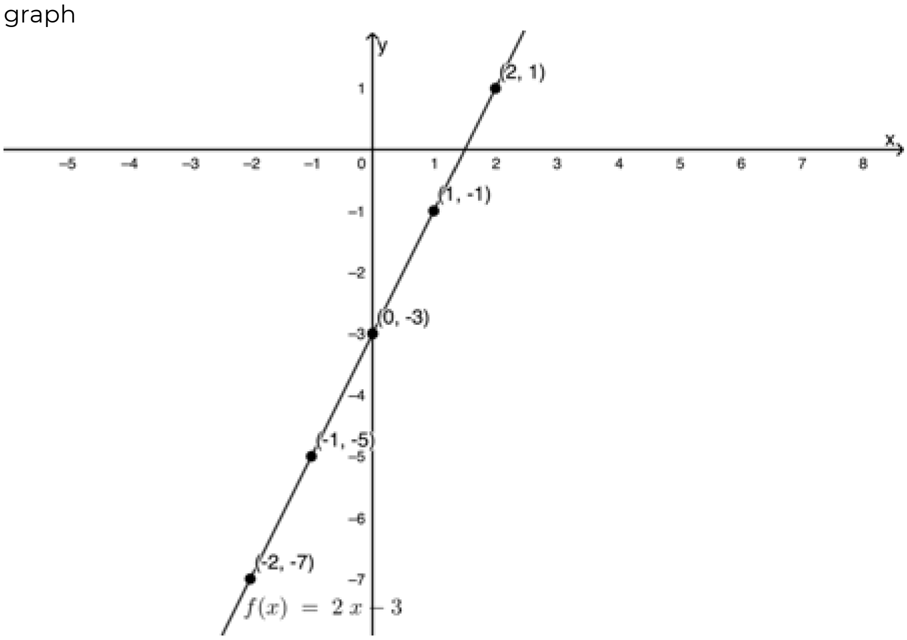
   a. table of values

x	-2	-1	0	1	2
y	-7	-5	-3	-1	1

   b. mapping diagram



c. graph



2. $f(x) = x^2 - 2$, $g(x) = x - 3$ and $k(x) = 3$.
   a. $f(-1) = (-1)^2 - 2 = 1 - 2 = -1$
   b. $f(-6) = (-6)^2) - 2 = 36 - 2 = 34$
   c. $k(2) = 3$
   d. $k(-1) + f(3) - g(7) = 3 + [(3)^2 - 2] - [(7) - 3] = 3 + 7 - 4 = 6$
   e. $g(f(4))$
   $f(4) = (4)^2 - 2 = 16 - 2 = 14$
   $g(14) = 14 - 3 = 11$
3. Cost of petrol: $P = 15.61L$; cost of diesel: $D = 13.47L$ where L is the amount in litres:
   a. $P = 15.61 \times 30 = \text{R}468.30$
   b. $D(35) = 13.47 \times 35 = \text{R}471.45$
   c.

$$P = 15.61\ L = 300$$

$$\therefore L = \frac{300}{15.61}$$

$$\therefore L = 19.11 \text{ l}$$

d.

$$D = 13.47\ L = 450$$

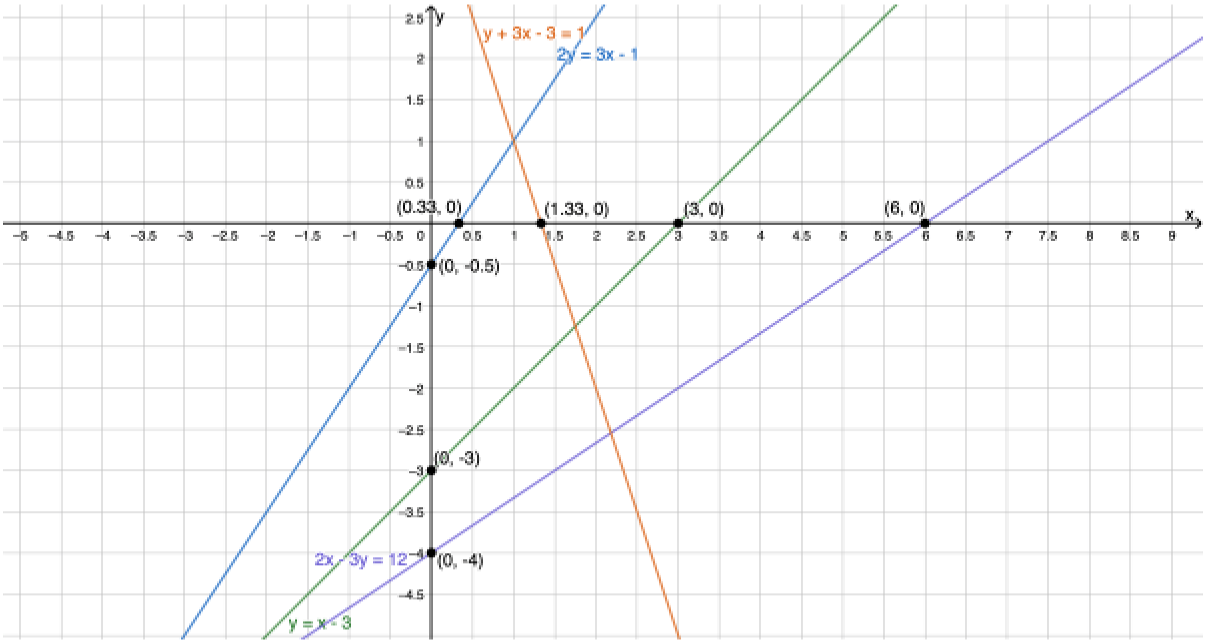
$$\therefore L = \frac{450}{13.47}$$

$$\therefore L = 33.41 \text{ l}$$

e. Let price difference be the function Q. Then
$Q = P - D = 15.61L - 13.47L = L(15.61 - 13.47) = 2.14L$.

Back to Exercise 1.2

Exercise 1.3



Back to Exercise 1.3

Exercise 1.4

1.
   a. $\{x : x \in \mathbb{R}, x < 6\}$
   b. $\{x : x \in \mathbb{R}, -12 < x < 13\}$
   c. $\{x : x \in \mathbb{R}, x > -\sqrt{3}\}$
2.
   a. $(-\infty, \infty)$
   b. $(\frac{2}{5}, infty)$
   c. $(13, 42]$

Back to Exercise 1.4

1.

a.

$$2y + 3x = 1$$
$$\therefore 2y = -3x + 1$$
$$\therefore y = -\frac{3}{2}x + \frac{1}{2}$$

b.

$$y + 2x - 3 = 1$$
$$\therefore y = -2x + 4$$

2.

$x + 2y - 2 = 0$
x-intercept (let $y = 0$):
$$x + 2(0) - 2 = 0$$
$$\therefore x = 2$$
$\therefore \text{the x-intercept is } (2, 0)$

y-intercept (let $x = 0$):
$$0 + 2y - 2 = 0$$
$$\therefore 2y = 2$$
$$\therefore y = 1$$
$\therefore \text{the x-intercept is } (0, 1)$

3. The graph passes through the point $(0, -1)$ and $(1, 2)$. Therefore, the y-intercept (the value of c is -1.
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - (-1)}{1 - 0} = \frac{3}{1} = 3$$
Therefore, the equation of the line is $y = 3x - 1$.

4.

a.

$$2y = 3x - 1$$
$$\therefore y = \frac{3}{2} - \frac{1}{2}$$
$$\therefore m = \frac{3}{2}$$
Therefore, the statement is false.

b.

$$2 - y = 2x - 1$$
$$\therefore y = -2x + 3$$
$$\therefore m = -2$$
Therefore, the statement is true.

c.

$$2y + 3x = 6$$
$$\therefore 2y = -3x + 6$$
$$\therefore y = -\frac{3}{2} + 3 \therefore c = 3$$
Therefore, the statement is false.

5.

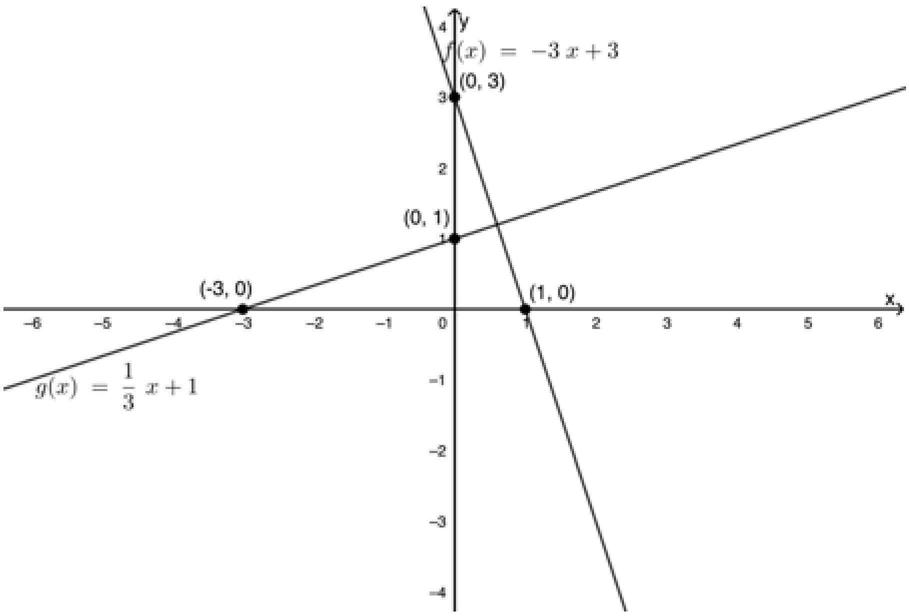
a.

Given $f(x) = 3 - 3x$ and $g(x) - 1 = \frac{1}{3}x$.

$f(x) = 3 - 3x$
x-intercept (let $y = 0$):

$0 = 3 - 3x$
$\therefore 3x = 3$
$\therefore x = 1$
y-intercept (let $x = 0$):
$y = 3 - 3(0)$
$\therefore y = 3$

$g(x) - 1 = \frac{1}{3}x$
x-intercept (let $y = 0$):
$0 - 1 = \frac{1}{3}x$
$\therefore x = -3$
y-intercept (let $x = 0$):
$y - 1 = \frac{1}{3}(0)$
$\therefore y = 1$



b.

To find the point where the graphs intersect each other, we need to set $f(x) = g(x)$.

$f(x) = 3 - 3x$ and $g(x) = \frac{1}{3}x + 1$

$$\begin{aligned} f(x) &= g(x) && \text{Setting the functions equal to each other will allow us find the } x \text{ co-ordinate of the point of intersection} \\ \therefore 3 - 3x &= \frac{1}{3}x + 1 \\ \therefore 9 - 9x &= x + 3 \\ \therefore -10x &= -6 \\ \therefore x &= \frac{6}{10} = \frac{3}{5} \end{aligned}$$

Now substitute $x = \frac{4}{5}$ in either function to find the y-coordinate of the point.

$f(\frac{3}{5}) = 3 - 3(\frac{3}{5}) = 3 - \frac{9}{5}$

$\therefore f(\frac{3}{5}) = \frac{15 - 9}{5} = \frac{6}{5}$

Therefore, the point of intersection is $(\frac{3}{5}, \frac{6}{5})$.

Back to Unit 1: Assessment

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Unit 2: Quadratic functions

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Identify the following characteristics of quadratic functions:
    - turning points
    - minima and maxima
    - shape and symmetry.
- Sketch and find the equation of the graph $y = ax^2 + q$.
- Investigate and generalise the impact of a and q on $y = ax^2 + q$.

What you should know

Before you start this unit, make sure you can

- Manipulate and simplify algebraic expressions. Go over Subject outcome 2.2, Unit 1: Simplifying algebraic expressions if you need more help with the basics.
- Solve quadratic equations. Go over Subject outcome 2.3, Unit 1: Solve linear and quadratic equations if you need more help with the basics.
- Plot points on the Cartesian plane. If you do not know how to plot points onto the Cartesian plane, then you should do Subject outcome 3.3, Unit 1: Plotting points on the Cartesian plane before continuing with this unit.

Here is a short self-assessment to make sure you have the skills you need to proceed with this unit.

Solve for x in each case.

1. $0 = 3x^2 - 6$
2. $2x^2 - 8 = 6$

Solutions

1.
$$\begin{aligned} 0 &= 3x^2 - 6 \\ \therefore x^2 - 2 &= 0 \\ \therefore x^2 &= 2 \\ \therefore x &= \pm\sqrt{2} \end{aligned}$$

2.
$$\begin{aligned} 2x^2 - 8 &= 6 \\ \therefore 2x^2 &= 14 \\ \therefore x^2 &= 7 \\ \therefore x &= \pm\sqrt{7} \end{aligned}$$

Introduction

In Unit 1 we learnt that functions are a kind of mathematical relationship between two variables that map each element of the **domain** (the set of input values) to **one, and only one,** element in the **range** (the set of output values).

We learnt that **linear functions** produce **straight lines** when we draw their graphs and that the standard form of linear functions is $y = mx + c$ or $y = ax + q$ where:

- m (or a) is the gradient or slope of the line, and
- c (or q) is the point where the line cuts the y-axis.

So the function $y = 3x - 1$ or $f(x) = 3x - 1$, if we write it in function notation, produces a straight line with a gradient of 3 that cuts the y-axis at the point $(0, -1)$.

But what kind of graph will the equation $g(x) = 4x^2 - 1$ produce? Will it also be a straight line? Is it even a function? What is the difference between the equations of $f(x)$ and $g(x)$?

In $f(x)$, the highest exponent on x is 1. However, in $g(x)$, the highest exponent on x is 2. You may, therefore, recognise that $y = 4x^2 - 1$ is a quadratic expression and so $g(x) = 4x^2 - 1$ is a quadratic function.

Of all the functions we will look at, quadratic functions are probably the most important. They produce graphs that we call **parabolas**. Parabolas crop up everywhere. If you cut a satellite dish in half through the centre, you get a parabola. The same is true for nearly all mirrors used in telescopes, headlamps and some light bulbs.

Figure 1: A satellite dish

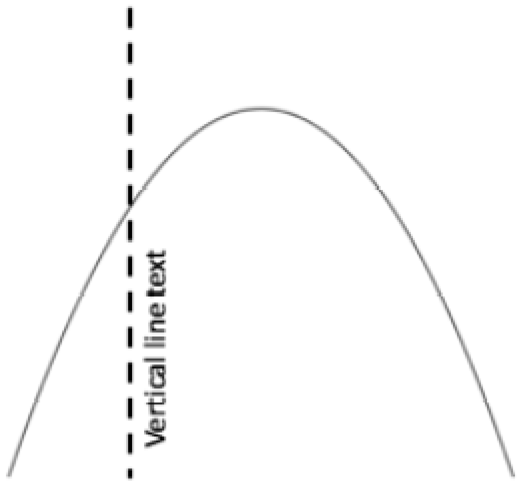
The ropes or chains hanging from the pillars of suspension bridges follow a parabolic curve.

Figure 2: Golden Gate Bridge

When you throw a ball (or any object) through the air, its path is a parabola.

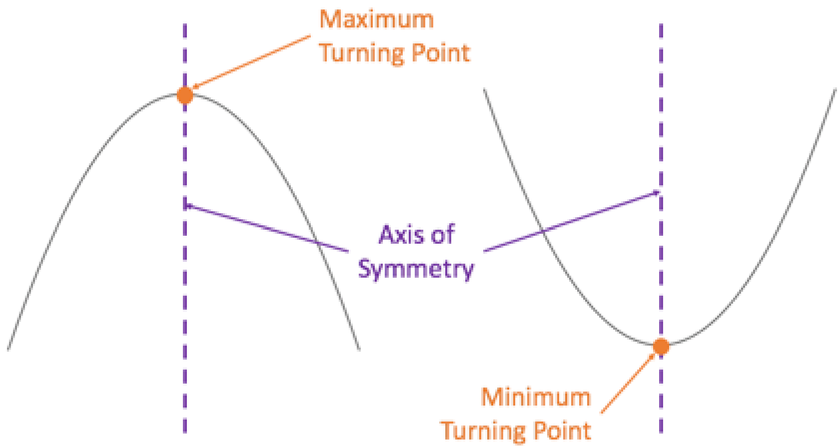
Figure 3: Bouncing ball parabolas

All of these examples show us the general shape of the parabola. We can see that parabolas are the graphs of functions. At no point does an input value give more than one output value.



There are two important characteristics of parabolas. The first is that they have a **turning point**; a point where they turn or change direction. The second is that they are symmetrical about the vertical line that goes through their turning point. This means that, if you folded a parabola along this vertical line, the two arms would be exactly on top of each other. We call this line the **axis of symmetry**.

The turning point of a parabola can either be its highest point or maximum value (like the top of a hill) or its lowest point or minimum value (like the bottom of a valley). When the turning point is the highest point, we call it a **maximum turning point**. When the turning point is the lowest point, we call it a **minimum turning point**.



Note

If you have access to the internet, watch the video called "Visual introduction to parabolas" for a good introduction to parabolas.

Visual introduction to parabolas (Duration: 08.15)

In this video the turning point is called the vertex. **Turning point** and **Vertex** mean the same thing when we are talking about parabolas.

The quadratic function

We will now take a closer look at quadratic functions.

Activity 2.1: Investigate the quadratic function

Time required: 30 minutes

What you need:

- a pen or pencil

- a calculator
- blank paper or a notebook

What to do:

Have a look at these two functions:

- $h(x) = 2x - 3$
- $k(x) = 2x^2 - 3$

1. Which function is the linear function? Why? Which function is the quadratic function? Why?
2. Plot both functions on the same set of axes. You can use any method to plot the linear function. Use the following table of values to help you plot the quadratic function. Plot the points and then join them with as smooth a curve as possible.

x	-3	-2	-1	0	1	2	3
$k(x)$							

3. What is the **same** about each function? What is the **same** about each graph?
4. What is different about each function? What is **different** about each graph? Think particularly about the x-intercepts and what you know about the number of answers we get for linear and quadratic equations.
5. If vertical lines have the general form of $y = k$ where k is a constant, what is the equation of the vertical line that is an axis of symmetry of the quadratic function? Remember, the axis of symmetry cuts the graph into two mirror image halves.
6. What is the turning point of the quadratic function?
7. What is the domain and range of the linear function?
8. What is the domain and range of the quadratic function?

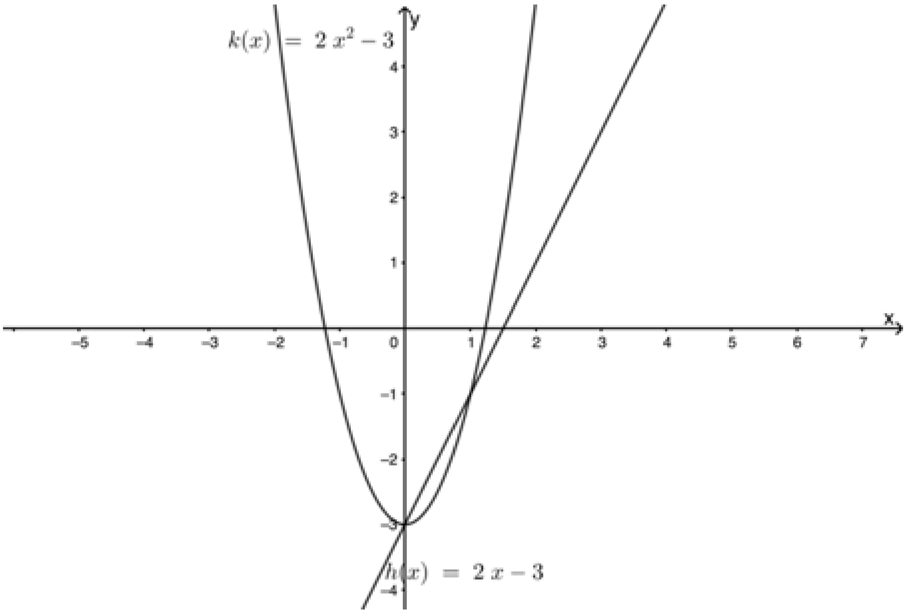
What did you find?

1. We were given two similar looking but different functions.
   $h(x)$ is the linear function. The highest power on x is 1. The function is in the form of a linear expression.

   $k(x)$ is the quadratic function. The highest power on x is 2. The function is in the form of a quadratic expression.
2. Here is the completed table of values for $k(x)$.

x	-3	-2	-1	0	1	2	3
$k(x)$	15	5	1	3	1	5	15

Here are the two functions sketched on the same set of axes.



3. Each function has a constant term of -3 and each graph has a y-intercept of -3. Each graph also has a coefficient on the leading term of 2.
4. $h(x)$ is a linear function with a highest power of 1. It has one intercept with the x-axis (when $y = 0$) corresponding to the fact that linear equations only have one root or answer. However, $k(x)$ is a quadratic function with a highest power of 2. It has two intercepts with the x-axis (when $y = 0$) corresponding to the fact that quadratic equations can have two roots or answers.
5. The parabola is symmetrical about the y-axis. All the x values on the y-axis are zero. Therefore, the axis of symmetry is the line ($x = 0$).
6. The turning point (or TP) of the quadratic function is the point $(0, -3)$.
7. The domain and range of the linear function are both unrestricted. They are both all real numbers.
Domain: $\{x \mid x \in \mathbb{R}\}$
Range: $\{h(x) \mid h(x) \in \mathbb{R}\}$
8. The domain of the quadratic function is all real numbers. There are no input values (values of x) that we cannot use.

   The range, however, is different. We can see from the graph that the y values are never less than -3. This is the minimum value of the graph. At this point the graph turns around.

   So, we can write the domain and range as follows:

   Domain: $\{x \mid x \in \mathbb{R}\}$
   Range: $\{k(x) \mid k(x) \in \mathbb{R}, k(x) \geq -3\}$

In Activity 2.1 we saw that the value of q in $y = ax + q$ and the quadratic function $y = ax^2 + q$ do the same thing – they tell us where the graph cuts the y-axis. This makes sense. Remember that one of the ways to sketch a linear function is the dual-intercept method where we find where the graph intercepts both axes.

- x-intercept (let $y = 0$)
- y-intercept (let $x = 0$)

Well, the same principle applies to the quadratic function. We can find the y-intercept by letting $x = 0$ and we can find the x-intercept by letting $y = 0$. Have a look at this example:

Example 2.1

Suppose we have $y = 3x^2 - 1$. Let's find the intercepts with the axes.

- y-intercept (let $x = 0$): $y = 3(0)^2 - 1 = -1$. As expected, the y-intercept is simply the value of q.
- x-intercept (let $y = 0$):

$$\begin{aligned} 0 &= 3x^2 - 1 \\ \therefore 3x^2 &= 1 \\ \therefore x^2 &= \frac{1}{3} \\ \therefore x &= \pm\sqrt{\frac{1}{3}} \end{aligned}$$

As expected, we have two points where the graph intercepts the x-axis.

The following activity further explores quadratic functions.

Activity 2.2: The effect of *a* in *y=ax²+q*

Time required: 30 minutes

What you need:

- a pen or pencil
- a calculator
- blank paper or a notebook
- an internet connection (optional)

What to do:

1. Work through these questions:
   a. On the same set of axes, plot the functions $f(x) = 2x^2 - 2$ and $g(x) = \frac{1}{2}x^2 - 2$. Use a table of values.
   b. How does the value of a in $y = ax^2 + q$ change the shape of the parabola?
   c. What are the turning points of each function? Are these maximum or minimum points?
   d. What is the range of each function?
   e. Now, on the same set of axes, plot the function $h(x) = -2x^2 + 2$ and $j(x) = -\frac{1}{2}x^2 + 2$. Use a table of values.
   f. How does changing the sign of a in $y = ax^2 + q$ change the shape of the parabola?
   g. What are the turning points of each function? Are these maximum or minimum points?
   h. What is the range of each function?
2. Now, if you have access to the internet, visit the quadratic function interactive simulation.

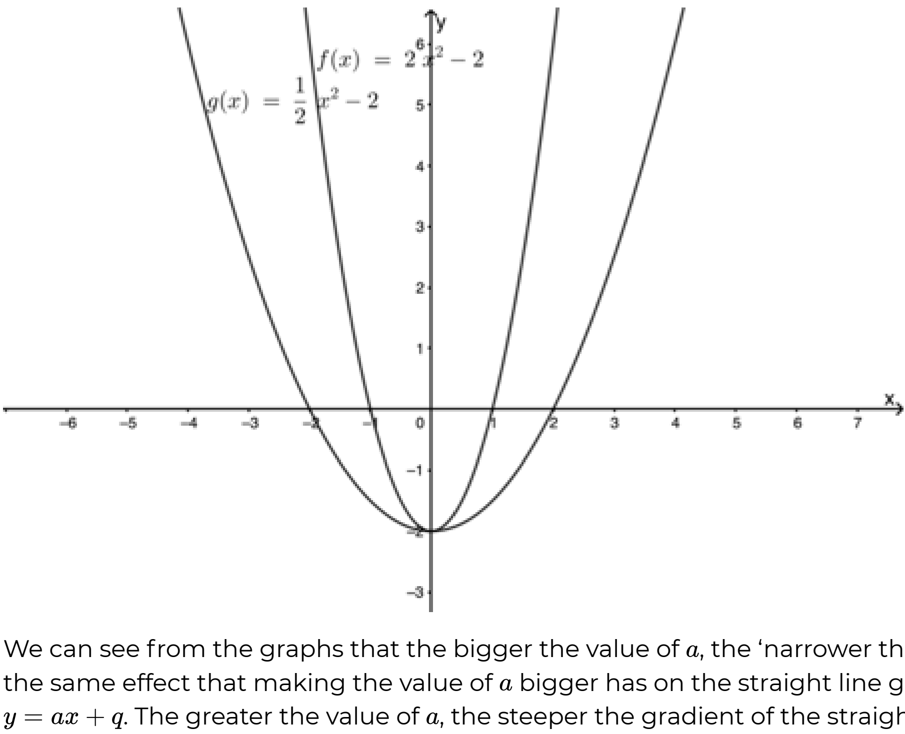
Here you will find a quadratic function of the form $y = ax^2 + q$ with sliders to change the values of a and q.

a. Change the value of q to confirm what you know about how the value of q affects the parabola. How does this relate to how q affects the straight line $y = ax + q$?
b. Change the value of a to confirm what you know about how the value of a affects the shape of the parabola. How does this relate to how a affects the straight line $y = ax + q$?
c. What is the difference between a large value of a like 5 and a small value of a like $\frac{1}{5}$ on the shape of the parabola?
d. What is the difference between positive and negative values of a on the shape of the parabola?
e. What does a value of $a = 0$ mean? Why is this?

What did you find?

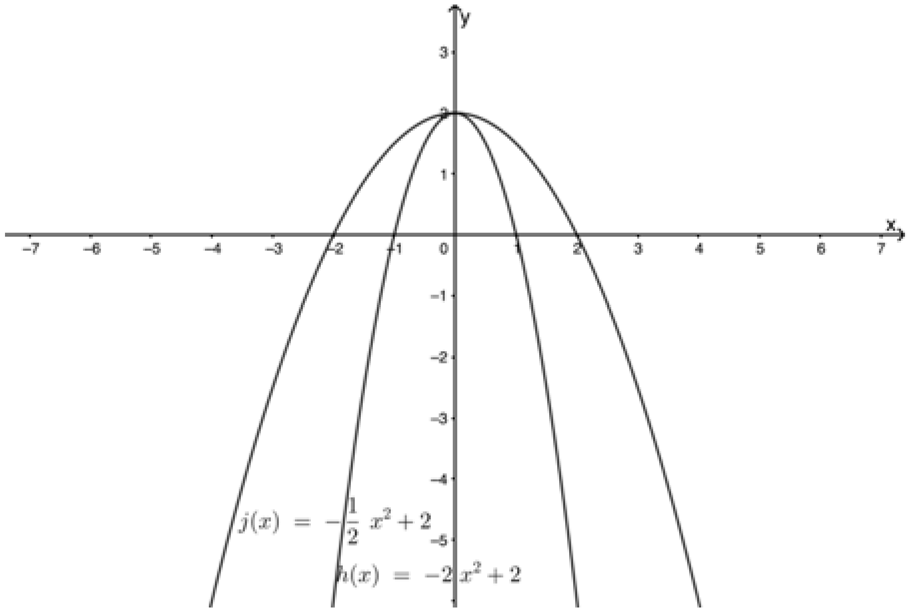
1.

a. Here are functions $f(x) = 2x^2 - 2$ and $g(x) = \frac{1}{2}x^2 - 2$ plotted on the same set of axes.



b. We can see from the graphs that the bigger the value of a, the 'narrower the parabola. This is the same effect that making the value of a bigger has on the straight line graph given by $y = ax + q$. The greater the value of a, the steeper the gradient of the straight line.
c. For each function the turning point is the point $(0, -2)$ (which just so happens to be the y-intercept as well). In both cases, the turning point is a **minimum** value.
d. The range of each function is all real values greater than or equal to -2. We can write the ranges as $\{f(x) \mid f(x) \in \mathbb{R}, f(x) \geq -2\}$ and $\{g(x) \mid g(x) \in \mathbb{R}, g(x) \geq -2\}$.

e. Here are the functions $h(x) = -2x^2 + 2$ and $j(x) = -\frac{1}{2}x^2 + 2$ plotted on the same set of axes.



f. We can see from the graphs that when the sign of a is changed from positive to negative, the graphs 'flip over'. Instead of curving upwards they curve downwards. The **absolute value** of a (the value of a irrespective of the sign) still determines the steepness of the graph. The bigger a is, the steeper the graph is.

g. For each function the turning point is the point $(0, 2)$ (which happens to be the y-intercept as well). In both cases, the turning point is a **maximum** value.

h. The range of each function is all real values less than or equal to 2. We can write the ranges as $\{f(x) \mid f(x) \in \mathbb{R}, f(x) \leq 2\}$ and $\{g(x) \mid g(x) \in \mathbb{R}, g(x) \leq 2\}$.

2.

a. In both the quadratic and linear function, the value of q tells us where the graph cuts the y-axis. It gives us the y-intercept.

b. In both the quadratic and linear function, the value of a determines how steep the graph is. A big value of a gives a steep straight line and a steep and narrow parabola.

c. A large value of a gives a very steep and narrow parabola, while a small value of a gives a parabola which is less steep and wider.

d. A positive value of a means that the parabola has a minimum TP and is a 'smiley face'. A negative value of a gives a parabola which is a 'sad face' and has a maximum turning point.

e. When $a = 0$, the quadratic function $y = ax + q$ becomes $y = q$. This is a horizontal straight line where every point on the line has a y-coordinate of q.

We learnt from Activity 2.2 that the value of a in $y = ax^2 + q$ determines the steepness of the parabola. The greater the value of a, the more steep the parabola is. We also discovered that the sign of a (whether it is positive or negative) determines whether the graph bends up into a 'smiley face' or down into a 'sad face'. The absolute value of a (its size irrespective of its sign) still determines how steep or shallow the curve is.

Figure 4 shows a summary of what we have learnt so far for $y = ax^2 + q$.

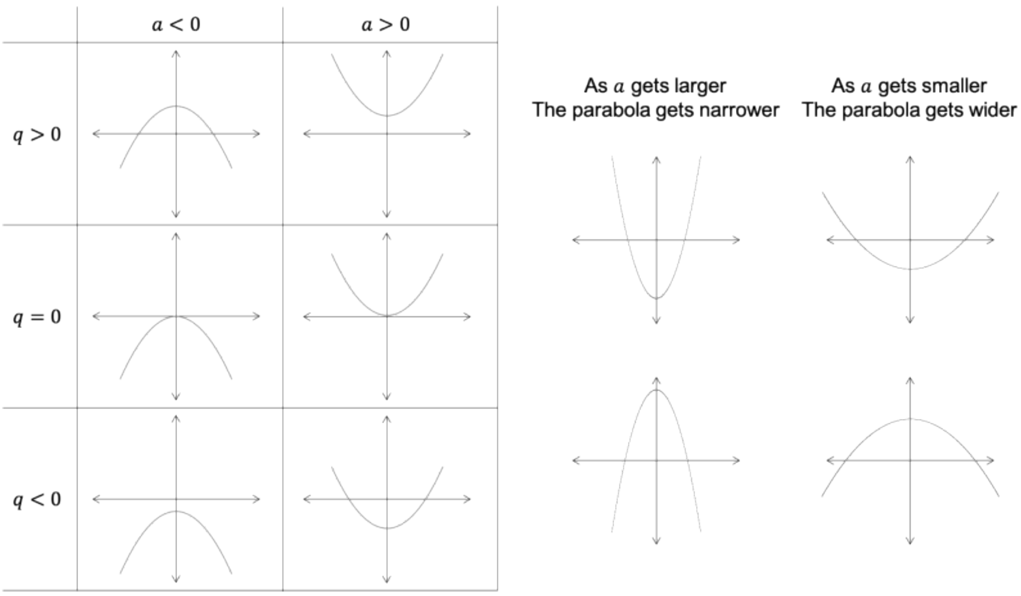


Figure 4: Summary of $aq$the effects of and on the function . $y = ax^2 = q$

Look at the summary in Figure 4. Whether the graph has two, one or zero x-intercepts tells us some very important things about the roots of the equation. Remember that 'roots' is just another word for the answers to an equation.

In general, we know that quadratic equations have two roots and, therefore, quadratic functions have two x-intercepts. But we can see, for example, when the value of $q = 0$ that the TP is on the x-axis and, rather than cutting through the x-axis, the graph just touches the x-axis. In this case, we say the function has one root.

However, when $a > 0$ and $q > 0$, we can see that the graph does not even touch the x-axis and there are definitely no x-intercepts. In this case, the quadratic equation has no real roots. The same is true when $a < 0$ and $q < 0$.

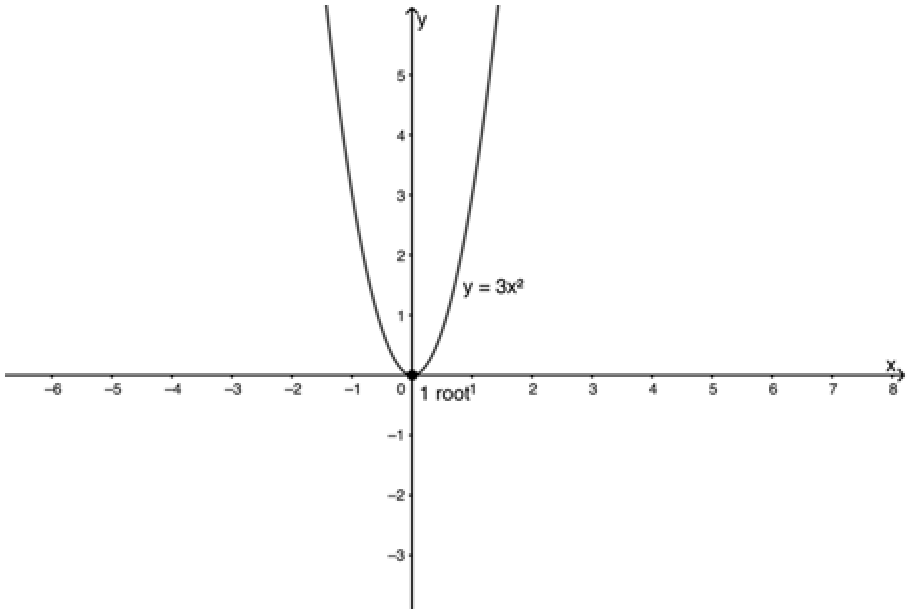
Did you know?

When a parabola has no x-intercepts we say that the function has no real roots. The word 'real' is important because the function does actually have roots, they are just non-real numbers. The graph will have x-intercepts in the non-real plane. You will learn all about non-real roots in levels three and four.

So the equation $y = 3x^2$ (where $q = 0$) has one root and the TP of its graph will lie on the x-axis. If we make $y = 0$ and solve for x we get:

$$
\begin{aligned}
&0 = 3x^2 \\
\therefore\ &0 = x^2 \\
\therefore\ &x = 0
\end{aligned}
$$

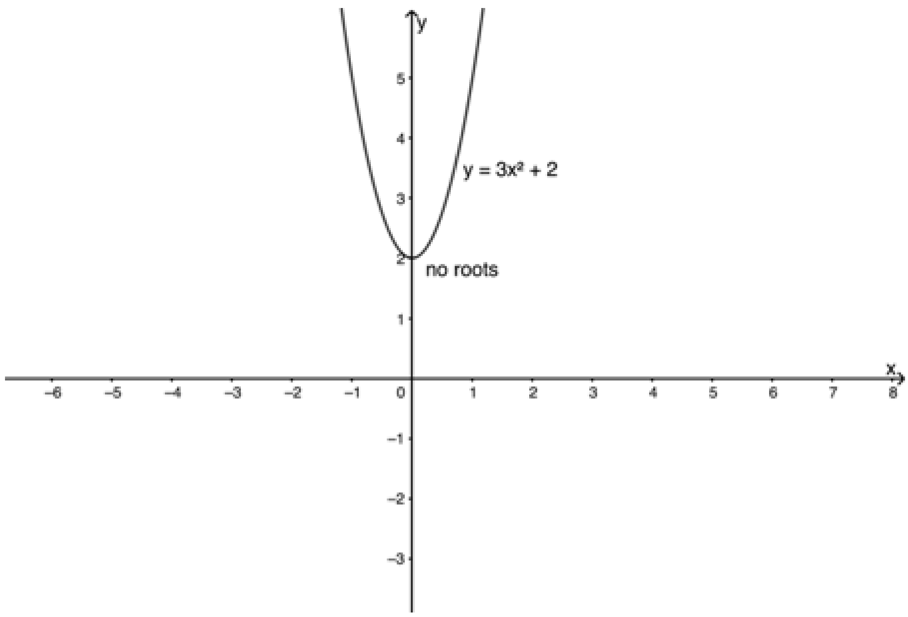
There is only one root (zero) and the TP, y-intercepts and x-intercept are all on the origin.



The equation $y = 3x^2 + 2$ (where $a > 0$ and $q > 0$) has no real roots and its graph will have no x-intercepts. If we make $y = 0$ and solve for x we get:

$$\begin{aligned} 0 &= 3x^2 + 2 \\ \therefore 3x^2 &= -2 \\ \therefore x^2 &= -\frac{2}{3} \end{aligned}$$

But there is no number we know of that, when we multiply it by itself, will give us a negative number ($-\frac{2}{3}$ in this case) so there are no real solutions to the equation and the graph will not cut or even touch the x-axis.



Example 2.2

State whether the graphs of the following quadratic equations will have two, one or zero x-intercepts.

1. $y = 2x^2 - 7$
2. $y = -2x^2 - 7$
3. $-7y = -2x^2 - 7$
4. $3y - 4x^2 = 0$

Solution

1. The value of a is positive and the value of q is negative. Therefore, the graph will have 2 x-intercepts.
2. The value of a is negative and the value of q is negative. Therefore, the graph will have no x-intercepts.
3. We need to get the equation into standard form:

$$\begin{aligned}-7y &= -2x^2 - 7\\ \therefore y &= \frac{-2}{-7}x^2 + 1\\ \therefore y &= \frac{2}{7}x^2 + 1\end{aligned}$$

   The value of a is positive and the value of q is positive. Therefore, the graph will have no x-intercepts.
4. We need to get the equation into standard form.

$$\begin{aligned}3y - 4x^2 &= 0\\ \therefore 3y &= 4x^2\\ \therefore y &= \frac{4}{3}x^2\end{aligned}$$

   The value of a is positive and $q = 0$. Therefore, the graph will have 1 x-intercept. In other words, it will touch the x-axis.

Now, have a look at the next example to get some practise sketching quadratic functions of the form $y = ax^2 + q$.

Example 2.3

1. Sketch the function $f(x) = 3x^2 - 12$. Mark the TP and intercepts.
   a. Is the TP a maximum or minimum?
   b. State the domain and range of the function.
   c. What is the axis of symmetry?
2. Sketch the function $h(x) = -\frac{1}{2}x^2 - 3$. Mark the intercepts and TP.
   a. Is the TP a maximum of minimum?
   b. State the domain and range of the function.

c. What is the axis of symmetry?

3. Given $s(x) = -3x^2 + 7$ and $t(x) = \frac{1}{3}x^2 + 3$. Without sketching the functions, determine:

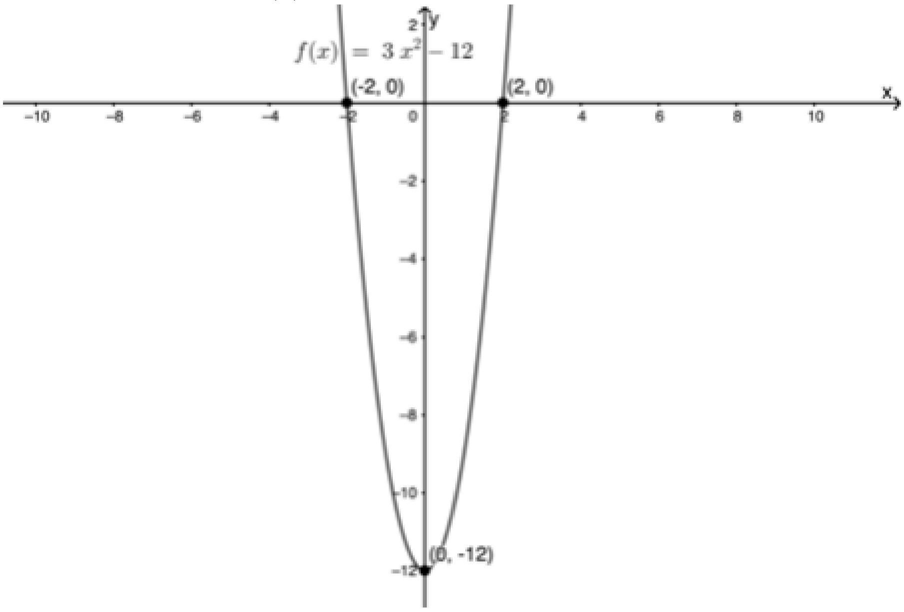
   a. Which graph will be steeper?

   b. Which function has no real roots?

   c. Which function has a minimum TP? What is the TP?

Solution

1. Here is the sketch of $f(x) = 3x^2 - 12$ with the TP and intercepts marked

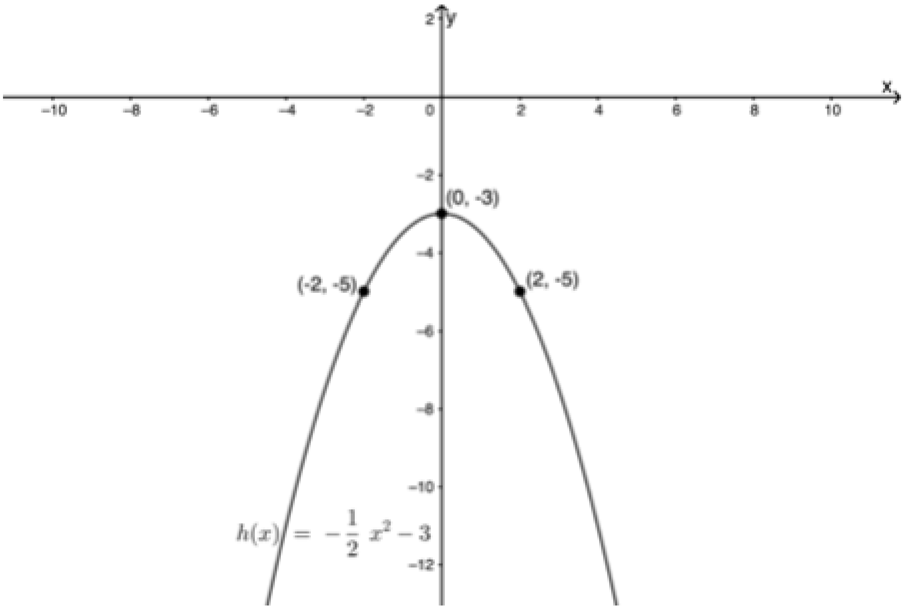


   a. The TP is a minimum.

   b. Domain: $\{x \mid x \in \mathbb{R}\}$
      Range: $\{f(x) \mid f(x) \in \mathbb{R}, f(x) \geq -12\}$

   c. The axis of symmetry is the y-axis or the line $x = 0$.

2. Here is the sketch of $h(x) = -\frac{1}{2}x^2 - 3$. Note that because the graph has no x-intercepts, we needed to find other points to help us accurately draw the shape of the graph.



a. The TP is a maximum.

b. Domain: $\{x \mid x \in \mathbb{R}\}$
Range: $\{h(x) \mid h(x) \in \mathbb{R}, h(x) \leq -3\}$

c. The axis of symmetry is the y-axis or the line $x = 0$.

3. The functions are $s(x) = -3x^2 + 7$ and $t(x) = \frac{1}{3}x^2 + 3$

a. The absolute value of a in $s(x)$ is 3 while in $t(x)$ it is $\frac{1}{3}$. Therefore $s(x)$ will be the steeper curve.

b. $t(x)$ will have no real roots because the value of a is positive and the value of q is positive.

c. In $t(x)$, the value of a is positive, therefore the graph curves upwards and has a minimum TP. Because the function is of the form $y = ax^2 + q$, the TP will be the same as the y-intercept which is 3.

Exercise 2.1

1. Look at the Cartesian plane below and match each of the equations to one of the graphed functions.

a. $y = -2x^2$

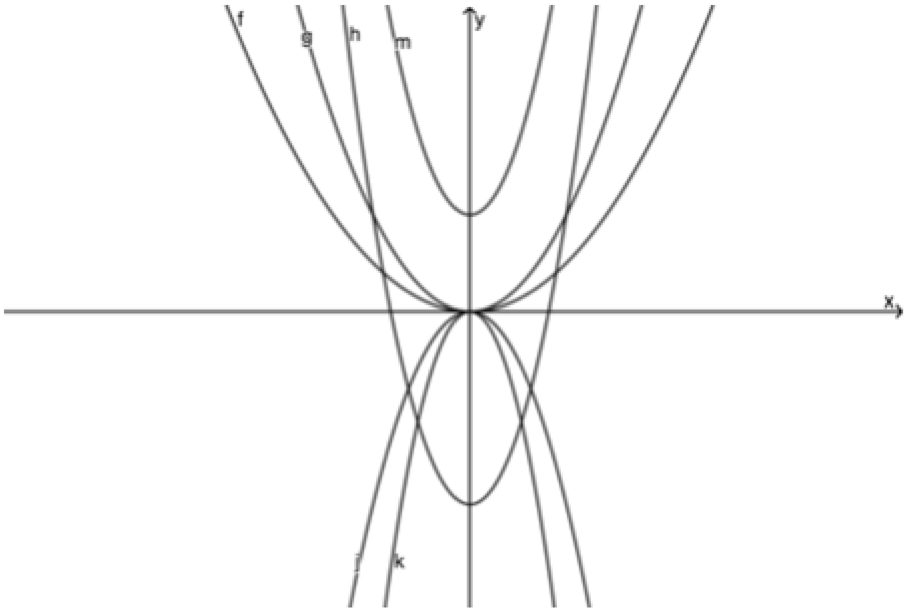
b. $y = 3x^2 + 1$

c. $y = \frac{1}{2}x^2$

d. $y = x^2$

e. $y = -4x^2$

f. $y = 3x^2 - 2$



2. Sketch the function $q(x) = -\frac{2}{3}x^2 + 6$. Mark the TP and intercepts.
   a. State the domain and range of $q(x)$.
   b. What is the equation of the axis of symmetry?
3. Without sketching the graph, determine what the TP of $4y + 3x^2 = -48$ is and whether it is a maximum or minimum TP.
4. Sketch the function given by $c(x) = 3x^2 - 4$ as accurately as possible showing the intercepts, TP and axis of symmetry on your sketch.

The full solutions are at the end of the unit.

Let's quickly summarise what we know about quadratic functions of the form $y = ax^2 + q$ and how to sketch them.

In order to sketch graphs of the form $y = ax^2 + q$, we need to determine the following:

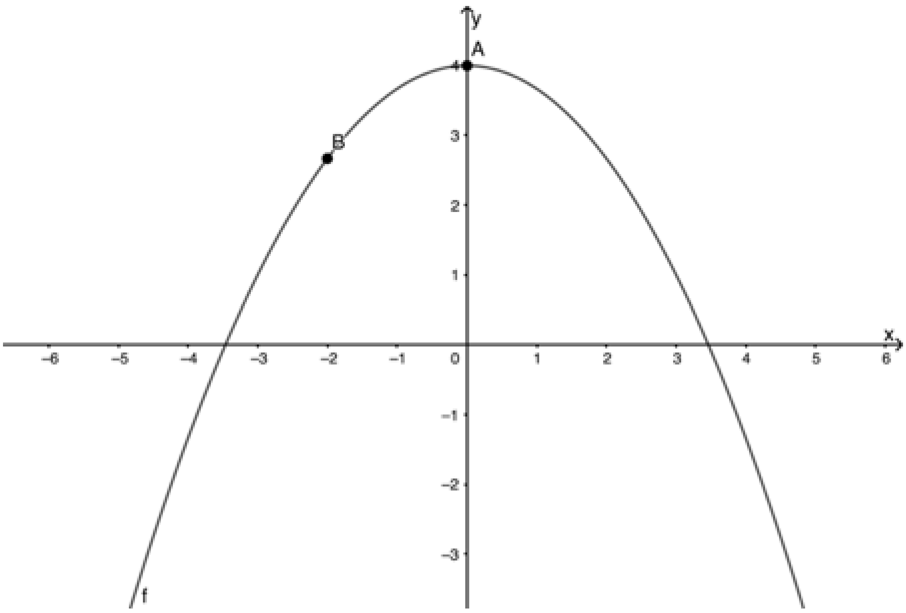
- The sign of a: this tells us if the graph curves upwards and has a minimum TP or whether it curves downwards and has a maximum TP.
- The y-intercept: we find this by letting $x = 0$.
- The x-intercepts: we find these by letting $y = 0$ (Note: there may be no x-intercepts. In this case, it is necessary to find one or two other points on the graph to help us get the shape right.)
- The TP: for $y = ax^2 + q$ this is always the same as the y-intercept.

Find the equation of quadratic functions

Now that we can sketch quadratic functions, we need to see how to find their equations from their graphs.

Example 2.4

1. The image below shows a graph of a function of the form $y = ax^2 + q$. Point A is the turning point with coordinates $(0, 4)$ and point B is $(-2, \frac{8}{3})$.



   Find the equation of the graph.

2. Find the equation of the function passing through the point $(-3, -3\frac{3}{5})$ with a TP on the origin and an axis of symmetry of $x = 0$.

Solution

1. We are told that the function is of the form $y = ax^2 + q$. Therefore, we know that the TP and the y-intercept are the same point. We are told that the TP is the point $\text{A}(0, 4)$.
   Therefore $q = 4$.

   So far, we can write the function as $y = ax^2 + 4$. Now substitute the coordinates of B into the equation and solve for a.

$$\begin{aligned} \frac{8}{3} &= a(-2)^2 + 4 \\ \therefore 8 &= 12a + 12 \\ \therefore 12a &= -4 \\ \therefore a &= -\frac{1}{3} \end{aligned}$$

   So the equation of the graph is $y = -\frac{1}{3}x^2 + 4$.

2. We are told that the axis of symmetry is the line $x = 0$. Therefore, the function is of the form $y = ax^2 + q$. We are also told that the TP is on the origin. This means that the y-intercept is 0 and, therefore, $q = 0$. So, the equation of the function is $y = ax^2$.

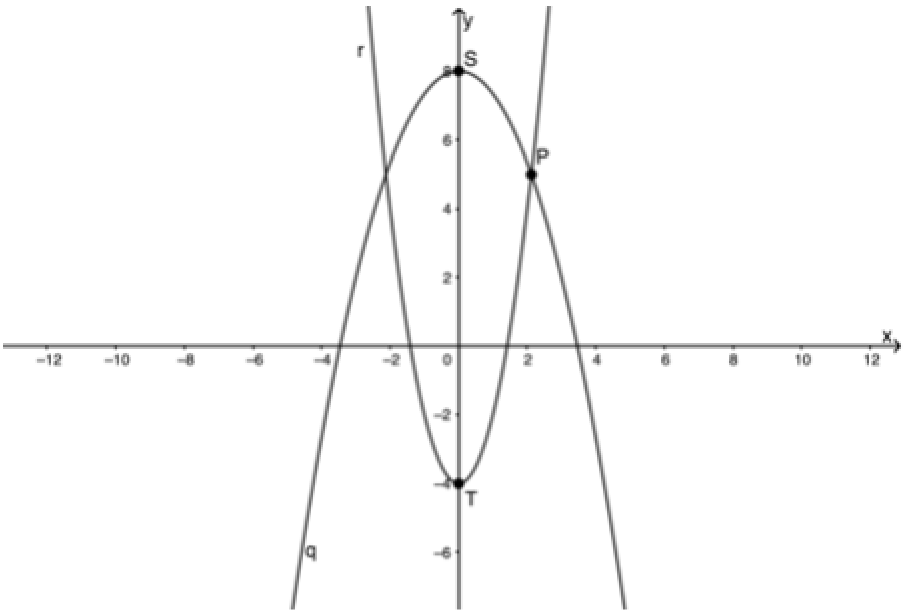
We can substitute $(-3, -3\frac{3}{5})$ into this equation for x and y and solve for a.

$$
\begin{aligned}
-3\frac{3}{5} &= a(-3)^2 \\
\therefore \frac{-18}{5} &= 9a \\
\therefore -18 &= 45a \\
\therefore a &= -\frac{18}{45} \\
\therefore a &= -\frac{2}{5}
\end{aligned}
$$

So the equation of the function is $y = -\frac{2}{5}x^2$.

Exercise 2.2

1. Find the equation of the graph of the form $y = ax^2 + q$ that passes through the origin and the point $(3, -\frac{4}{3})$.
2. The functions $r(x)$ and $q(x)$, both of the form $y = ax^2 + q$, are plotted as shown. They intersect each other at $\mathrm{P}(\frac{3}{\sqrt{2}}, 5)$. S is the point $(0, 8)$ and T is the point $(0, -4)$.



   a. Find the equations of both functions.
   b. For which values of x is r increasing? In other words, for which values of x are the y-values of r getting bigger and bigger?

c. For which values of x is $q(x) > r(x)$?

3. The range of $d(x)$ is given as $[-4, \infty]$. If $d(x)$ is a quadratic function of the form $y = ax^2 + q$ and passes through the point $(1, 3)$, what is the equation of $d(x)$?

The full solutions are at the end of the unit.

Summary

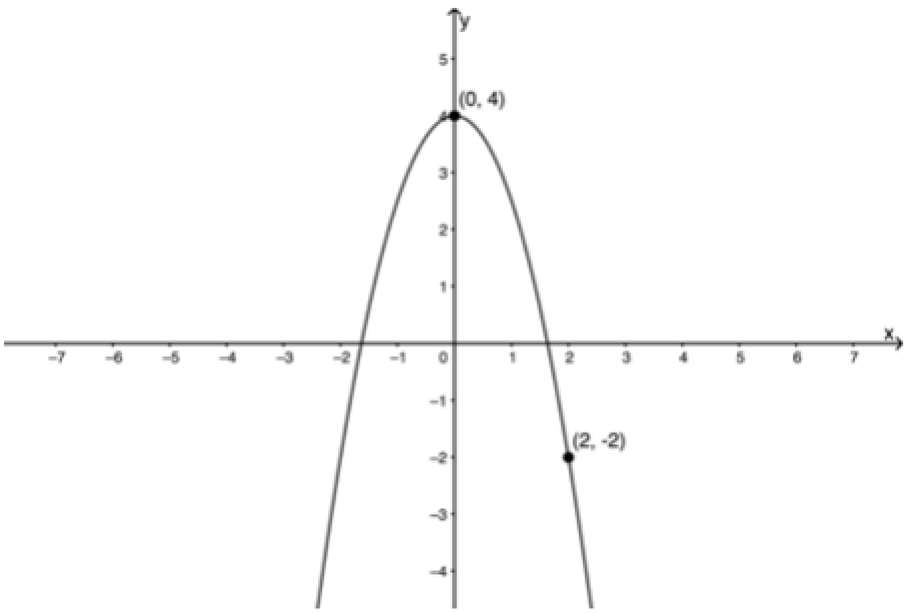
In this unit you have learnt the following:

- The function of the form $y = ax^2 + q$ is a parabola with an axis of symmetry of the y-axis or the line $x = 0$ and a y-intercept of $(0, q)$.
- Because the axis of symmetry is the line $x = 0$, the turning point (TP) and the y-intercept are the same point; therefore the TP is the point $(0, q)$.
- If $a > 0$, the graph curves upwards and has a minimum TP.
- If $a < 0$, the graph curves downwards and has a maximum TP.
- If $a < 0$ and $q < 0$, or $a > 0$ and $q > 0$, the graph has no x-intercepts and the equation has no real roots.
- If $q = 0$, the graph touches the x-axis and the equation has one real root.

Unit 2: Assessment

Suggested time to complete: 30 minutes

1. Determine whether $2x^2 - 3y = 4$, has a minimum or maximum value and give the corresponding value of x.
2. From the following sketch:



a. Determine the range for the parabola $f(x)$.

b. Determine the domain for the parabola $f(x)$.

c. Determine the equation for the parabola $f(x)$.

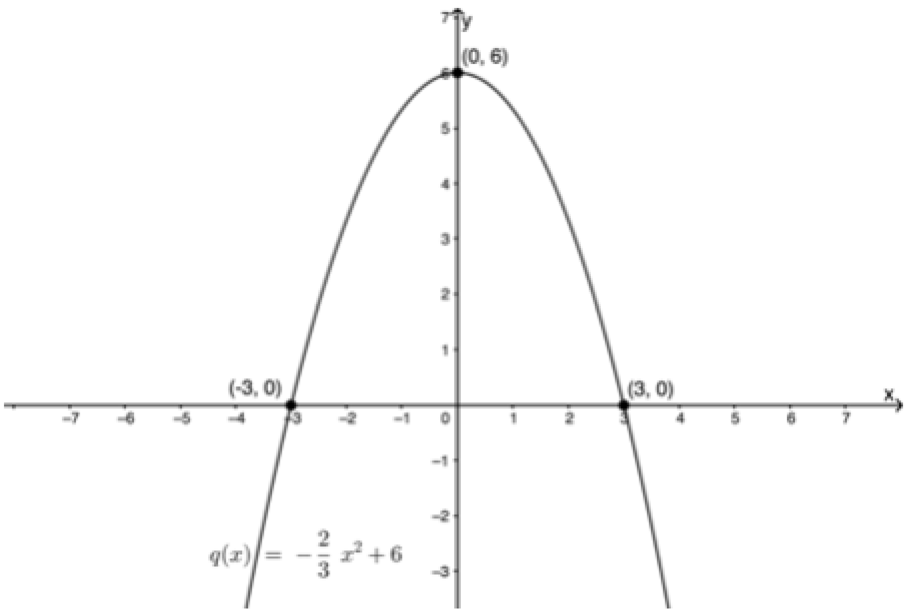
3. Given $y = 2 - 2x^2$, determine the following:
    a. The x-intercepts.
    b. The y-intercepts.
    c. The turning point.
    d. The axis of symmetry
    e. The range of the graph.
    f. The domain of the graph.
    g. Sketch the graph, showing the TP and all intercepts.

The full solutions are at the end of the unit.

Unit 2: Solutions

Exercise 2.1

1.
    a. $y = -2x^2$: $j(x)$
    b. $y = 3x^2 + 1$: $m(x)$
    c. $y = \frac{1}{2}x^2$: $f(x)$
    d. $y = x^2$: $g(x)$
    e. $y = -4x^2$: $k(x)$
    f. $y = 3x^2 - 2$: $h(x)$
2.



    a. Domain: $\{x \mid x \in \mathbb{R}\}$
    Range: $\{q(x) \mid q(x) \in \mathbb{R}, q(x) \leq 6\}$
    b. The axis of symmetry is the y-axis or the line $x = 0$.

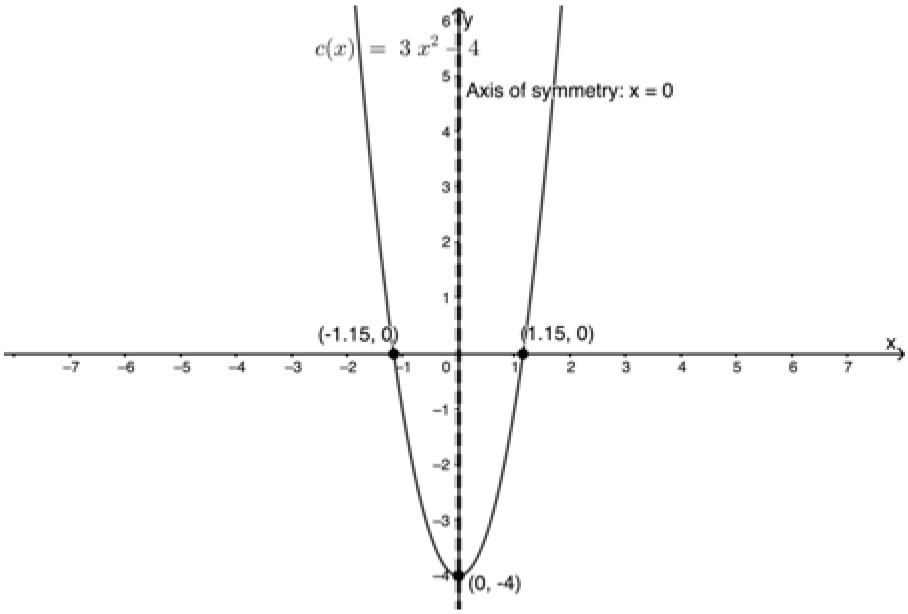
3.

$$4y + 3x^2 = -48$$
$$\therefore 4y = -3x^2 - 48$$
$$\therefore y = -\frac{3}{4}x^2 - 12$$

Because the function is of the form $y = ax^2 + q$ the TP is the same as the y-intercept. Therefore, the TP is the point $(0, -12)$.

The value of a is negative, therefore the graph curves downwards and the TP is a maximum.

4.



Back to Exercise 2.1

Exercise 2.2

1. The graph passes through the origin. Therefore $q = 0$.

   Substitute $(3, -\frac{4}{3})$ into $y = ax^2$ to solve for a.

$$-\frac{4}{3} = a(3)^2$$
$$\therefore -4 = 27a$$
$$\therefore a = -\frac{4}{27}$$

   The function is $y = -\frac{4}{27}x^2$.

2.

   a. $r(x)$:
   The y-intercept of the function is $(0, -4)$. Therefore $q = -4$.
   Substitute $\mathrm{P}(\frac{3}{\sqrt{2}}, 5)$ into $y = ax^2 - 4$.

$$5 = a\left(\frac{3}{\sqrt{2}}\right)^2 - 4$$
$$\therefore 5 = \left(\frac{9}{2}\right)a - 4$$
$$\therefore 10 = 9a - 8$$
$$\therefore 18 = 9a$$
$$\therefore a = 2$$
$$\therefore r(x) = 2x^2 - 4$$

$q(x)$:
The y-intercept of the function is $(0, 8)$. Therefore $q = -8$.
Substitute $\mathrm{P}(\frac{3}{\sqrt{2}}, 5)$ into $y = ax^2 + 8$.

$$5 = a\left(\frac{3}{\sqrt{2}}\right)^2 + 8$$
$$\therefore 5 = \left(\frac{9}{2}\right)a + 8$$
$$\therefore 10 = 9a + 16$$
$$\therefore -6 = 9a$$
$$\therefore a = -\frac{2}{3}$$
$$\therefore q(x) = -\frac{2}{3}x^2 + 8$$

b. $r(x)$ increases for all values to the right of the minimum TP. Therefore, $r(x)$ increases for $x \geq 0$.

c. $q(x) > r(x)$ between the points where the graphs intersect. Because the graphs have an axis of symmetry of $x = 0$, they intersect at the points $(\frac{3}{\sqrt{2}}, 5)$ and $(-\frac{3}{\sqrt{2}}, 5)$. Therefore, $q(x) > r(x)$ for $\left(-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$.

Note: We need to use round brackets to indicate that the end points of the interval are not included because we are only looking for where $q(x) > r(x)$.

3. The range of $d(x)$ is given as $[-4, \infty]$ and the function is of the form $y = ax^2 + q$. Therefore, it has a minimum TP and y-intercept of $(-4, 0)$. Therefore $q = -4$.
Substitute $(1, 3)$ into $y = ax^2 - 4$ and solve for a.
$$3 = a(1)^2 - 4$$
$$\therefore 3 = a - 4$$
$$\therefore a = 7$$

$$\therefore d(x) = 7x^2 - 4$$

Back to Exercise 2.2

Unit 2: Assessment

1.
$$2x^2 - 3y = 4$$
$$\therefore -3y = -2x^2 + 4$$
$$\therefore y = \frac{2}{3} - \frac{4}{3}$$

$\therefore a > 0$ so the TP is a minimum and the corresponding x value of the TP is $x = 0$ (the axis of symmetry is the line $x = 0$).

2.

a. Range: $\{f(x) \mid f(x) \in \mathbb{R}, f(x) \leq 4\}$

b. Domain: $\{x \mid x \in \mathbb{R}\}$

c. $f(x) = ax^2 + 4$. Substitute $(2, -2)$ into $f(x)$ to solve for a:

$$-2 = a(2)^2 + 4$$
$$\therefore -2 = 4a + 4$$
$$\therefore 4a = -6$$
$$\therefore a = -\frac{3}{2}$$
$$\therefore f(x) = -\frac{3}{2}x^2 + 4.$$

3.

a. x-intercepts (let $y = 0$):

$$0 = 2 - 2x^2$$
$$\therefore 2x^2 = 2$$
$$\therefore x^2 = 1$$
$$\therefore x = \pm 1$$

Therefore x-intercepts are $(-1, 0)$ and $(1, 0)$.

b. y-intercepts (let $x = 0$):

$$y = 2 - 2(0)^2$$
$$\therefore y = 2$$

Therefore y-intercept is the point $(0, 2)$.

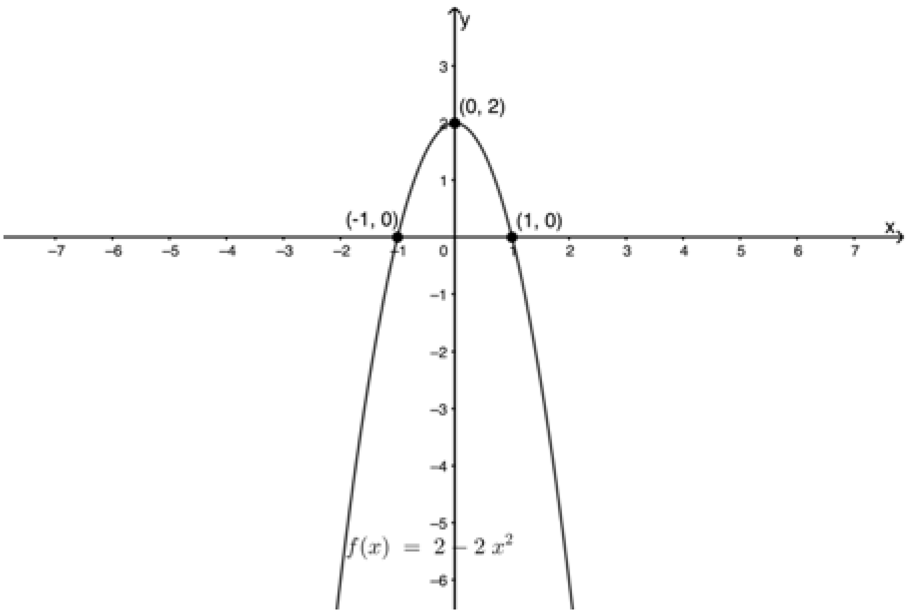
c. TP is the point $(0, 2)$.

d. The axis of symmetry is the line $x = 0$.

e. $a < 0$ therefore the TP is a maximum.
Range: $\{y \mid y \in \mathbb{R}, y \leq 2\}$

f. Domain: $\{x \mid x \in \mathbb{R}\}$

g.



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Unit 3: Hyperbolic functions

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Identify the following characteristics of functions:
  - continuous or discontinuous
  - asymptotes.
- Sketch and find the equation of the graph $y = \dfrac{a}{x} + q$.
- Investigate and generalise the impact of a and q on $y = \dfrac{a}{x} + q$.

What you should know

Before you start this unit, make sure you can

- Determine the domain and range of a function by looking at its graph. Go over Subject outcome 2.1 Unit 1 and Unit 2 if you need to.
- Manipulate and simplify algebraic expressions. Go over Subject outcome 2.2 Unit 1: Simplifying algebraic expressions if you need more help with the basics.
- Solve linear equations. Go over Subject outcome 2.3 Unit 1: Solve linear and quadratic equations if you need more help with the basics.
- Plot points on the Cartesian plane. If you do not know how to plot points onto the Cartesian plane, then you should do Subject outcome 3.3 Unit 1: Plotting points on the Cartesian plane before continuing with this unit.

Introduction

In Unit 1: Linear functions we learnt that functions are mathematical relations between two variables that map each element of the **domain** (the set of input values) to **one and only one** element in the **range** (the set of output values). We also saw that linear functions (functions of the form $y = ax + q$) produce straight line graphs.

In Unit 2: Quadratic functions we learnt that quadratic functions (functions of the form $y = ax^2 + q$) produce parabolas when we sketch them. We learnt that parabolas are widely used in satellite dishes and mirrors in telescopes.

The hyperbolic function, that produces a graph we call a **hyperbola**, is also widely used in lenses and mirrors because it also focuses light to a single point. In this unit, we will explore the hyperbolic function in more detail and discover some very important properties that make it special.

The hyperbolic function

Activity 3.1: Investigate the hyperbolic function

Time required: 30 minutes

What you need:

- a pen or pencil
- a calculator
- blank paper or a notebook

What to do:

Work through this scenario and answer the following questions:

A scientist was doing some experiments with helium gas. He measured the pressure a fixed amount of the gas was under (in kPa) for different volumes (in m^3) while making sure to keep the temperature of the gas constant at $25\ ^\circ\text{C}$. Here are his results.

Volume (in m^3**)**	320	300	160	150	80	64	40
Pressure (in kPa**)**	50	53.33	100	106.67	200	250	400

1. Plot these points on a Cartesian plane and join them with as smooth a curve as possible. Let volume be x and pressure be y.
2. What happens to the pressure as the volume increases? Will the pressure ever reach zero?
3. What happens to the pressure as the volume decreases? Can the volume ever be zero?
4. Write a mathematical expression that relates the pressure and volume of this gas at this temperature. Here's a hint: what is 320×50 equal to?
5. Is the graph symmetrical? About what line do you think the graph is symmetrical?
6. Is there a piece of the graph that is missing? What happens if we have $x = -320$? Can you plot this missing piece of the graph?

What did you find?

1. If we plot these points on a Cartesian plane and then join them with a smooth curve, we get the graph in Figure 1.

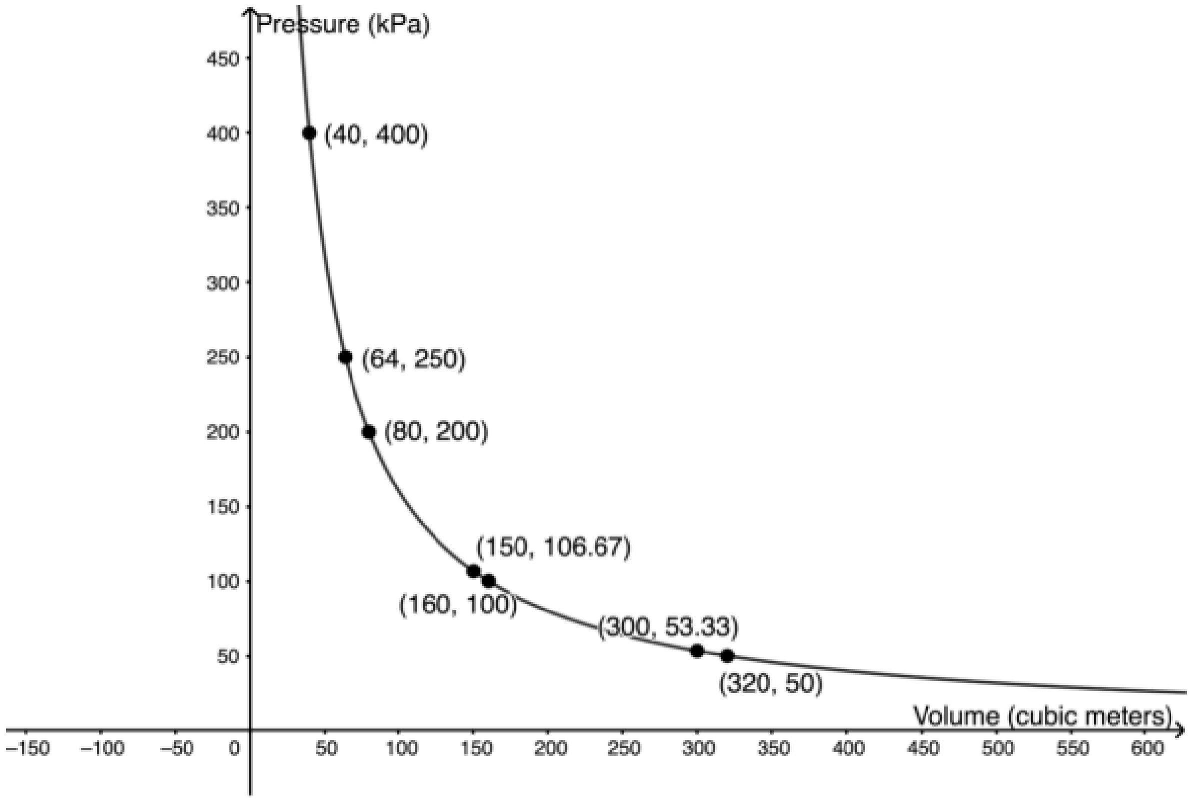


Figure 1: Plot of volume vs pressure of a gas

2. As the volume increases (the x values get bigger), the pressure decreases. It looks like eventually the pressure will be zero, but does this make any sense? Can the pressure of a gas ever be zero? No matter how big the volume gets, there will still be some pressure.
3. As the volume decreases (the x values get smaller), the pressure increases. We know that physically the volume can never be zero. Everything in the universe always takes up some space.
4. $320 \times 50 = 16\ 000$. But the interesting thing is that so does 300×53.33 (if we round up), and 160×100. In fact, if we multiple each x and y pair, the answer is always $16\ 000$. This means that this graph is given by the expression $xy = 16\ 000$.
5. It looks like the graph is symmetrical about the straight line that passes through the origin and has a gradient of 1. In other words, it looks like the graph is symmetrical about the line $y = x$ as shown in figure 2.

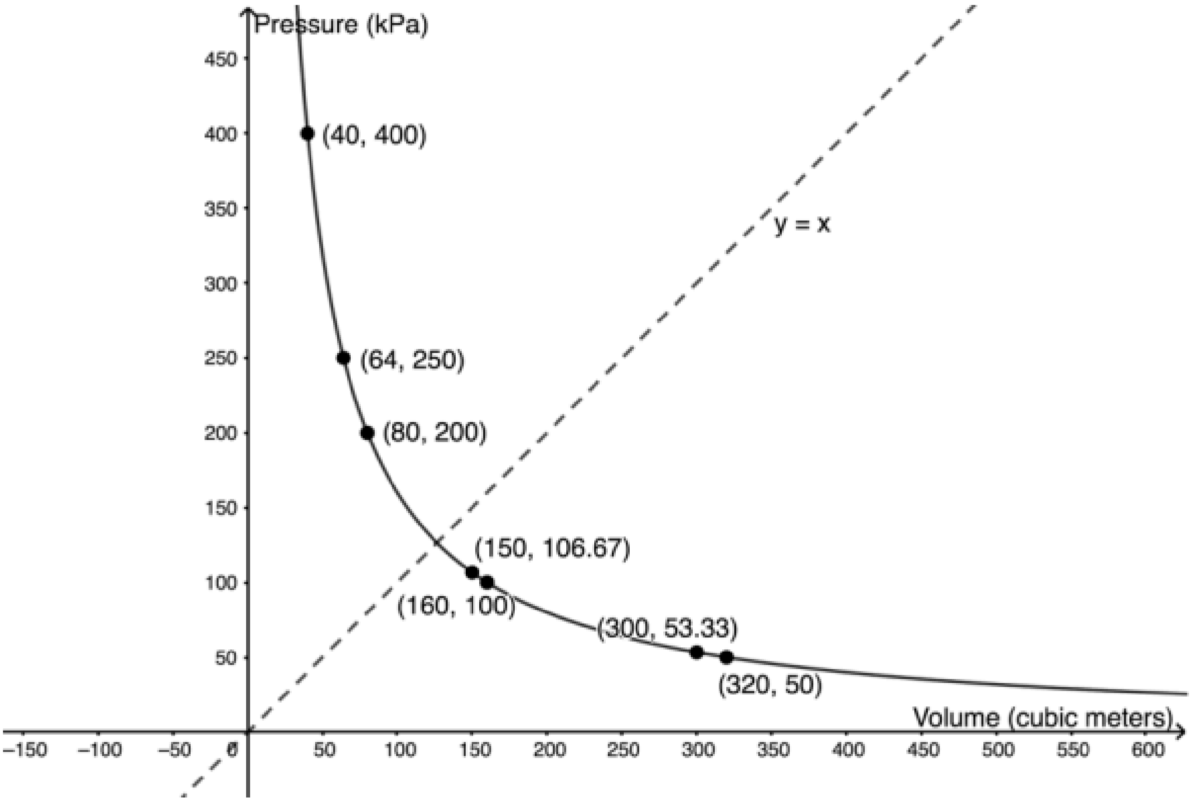


Figure 2: Plot of volume vs pressure of a gas showing symmetry about the line $y = x$

6. We know that the equation of the graph is given by $xy = 16\ 000$. Therefore, there is a piece of the graph missing because we know that $-320 \times -50 = 16\ 000$. We could make each of the values we have negative and add these new points to complete the graph. The complete graph would look like Figure 3.

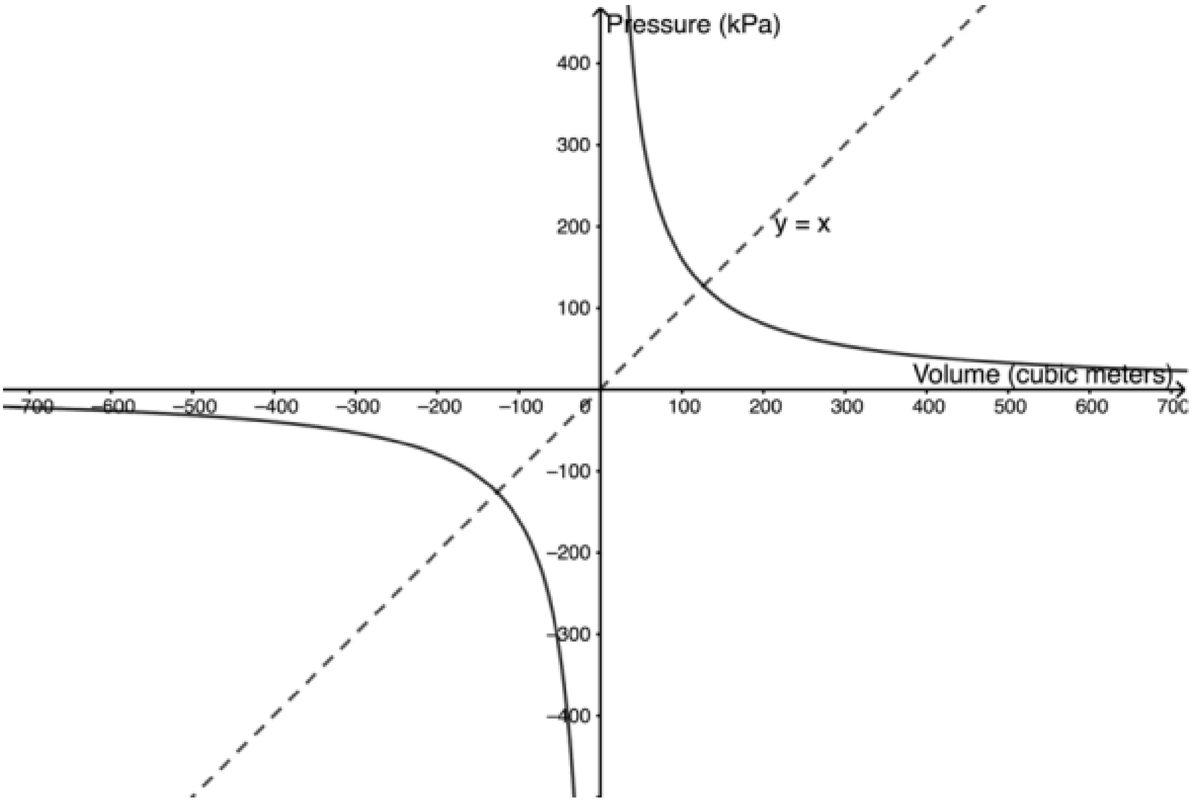


Figure 3: Graph of the function $xy = 16\ 000$

The hyperbola (the graph of the hyperbolic function) looks quite similar to a parabola. It almost looks like a parabola on its side. We can also see that the hyperbola, like the parabola, has an axis of symmetry. In Figure 3, this is the line $y = x$. But, can you see that there is another axis of symmetry – the line $y = -x$ (see Figure 4)?

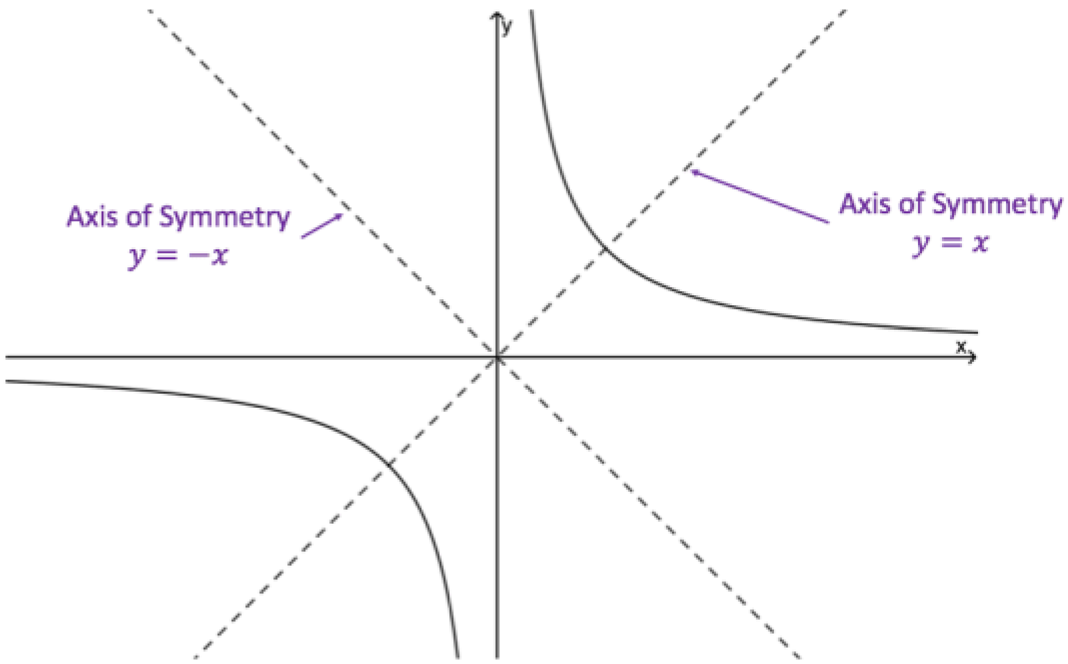


Figure 4: The axes of symmetry of the hyperbola

We saw that the hyperbolic function is given by $xy = k$ where k is some constant. We can rewrite this to look more like the other functions we have studied as $y = \frac{k}{x}$. This is called writing the equation in standard form.

In the case above, the function relating the volume and pressure of the gas is $y = \frac{16\ 000}{x}$.

The features of the hyperbolic function

Let's do another activity to explore the features of the hyperbola in more detail.

Activity 3.2: Discover the properties of the hyperbolic function

Time required: 30 minutes

What you need:

- a pen or pencil
- a calculator
- blank paper or a notebook

What to do:

You are given the hyperbolic function $f(x) = \frac{3}{x}$.

1. Complete the following table of values for this function.

x	-3	-2	-1	$-\frac{1}{3}$	0	$\frac{1}{3}$	1	2	3
y									

2. Plot these points and join them with a smooth curve.
3. What happens when $x = 0$?
4. What happens to y when the value of x gets very large?
5. What happens to y when the value of x gets very small?
6. Will the graph ever touch or cross either of the axes? Why or why not?
7. Is $y = \frac{3}{x}$ really a function? Test it and if so, write it in function notation.
8. What are the domain and range of this function?
9. About which two straight lines is the graph symmetrical?

What did you find?

1. The completed table of values.

x	-3	-2	-1	$-\frac{1}{3}$	0	$\frac{1}{3}$	1	2	3
y	-1	$-\frac{3}{2}$	-3	-9	Undefined	9	3	$\frac{3}{2}$	1

2. If we plot the points and join them with a smooth curve, we get a hyperbola (Figure 5).

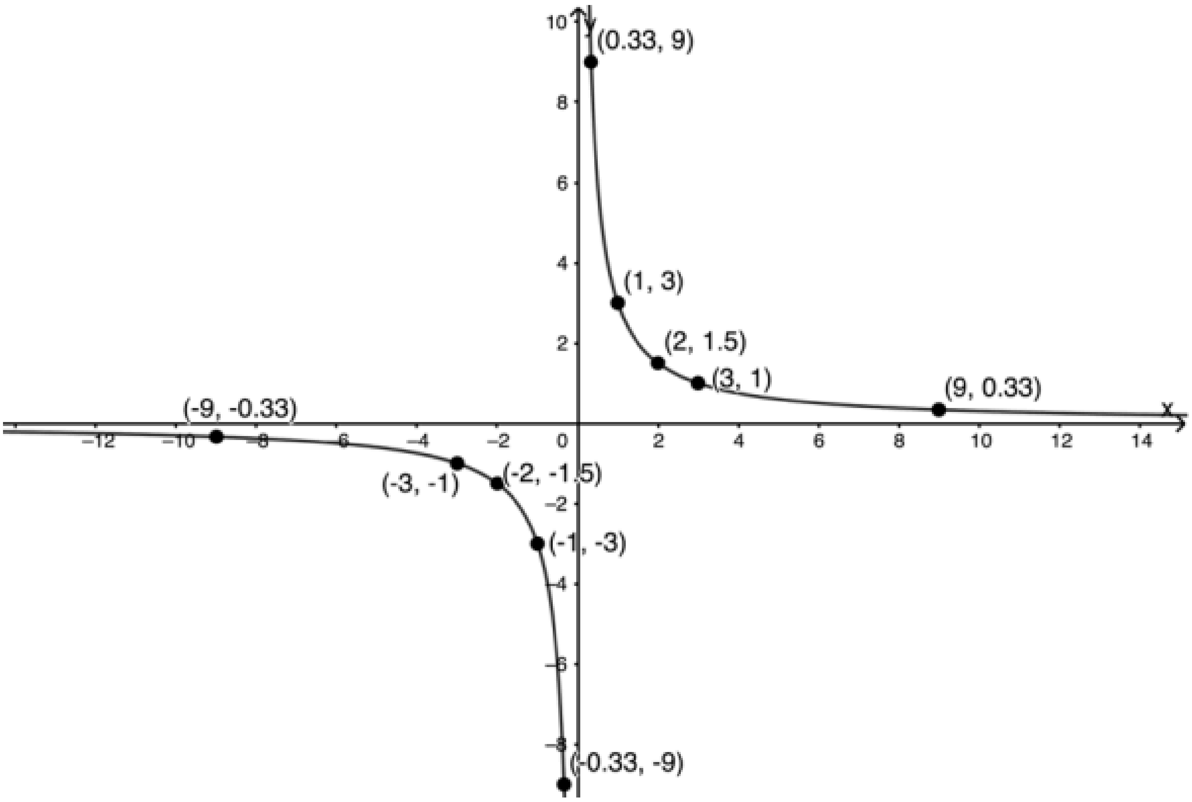


Figure 5: Plot of $y = \dfrac{3}{x}$

3. When we let $x = 0$, $\dfrac{3}{x}$ is undefined. We are never allowed to divide by zero. Therefore $x \neq 0$.
4. As x gets larger and larger, the value of $\dfrac{3}{x}$ gets smaller and smaller. We can see this in the graph. As the values of x move off to the right or the left, the graph gets closer and closer to the x-axis.
5. As x gets smaller and smaller, the value of $\dfrac{3}{x}$ gets bigger and bigger. We can see this in the graph. As the values of x get closer to the y-axis, the graph increases or decreases rapidly.
6. We know that $x \neq 0$. This means that the graph will never touch or cross the y-axis. We can rewrite the equation as $x = \dfrac{3}{y}$. Now we can also see that $y \neq 0$. This means that the graph will never touch or cross the x-axis either.
7. If we use the vertical line test, we can see that at no point does any input value give more than one output value as shown in Figure 6. At $x = 0$, however, the graph does not exist at all. We can rewrite it in function notation as $f(x) = \dfrac{3}{x}, x \neq 0$.

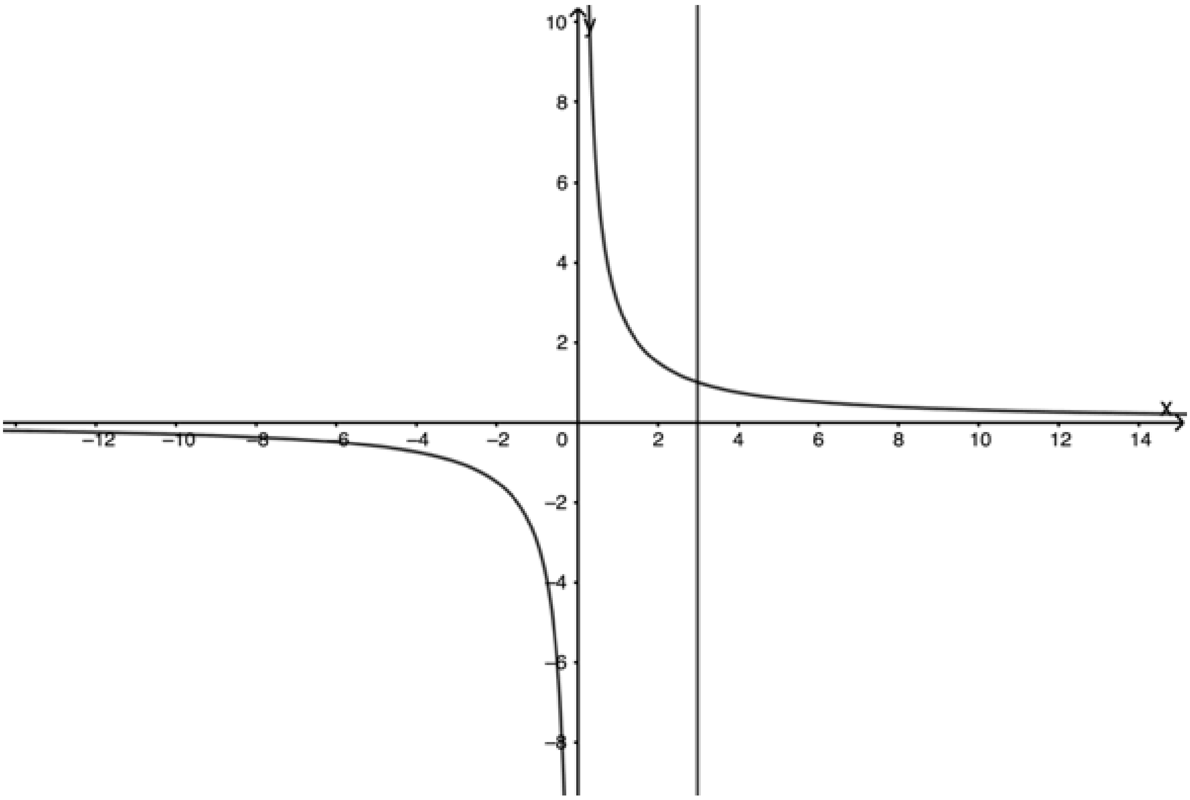


Figure 6: Vertical line test on a hyperbolic function

8. Domain: $x : x \in \mathbb{R}, x \neq 0$
   Range: $f(x) : f(x) \in \mathbb{R}, y \neq 0$
9. The graph is symmetrical about the lines $y = x$ and $y = -x$ (Figure 7).

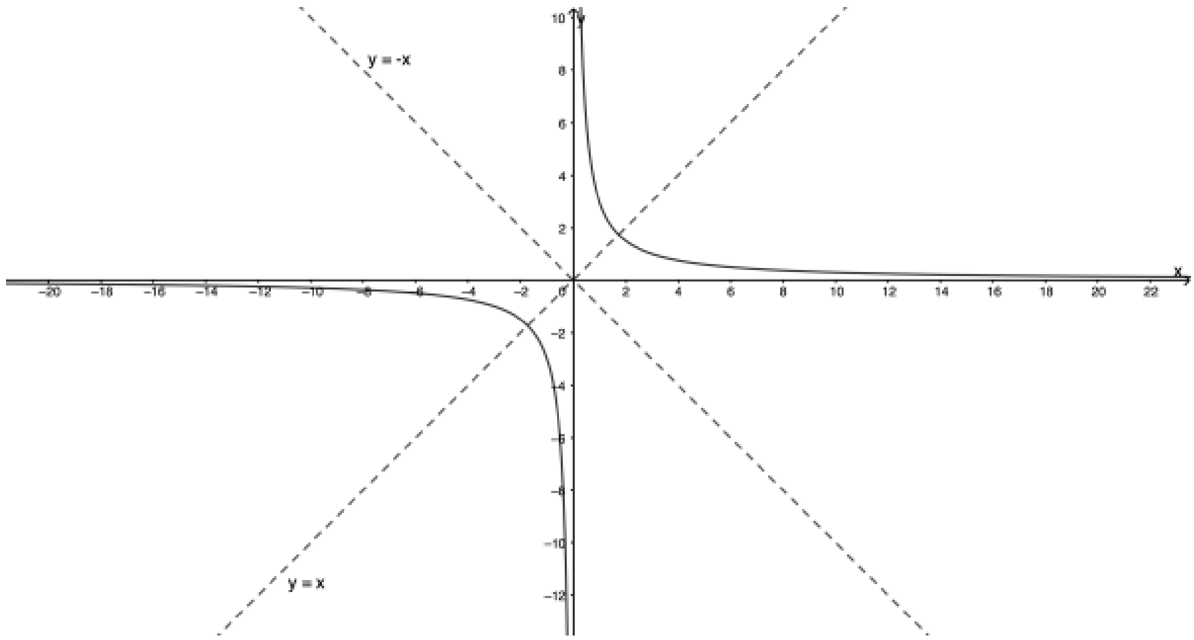


Figure 7: Axes of symmetry of a hyperbola $y = \dfrac{3}{x}$

Asymptotes and continuity

In Activity 3.2, we saw that for $y = \dfrac{3}{x}$, the value of x can never be zero because we are never allowed to divide by zero. The graph gets closer and closer to the y-axis (where $x = 0$) but never touches or crosses it. We call the y-axis an **asymptote** of the graph. An asymptote is a straight line that a curve gets closer and closer to without ever touching it.

But we also saw that we can rewrite the equation of the function as $x = \dfrac{3}{y}$. This shows us that $y \neq 0$. Therefore, the graph gets closer and closer to the x-axis (where $y = 0$) but never touches or crosses it. Therefore, the y-axis is also an asymptote of the graph.

We know that the hyperbola has two asymptotes. Look at the vertical asymptote in Figure 8, for example. The graph approaches the y-axis from the right. But to keep drawing the graph, we eventually have to pick up our pencil and start again on the left-hand side of the y-axis.

Because, we have to pick up our pencil to draw the whole graph, we say that the hyperbola is **discontinuous**. Look at the parabola and straight-line graphs in Figure 8. Do we ever need to pick up our pencil to draw these? Because we do not, we say that these functions are **continuous**.

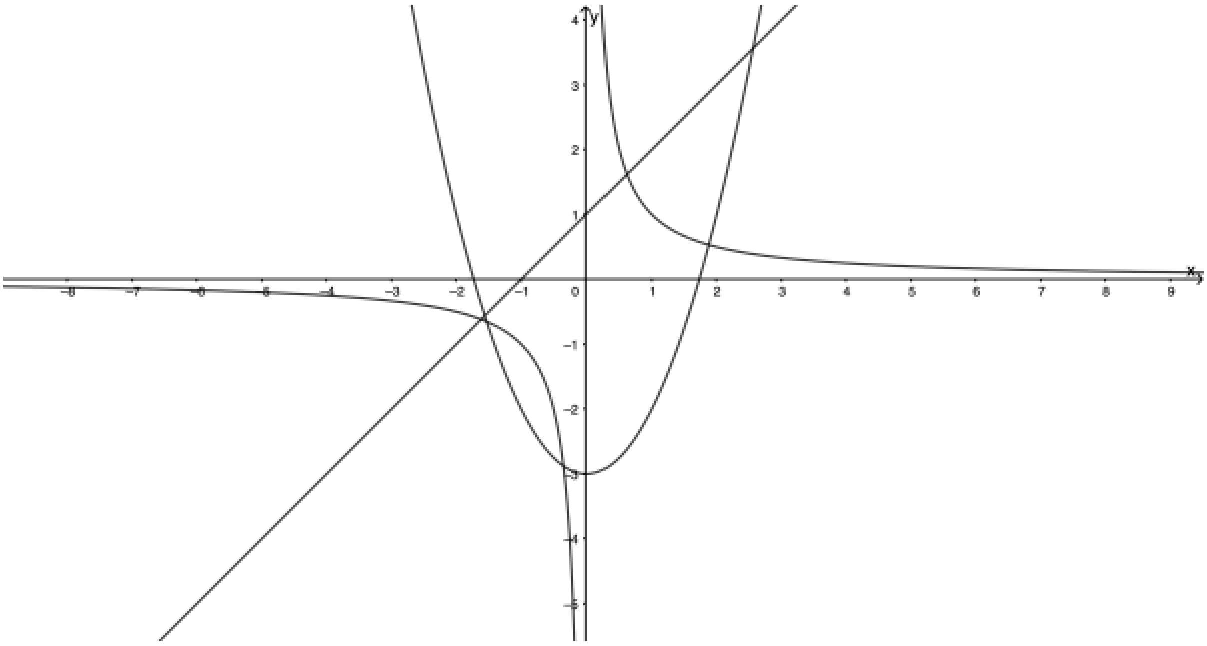


Figure 8: Continuous and discontinuous functions

Take note!

In general, hyperbolic functions of the form $f(x) = \frac{a}{x}$ have the following properties.

- The line $y = 0$ (the x-axis) is a **horizontal asymptote**.
- The line $x = 0$ (the y-axis) is a **vertical asymptote**.
- The function is **discontinuous** at the asymptotes.
- The domain of the function is $x : x \in \mathbb{R}, x \neq 0.$
- The range of the function is $f(x) : f(x) \in \mathbb{R}, y \neq 0.$
- The line $y = x$ is an axis of symmetry.
- The line $y = -x$ is an axis of symmetry.

But what happens if we have the function $g(x) = \frac{3}{x} + 1$? Now the function is of the form $y = \frac{a}{x} + q$. What does this graph look like?

The effect of q on the hyperbola of the form $y = \frac{a}{x} + q$?

Activity 3.3: Discover the effect of *q* on the hyperbola of the form *y= a/x +q*

Time required: 30 minutes

What you need:

- coloured pens or pencils
- a calculator
- blank paper or a notebook
- access to the internet

What to do:

Part A

1. Plot the following hyperbolic functions on the same set of axes using a table of values.
   a. $f(x) = \dfrac{3}{x}$
   b. $g(x) = \dfrac{3}{x} + 1$
   c. $h(x) = \dfrac{3}{x} - 2$
   You should sketch each graph in a different colour.
2. What are the vertical asymptotes of $g(x)$ and $h(x)$? Are they the same as for $f(x)$?
3. What are the horizontal asymptotes of $g(x)$ and $h(x)$? Are they the same as for $f(x)$?
4. What is the domain and range of $g(x)$ and $h(x)$?
5. What are the axes of symmetry of $g(x)$ and $h(x)$?
6. What is the effect of the value of q on the graph of $y = \dfrac{a}{x} + q$?

Part B

Now, if you have access to the internet, visit the hyperbolic function interactive simulation.

Here you will find a hyperbolic function of the form $y = \dfrac{a}{x} + q$ with a slider to change the values of q.

1. Change the value of q to confirm what you know about how the value of q affects the hyperbola. How does this relate to how q affects the straight line $y = ax + q$ and the parabola $y = ax^2 + q$?
2. How does changing q affect the vertical asymptote of the hyperbola?
3. How does changing q affect the horizontal asymptote of the hyperbola?
4. How does changing q affect the axes of symmetry of the hyperbola?

What did you find?

Part A

1. Here are plots of all three functions (Figure 9).

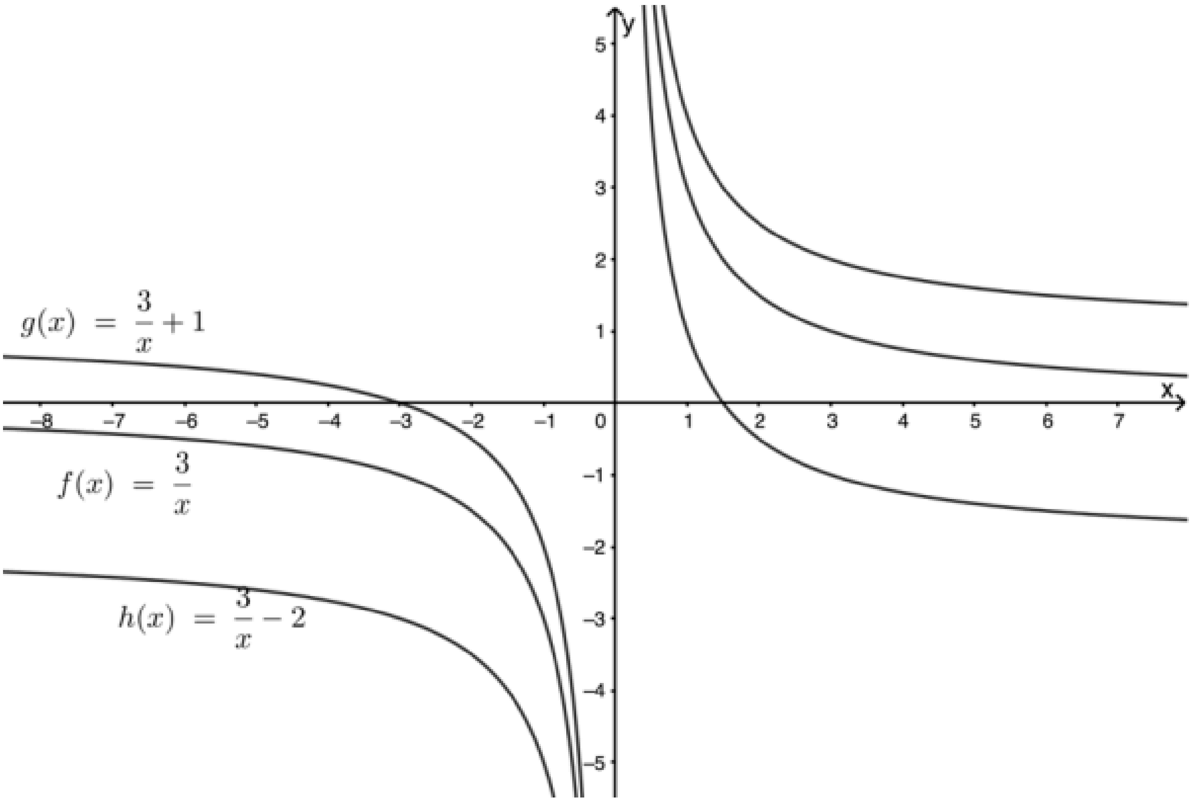


Figure 9: Plots of $f(x)$, $g(x)$ and $h(x)$

2. The vertical asymptotes of $g(x)$ and $h(x)$ are both the y-axis or the line $x = 0$. This is the same as $f(x)$.
3. The horizontal asymptote of $g(x) = \frac{3}{x} + 1$ is the line $y = 1$ (Figure 10).

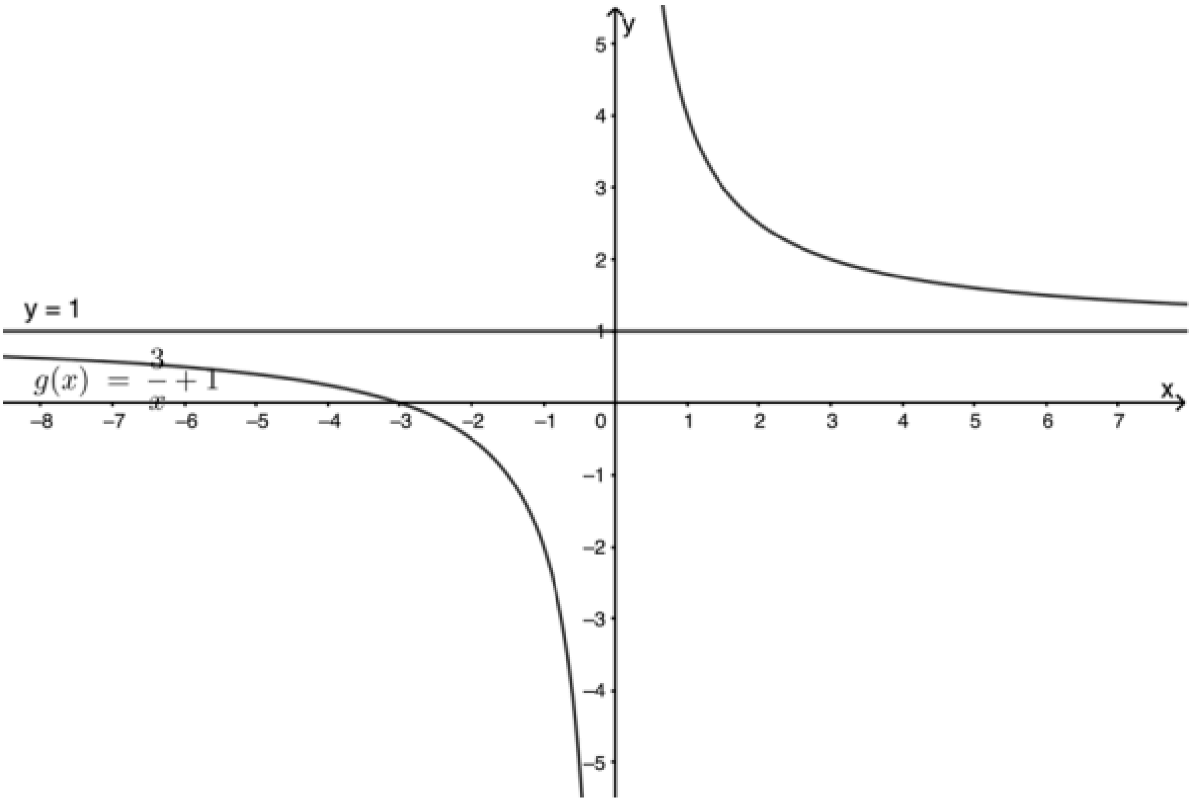


Figure 10: Horizontal asymptote of $g(x)$

The horizontal asymptote of $h(x) = \frac{3}{x} - 2$ is the line $y = -2$ (Figure 11).

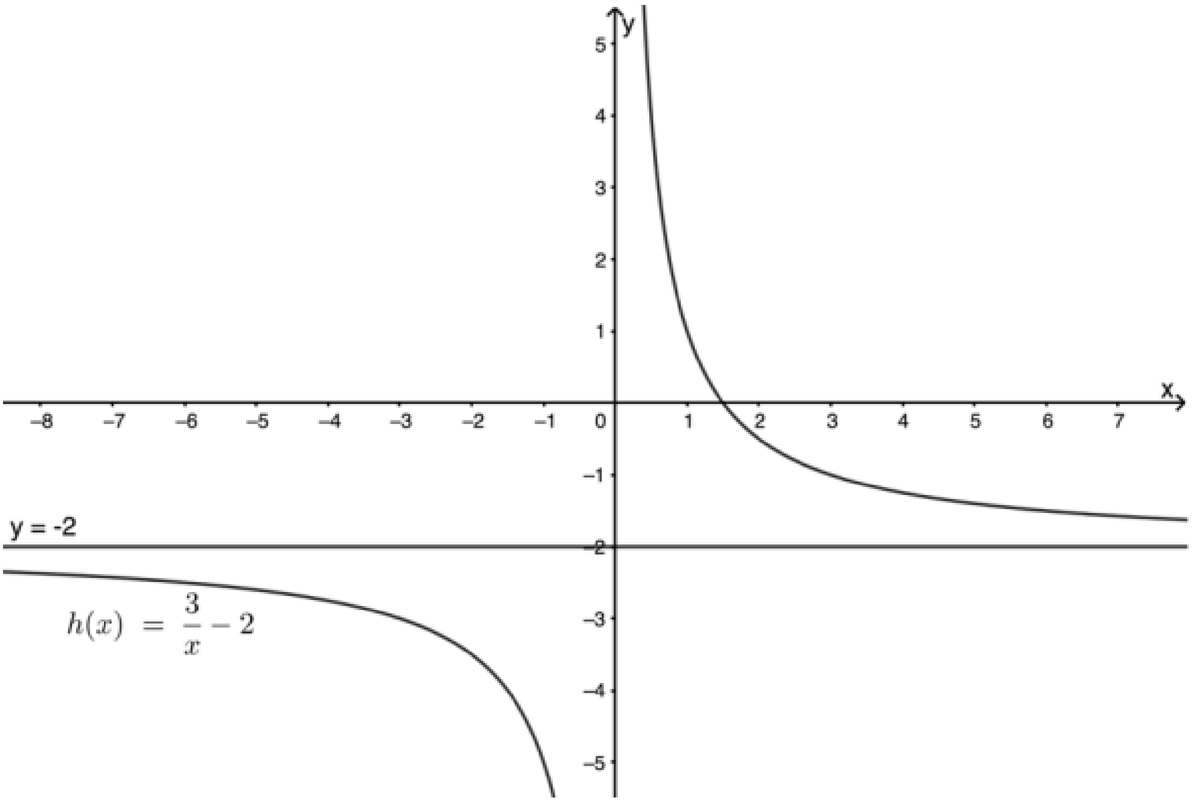


Figure 11: Horizontal asymptote of $h(x)$

4. The domain and range of $g(x)$ are:
   Domain: $x \mid x \in \mathbb{R}, x \neq 0$
   Range: $f(x) \mid f(x) \in \mathbb{R}, y \neq 1$

   The domain and range of $h(x)$ are:
   Domain: $x \mid x \in \mathbb{R}, x \neq 0$
   Range: $f(x) \mid f(x) \in \mathbb{R}, y \neq -2$
5. The axes of symmetry of $g(x) = \frac{3}{x} + 1$ are the lines $y = x + 1$ and $y = -x + 1$ (Figure 12).

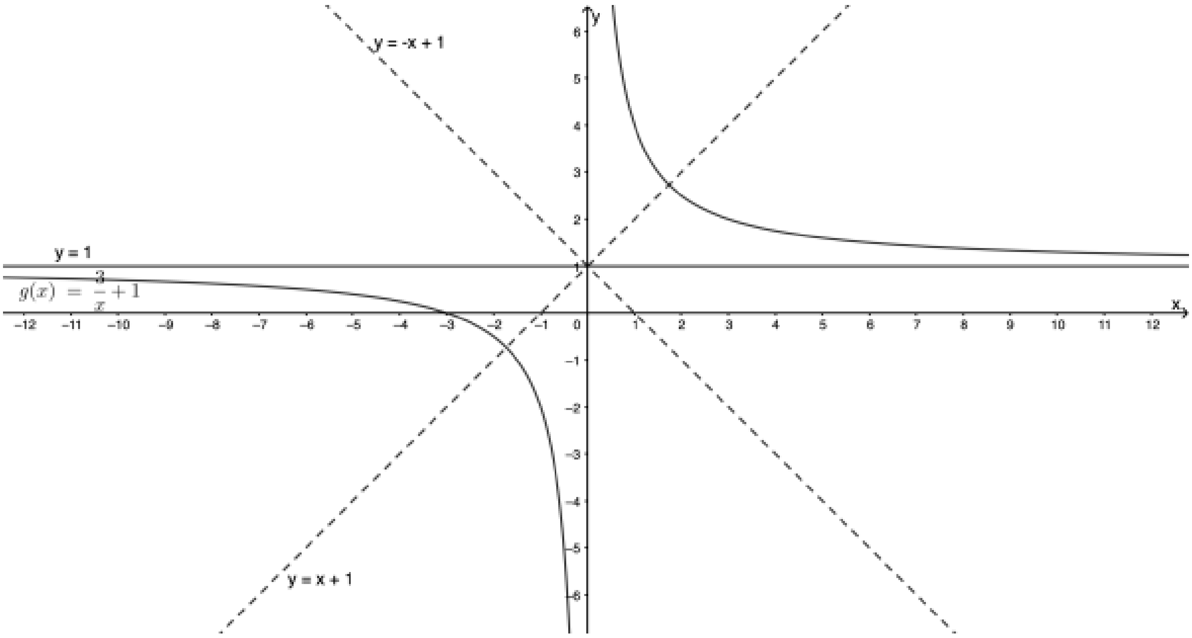


Figure 12: Axes of symmetry of $g(x)$

The axes of symmetry of $h(x) = \frac{3}{x} - 2$ are the lines $y = x - 2$ and $y = -x - 2$ (Figure 13).

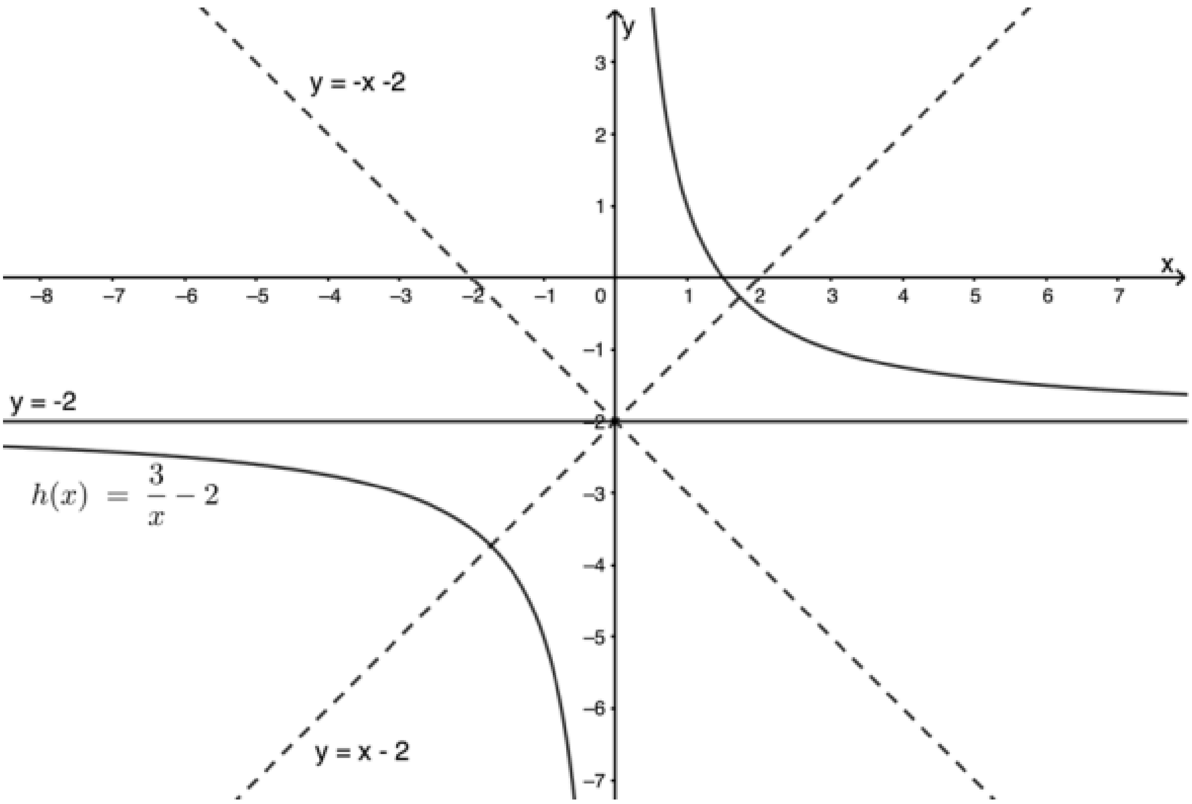


Figure 13: Axes of symmetry of $h(x)$

6. The effect of q on the graph of $y = \frac{a}{x} + q$ is to move the graph vertically up or down. This moves the graph itself up or down as well as moving the horizontal asymptote and lines of symmetry up or down as well.

Part B

1. As we increase the value of q, the hyperbola shifts up vertically. Decreasing the value of q shifts the graph down. This is exactly the same affect that changing the value of q has on the straight line $y = ax + q$ and the parabola $y = ax^2 + q$.
2. Changing the value of q has no effect on the vertical asymptote. The graph does not shift left or right at all.
3. As changing q shifts the hyperbola up and down, it also shifts the hyperbola's horizontal axis up or down by the same amount.
4. In the same way as changing q shifts the graph and the horizontal asymptote up or down, it also shifts the axes of symmetry up or down by the same amount.

We have discovered that the effect of q in $y = \frac{a}{x} + q$ is the same as in $y = ax + q$ and $y = ax^2 + q$. Changing q shifts the graph vertically up or down. The horizontal asymptote and axes of symmetry are also shifted up and down by the same amount.

Sketching hyperbolic functions of the form $y = \frac{a}{x} + q$

Have a look at this next example.

Example 3.1

Sketch the following hyperbolic functions by first finding their asymptotes and axes of symmetry and then by finding any two other points on the graph. These can include any intercepts with the axes if they exist.

1. $f(x) = \frac{1}{x} - 1$
2. $j(x) = \frac{3}{x} - \frac{3}{2}$

Solutions

1. $f(x) = \frac{1}{x} - 1$ is of the form $y = \frac{a}{x} + q$ but has been shifted one unit down. Therefore, the graph's horizontal asymptote is the line $y = -1$. The axes of symmetry are the lines $y = x - 1$ and $y = -x - 1$.

   We can find the x-intercept if we make $y = 0$.

$$0 = \frac{1}{x} - 1$$
$$\therefore \frac{1}{x} = 1$$
$$\therefore x = 1$$

We can find another point by substituting $x = 2$ into the function. Therefore $f(2) = \frac{1}{2} - 1 = -\frac{1}{2}$.

Here is the sketch of $f(x)$ (Figure 14).

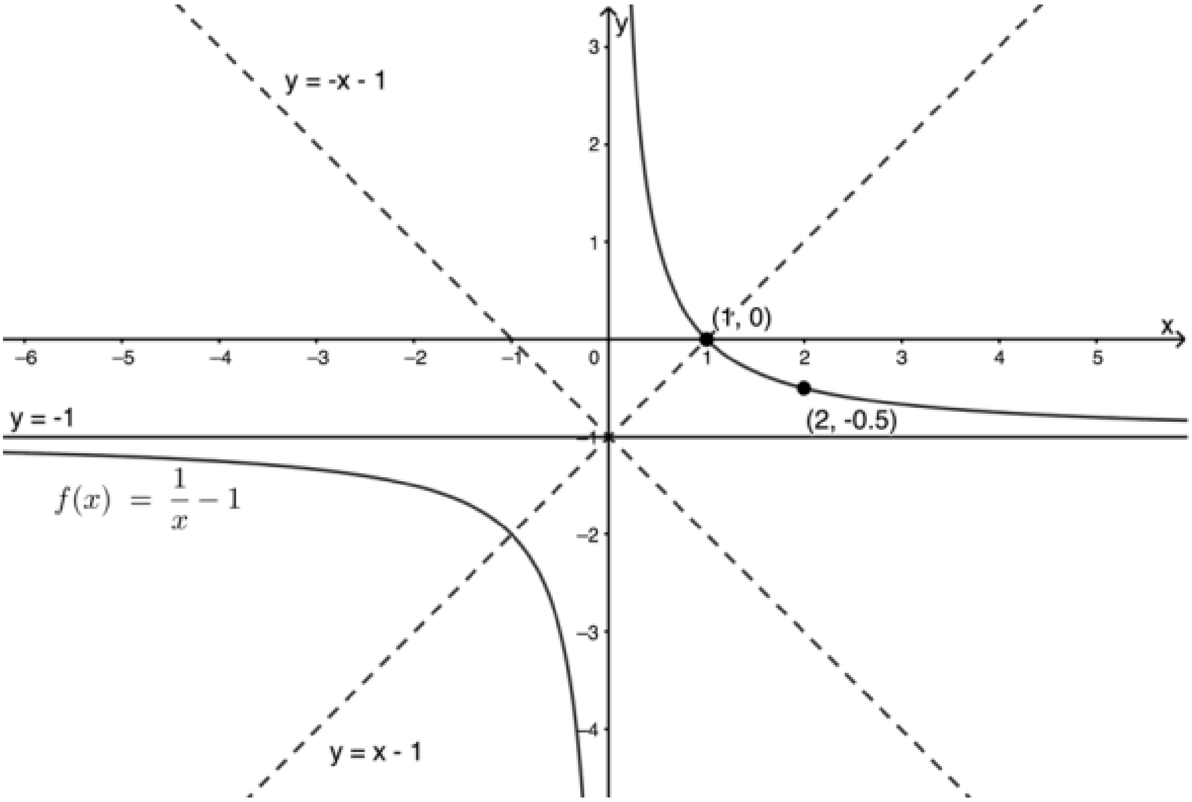


Figure 14: Sketch of $f(x)$

2. $j(x) = \frac{3}{x} - \frac{3}{2}$ is of the form $y = \frac{a}{x} + q$ but has been shifted down by one and a half units. Therefore, the graph's horizontal asymptote is the line $y = -\frac{3}{2}$. The axes of symmetry are the lines $y = x - \frac{3}{2}$ and $y = -x - \frac{3}{2}$.

   We can find the x-intercept if we make $y = 0$.
   $$0 = \frac{3}{x} - \frac{3}{2}$$
   $$\therefore \frac{3}{x} = \frac{3}{2}$$
   $$\therefore x = 2$$

   We can find another point by substituting $x = 1$ into the function. Therefore $j(1) = \frac{3}{1} - \frac{3}{2} = \frac{3}{2}$.

Here is the sketch of $j(x)$ (Figure 15).

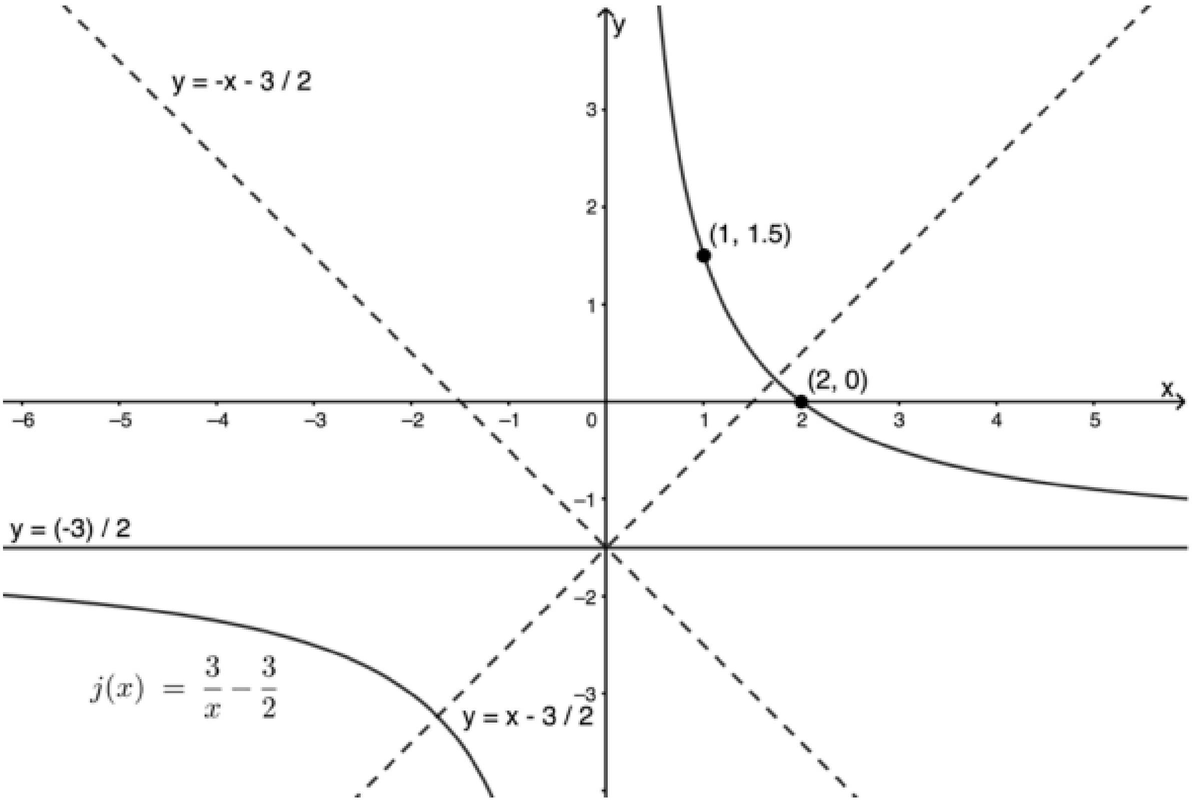


Figure 15: Sketch of $j(x)$

The effect of a on $y = \frac{a}{x} + q$

So far, we have discovered what effect q has on the hyperbolic function of the form $y = \frac{a}{x} + q$. But what is the effect of a? Work through this next example to find out.

Example 3.2

1. Sketch the following hyperbolic functions on the same set of axes by first finding their asymptotes and axes of symmetry and then by finding any two other points on the graph. These can include any intercepts with the axes if they exist.

   $f(x) = \frac{1}{x} + 1$

   $g(x) = \frac{3}{x} + 1$

$h(x) = -\frac{2}{x} + 1$

2. What affect does the value of a have on the shape and position of the hyperbolic function of the form $y = \frac{a}{x} + q$?

Solutions

1. All three functions have a value of $q = 1$. Therefore, all the functions will have the same horizontal asymptote and the same axes of symmetry. We just need to find two additional points on each graph to help us sketch them.

   First, we will find the x-intercepts.

$$
\begin{aligned}
f(x) &= 0: \\
\therefore \frac{1}{x} + 1 &= 0 \\
\therefore \frac{1}{x} &= -1 \\
\therefore x &= -1
\end{aligned}
$$

$$
\begin{aligned}
g(x) &= 0: \\
\therefore \frac{3}{x} + 1 &= 0 \\
\therefore \frac{3}{x} &= -1 \\
\therefore x &= -3
\end{aligned}
$$

$$
\begin{aligned}
h(x) &= 0: \\
\therefore -\frac{2}{x} + 1 &= 0 \\
\therefore \frac{2}{x} &= 1 \\
\therefore x &= 2
\end{aligned}
$$

   Now we can find any other point on each graph. We will choose convenient values of x to work with for each function.

$$
\begin{aligned}
f(-2) &= \frac{1}{(-2)} + 1 = \frac{1}{2} \\
g(-1) &= \frac{3}{(-1)} + 1 = -2 \\
h(1) &= -\frac{2}{1} + 1 = 3
\end{aligned}
$$

   With this information, we can plot each of the functions (Figure 16).

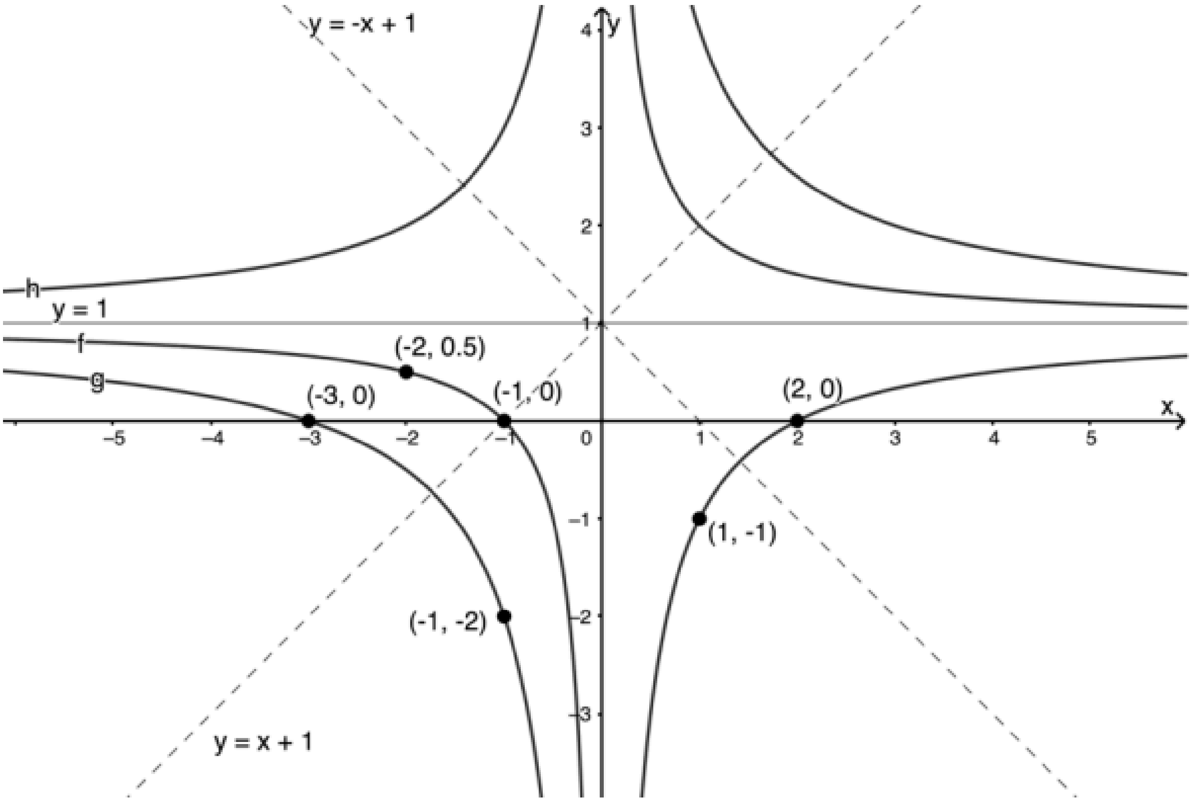


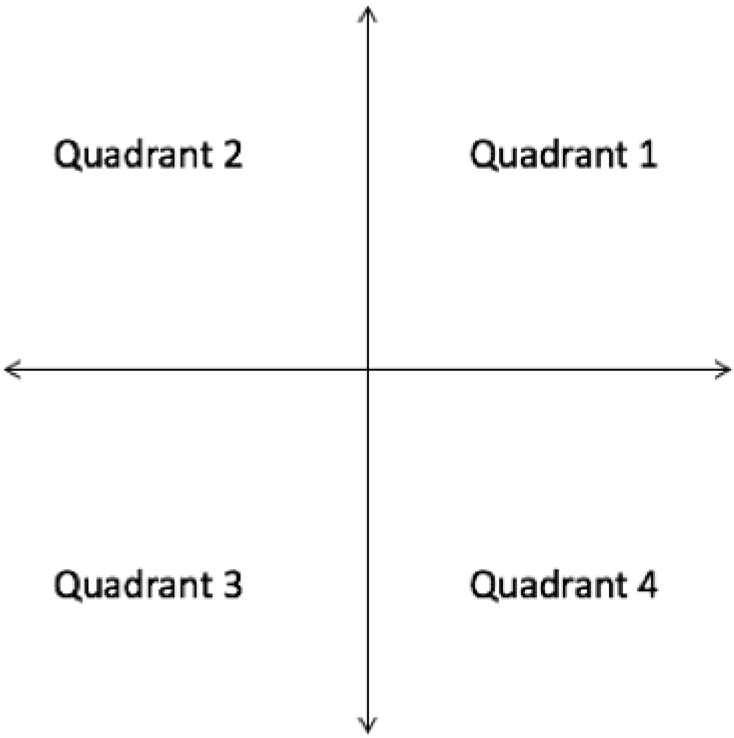
Figure 16: Sketches of $f(x)$, $g(x)$ and $h(x)$

2. From the graphs we plotted, we can see that the bigger the absolute value of a, the wider the hyperbola is. Look at the difference of the shape of $f(x) = \frac{1}{x} + 1$ and $g(x) = \frac{3}{x} + 1$ in Figure 16.

   Also, the sign of a determines where the graph is plotted. If $a > 0$, the graph is in the first and third quadrants of the Cartesian plane (see Figure 16). If $a < 0$, the graph is in the second and fourth quarters of the Cartesian plane (see Figure 16).

Note

We call each quarter of the Cartesian plane a **quadrant** and we number the quadrants from the top right in an **anti-clockwise direction**.



If you have access to the internet, visit this interactive simulation.

Here you will find a hyperbolic function of the form $y = \frac{a}{x} + q$ with sliders to change the values of a and q. Spend some time playing with it to make sure that you understand how changing the values of a and q affects the shape and location of the hyperbola of the form $y = \frac{a}{x} + q$.

Example 3.3

Sketch the function $q(x) = -\frac{3}{4x} + 2$ and label the intercepts, asymptotes and axes of symmetry.

Solution

First, we need to write the function in the form $y = \frac{a}{x} + q$. Therefore $q(x) = -\frac{\frac{3}{4}}{x} + 2$, $a = -\frac{3}{4}$ and $q = 2$.

Because $a < 0$, we know that the graph is plotted in quadrants 2 and 4. We also know that the horizontal asymptote is the line $y = 2$ and that the axes of symmetry are the lines $y = x + 2$ and $y = -x + 2$.

Because the x-axis is a vertical asymptote, we know there are no y-intercepts but we can find the x-intercepts by letting $y = 0$.

$$f(x) = -\frac{\frac{3}{4}}{x} + 2 = 0$$

$$\therefore -\frac{\frac{3}{4}}{x} = -2$$

$$\therefore -\frac{3}{4} = -2x \qquad \text{We multiplied both side of the equation by } x$$

$$\therefore x = \frac{3}{8}$$

Finally, we can sketch the graph as shown in Figure 17.

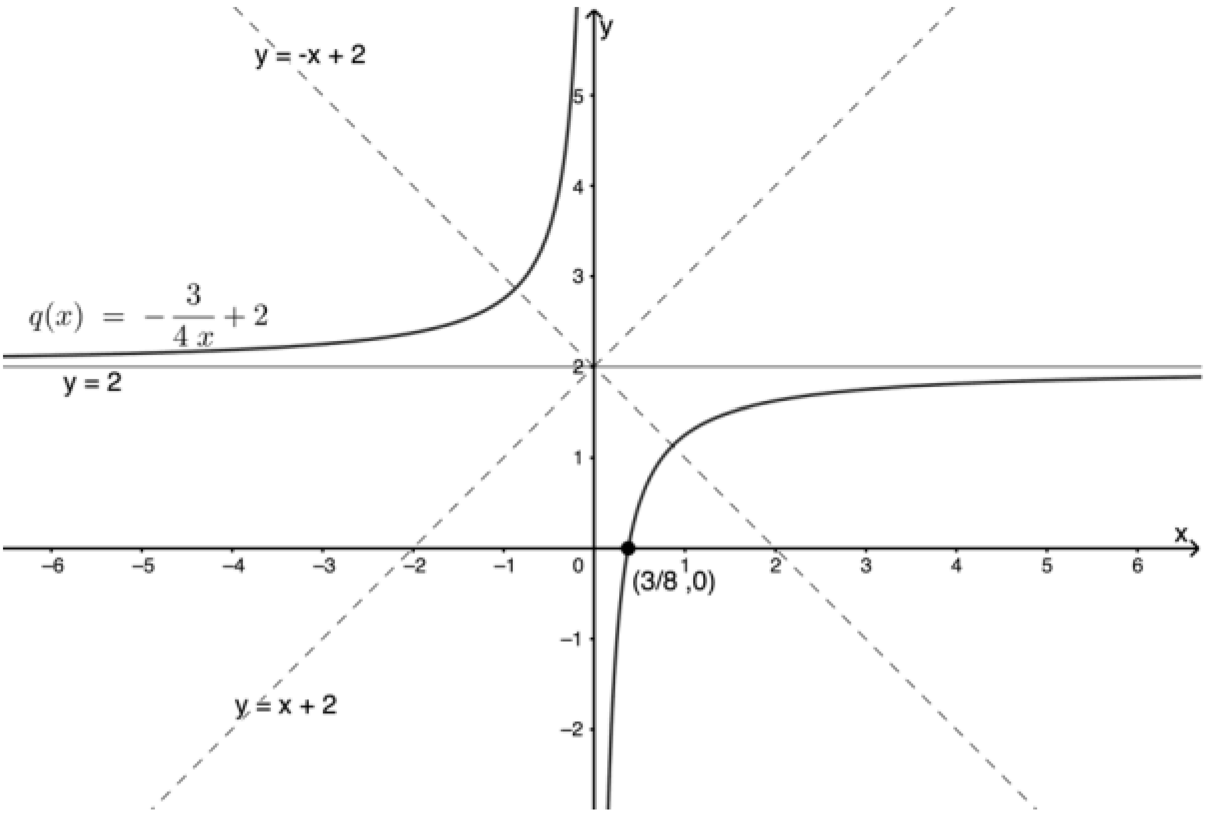


Figure 17: Sketch of $q(x) = -\frac{3}{4x} + 2$

Exercise 3.1

1. Sketch the function $g(x) = \frac{2}{x} + 4$ and label the intercepts, asymptotes and axes of symmetry.

2. Sketch the function $r(x) + 6 = -\frac{4}{x}$ and label the intercepts, asymptotes and axes of symmetry.

The full solutions are at the end of the unit.

Take note!

Figure 18 shows a summary of what we know about the hyperbolic function of the form $y = \frac{a}{x} + q$.

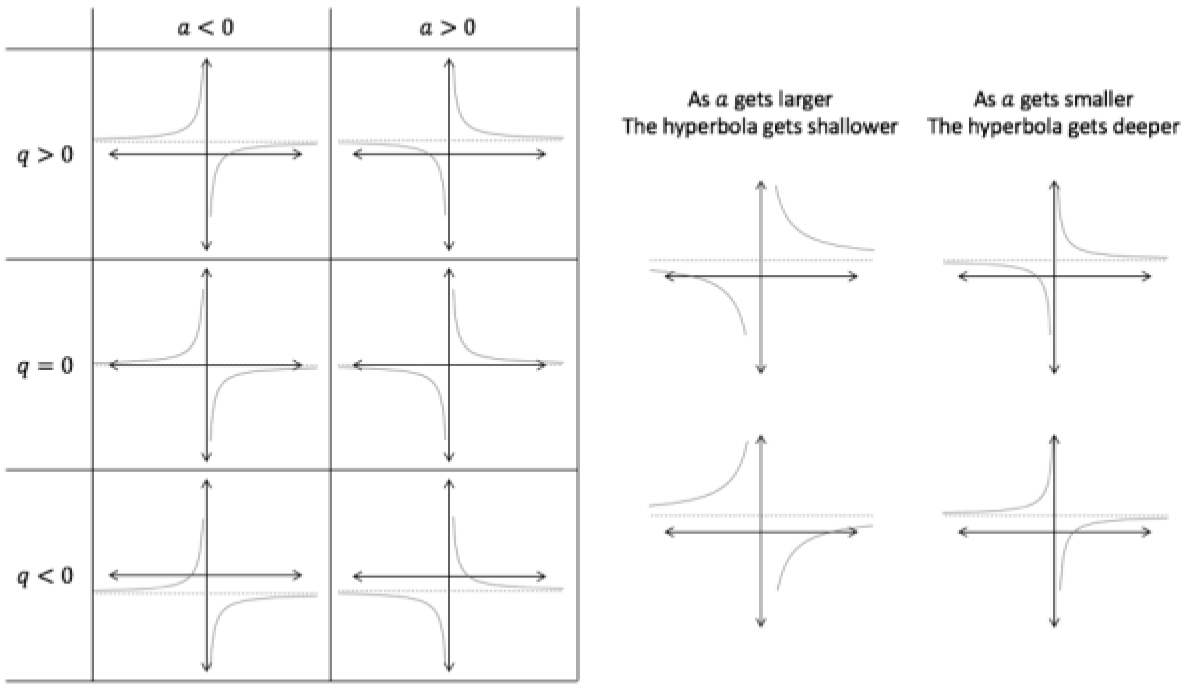


Figure 18: Summary of the hyperbolic function of the form $y = \frac{a}{x} + q$

In general, hyperbolic functions of the form $f(x) = \frac{a}{x} + q$ have the following properties:

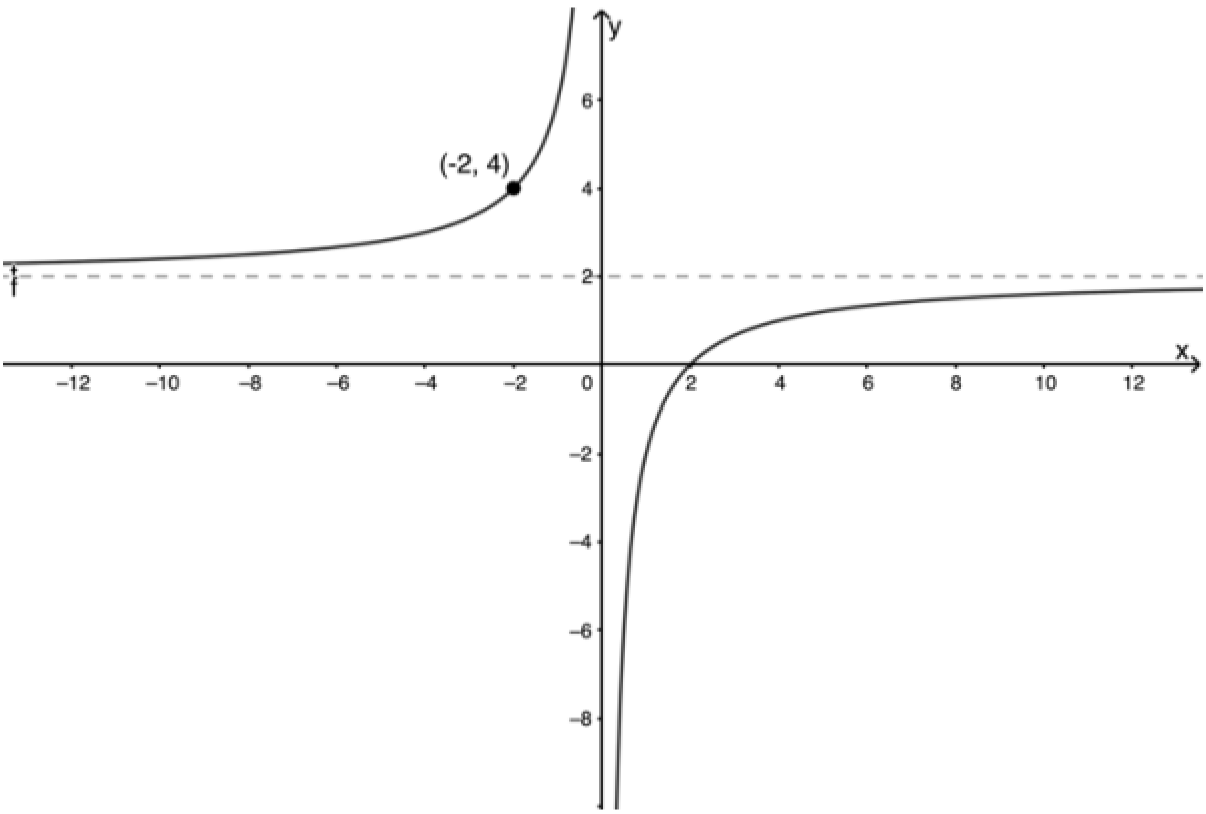
- The line $y = q$ is a horizontal asymptote.
- The line $x = 0$ (the y-axis) is a vertical asymptote.
- The function is discontinuous at the asymptotes.
- The domain of the function is $\{x \mid x \in \mathbb{R}, x \neq 0\}$.
- The range of the function is $\{f(x) \mid f(x) \in \mathbb{R}, y \neq q\}$.
- The line $y = x + q$ is an axis of symmetry.
- The line $y = -x + q$ is an axis of symmetry.

Find the equation of a hyperbolic function

So far, we know how to sketch the graph of hyperbolic functions of the form $y = \frac{a}{x} + q$. Have a look at the next example to see how to find the equation of the function from the graph.

Example 3.4

The graph below is of a hyperbola of the form $y = \frac{a}{x} + q$. A is the point $(-2, 4)$ and the horizontal asymptote is the line $y = 2$. Find the equation of the function.



Solution

The horizontal asymptote is $y = 2$. Therefore, we know that $q = 2$ and hence that $q(x) = \frac{a}{x} + 2$.

The graph is in the 2nd and 4th quadrants. Therefore, we know that $a < 0$

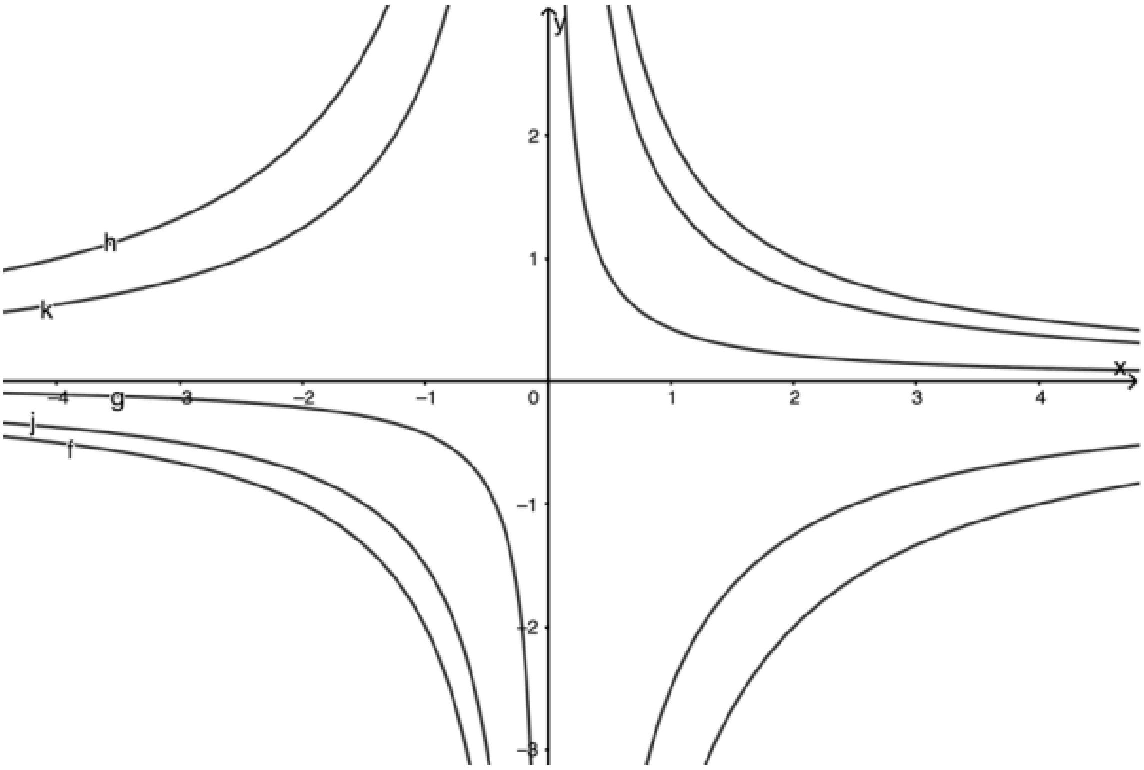
The point $(-2, 4)$ lies on the graph so we can substitute these values into the equation to solve for a.

$$f(-2) = 4$$
$$\therefore 4 = \frac{a}{-2} + 2$$
$$\therefore \frac{a}{-2} = 2$$
$$\therefore a = -4$$

$q(x) = -\frac{4}{x} + 2$

Exercise 3.2

1. A hyperbola of the form $\frac{a}{x} + q$ has a horizontal asymptote of $y = -\frac{2}{3}$ and an x-intercept of $(-\frac{5}{4}, 0)$. What is the equation of the function?
2. Given the following graph, match each function with the following equations:
   a. $y = \frac{3}{2x}$
   b. $y = \frac{3}{7x}$
   c. $y = \frac{2}{x}$
   d. $y = -\frac{4}{x}$
   e. $y = -\frac{5}{2x}$



3. A hyperbola has an equation of $2x(y - 2) = 5$. State:
   a. the equations of the asymptotes
   b. the equations of the axes of symmetry

c. the domain

d. the range.

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to sketch hyperbolic functions of the form $y = \frac{a}{x} + q$.
- How to find the equation of graphs of the form $y = \frac{a}{x} + q$.
- The effect of a and q on the shape and position of $y = \frac{a}{x} + q$.
- What an asymptote is and how to find the asymptotes of a hyperbola of the form $y = \frac{a}{x} + q$.
- What is meant by a continuous and a discontinuous function.

Unit 3: Assessment

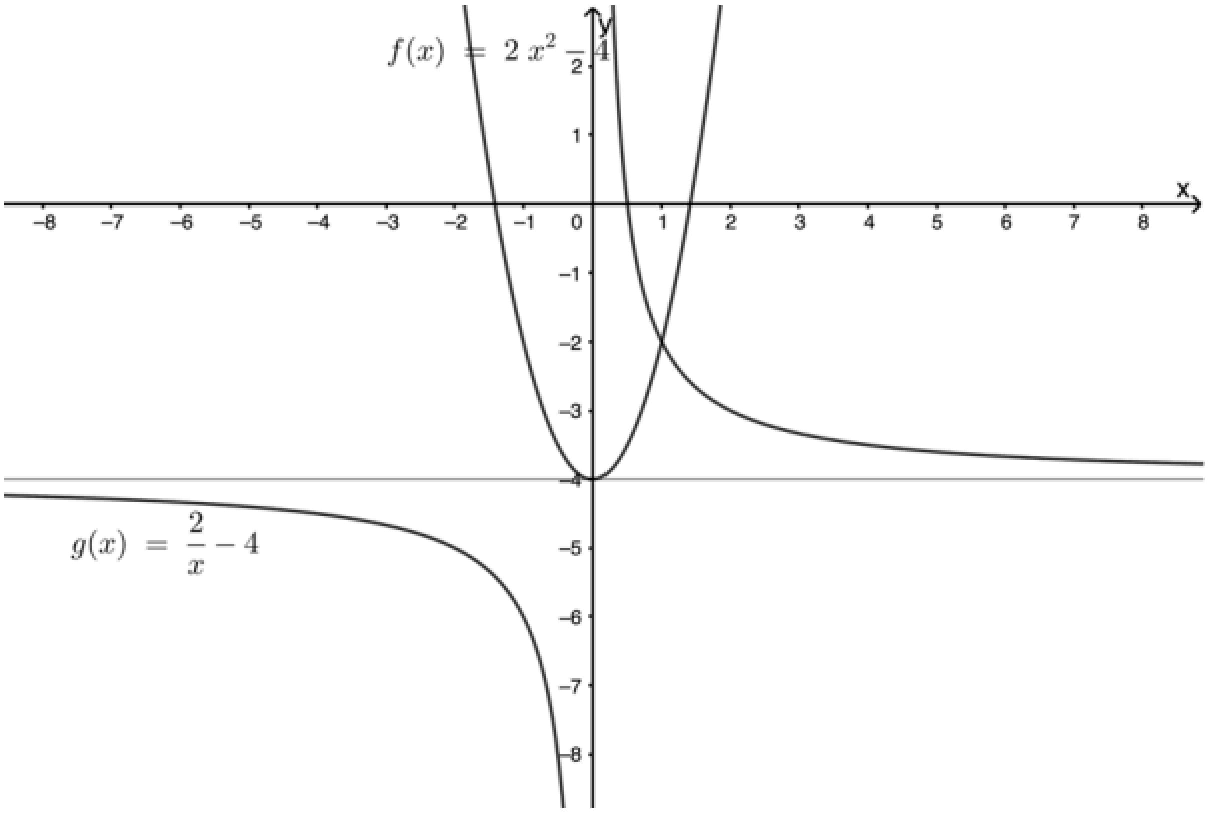
Suggested time to complete: 40 minutes

1. Given $y = -\frac{3}{x} + 2$, determine the intercepts with the axes.
2. Given the function $xy = -4$.
   a. Sketch the function.
   b. Give the equations of the asymptotes.
   c. Looking at the graph, what are the coordinates of the point on the graph symmetrical to $(1, 4)$ about the line $y = x$?
   d. Does the point $(\frac{1}{2}, -8)$ lie on the graph?
3. A hyperbola has an equation of $3x(y + 3) = 6$.
   a. State the equations of the asymptotes.
   b. State the equations of the axes of symmetry.
   c. State the domain.
   d. State the range.
   e. Sketch the function.
4. Without sketching the graphs, describe in words the differences between the following functions:
   a. $f(x) = \frac{2}{x}$ and $g(x) = \frac{2}{x} + 2$
   b. $s(x) = \frac{2}{x}$ and $t(x) = -\frac{2}{x}$
   c. $a(x) = \frac{2}{x}$ and $b(x) = -\frac{2}{x} + 2$

d. $q(x) = \frac{2}{x}$ and $r(x) = \frac{4}{x}$

5. What is the equation of the hyperbolic function of the form $y = \frac{a}{x} + q$ which has an axis of symmetry of $y = x - 2$ and an x-intercept of $(\frac{3}{2}, 0)$?

6. Look at the sketched functions $f(x) = 2x^2 - 4$ and $g(x) = -\frac{2}{x} - 4$.



a. At what point(s) do the graphs intersect (or cross each other)?

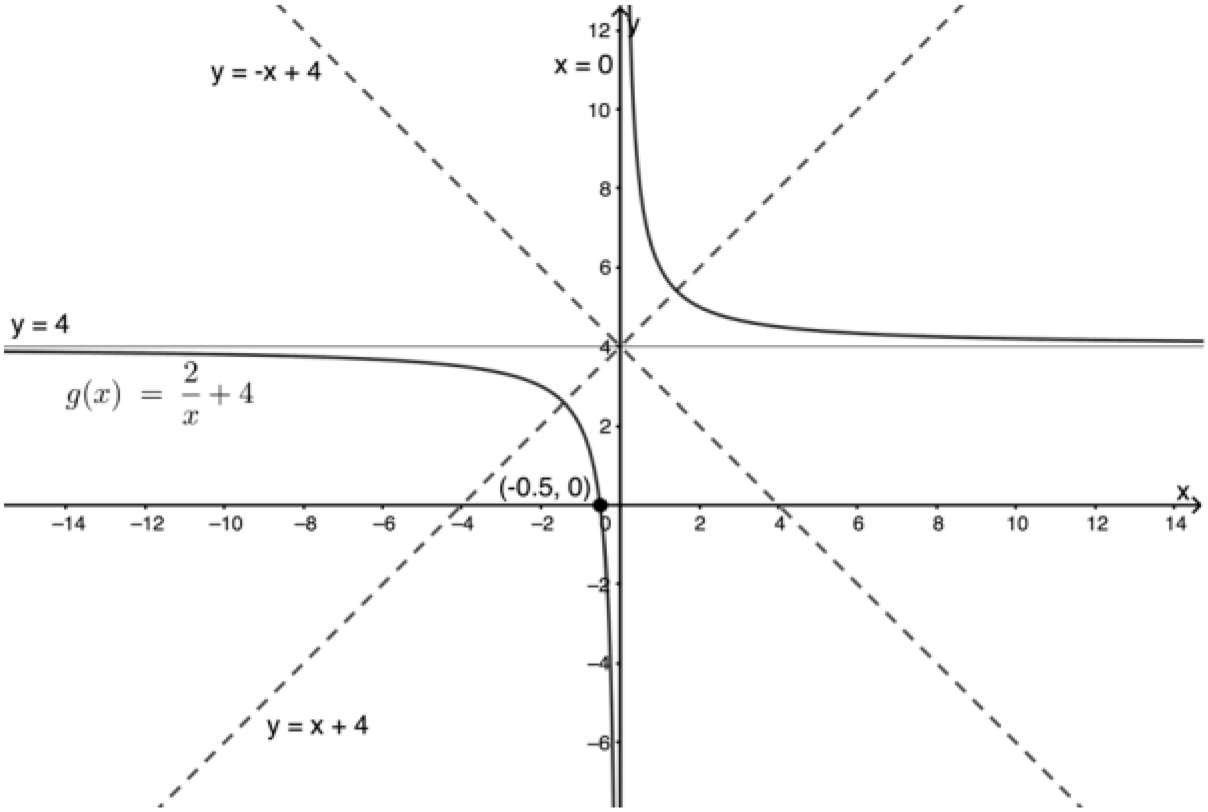
b. For which values of x is $g(x) > f(x)$. In other words, for which values of x is the graph of $g(x)$ above the graph of $f(x)$?

The full solutions are at the end of the unit.

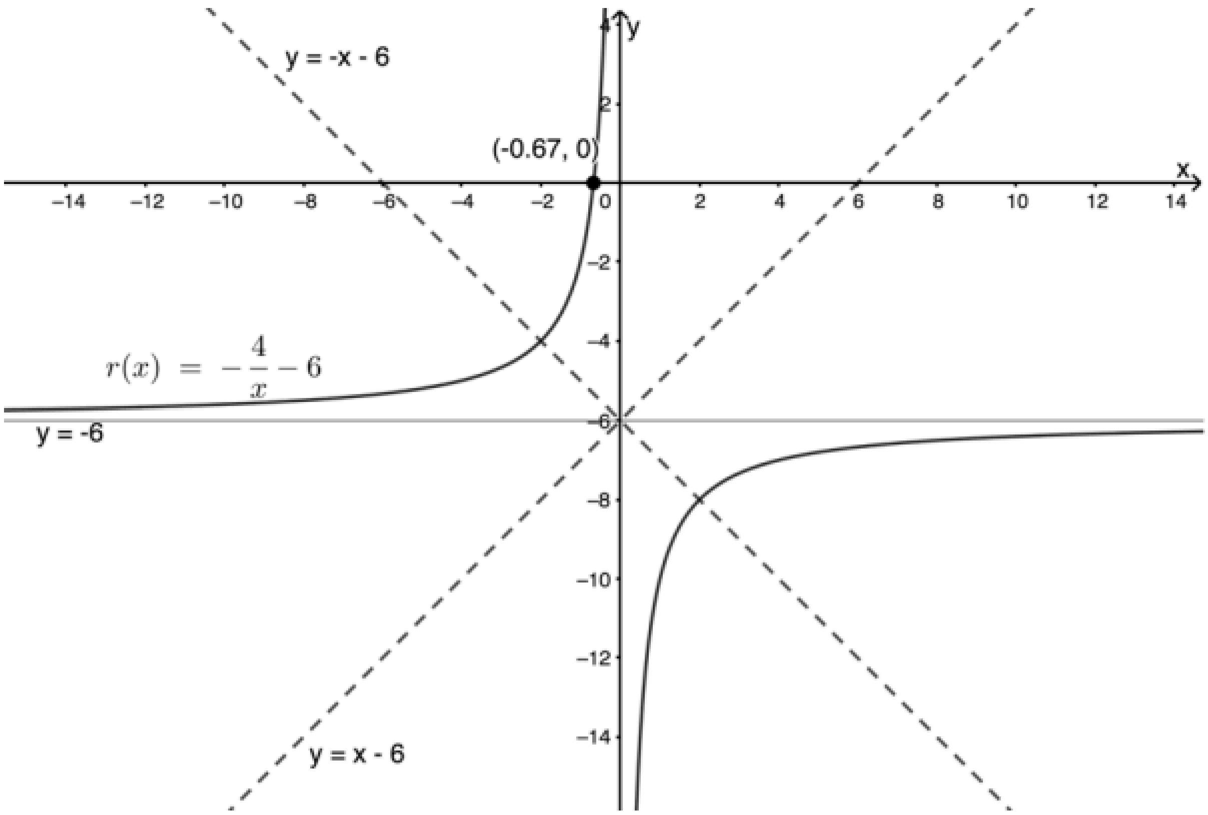
Unit 3: Solutions

Exercise 3.1

1.



2.



Back to Exercise 3.1

Exercise 3.2

1. The horizontal asymptote is $y = -\frac{2}{3}$. Therefore $q = -\frac{2}{3}$. Therefore the equation is $y = \frac{a}{x} - \frac{2}{3}$.

But the x-intercept is $(-\frac{5}{4}, 0)$. Substitute this point into the equation and solve for a.

$$\begin{aligned} y &= \frac{a}{x} - \frac{2}{3} \\ \therefore 0 &= \frac{a}{-\frac{5}{4}} - \frac{2}{3} \\ \therefore \frac{a}{\frac{5}{4}} &= -\frac{2}{3} \\ \therefore a &= \frac{2}{3} \times \frac{4}{5} = -\frac{8}{15} && \text{Multiply through by the inverse of the denominator} \end{aligned}$$

The equation is $y = -\frac{8}{15x} - \frac{2}{3}$.

2.

a. $j(x) = \frac{3}{2x}$

b. $g(x) = \frac{3}{7x}$

c. $f(x) = \frac{2}{x}$

d. $h(x) = -\frac{4}{x}$

e. $k(x) = -\frac{5}{2x}$

3.

$$\begin{aligned} 2x(y-2) &= 5 \\ \therefore y - 2 &= \frac{5}{2x} \\ \therefore y &= \frac{5}{2x} + 2 \end{aligned}$$

a. Horizontal asymptote:
$y = 2$
Vertical asymptote:
$x = 0$

b. Axes of symmetry:
$y = x + 2$ and $y = -x + 2$

c. Domain: $\{x \mid x \in \mathbb{R}, x \neq 0\}$

d. Range: $\{y \mid y \in \mathbb{R}, y \neq 2\}$

Back to Exercise 3.2

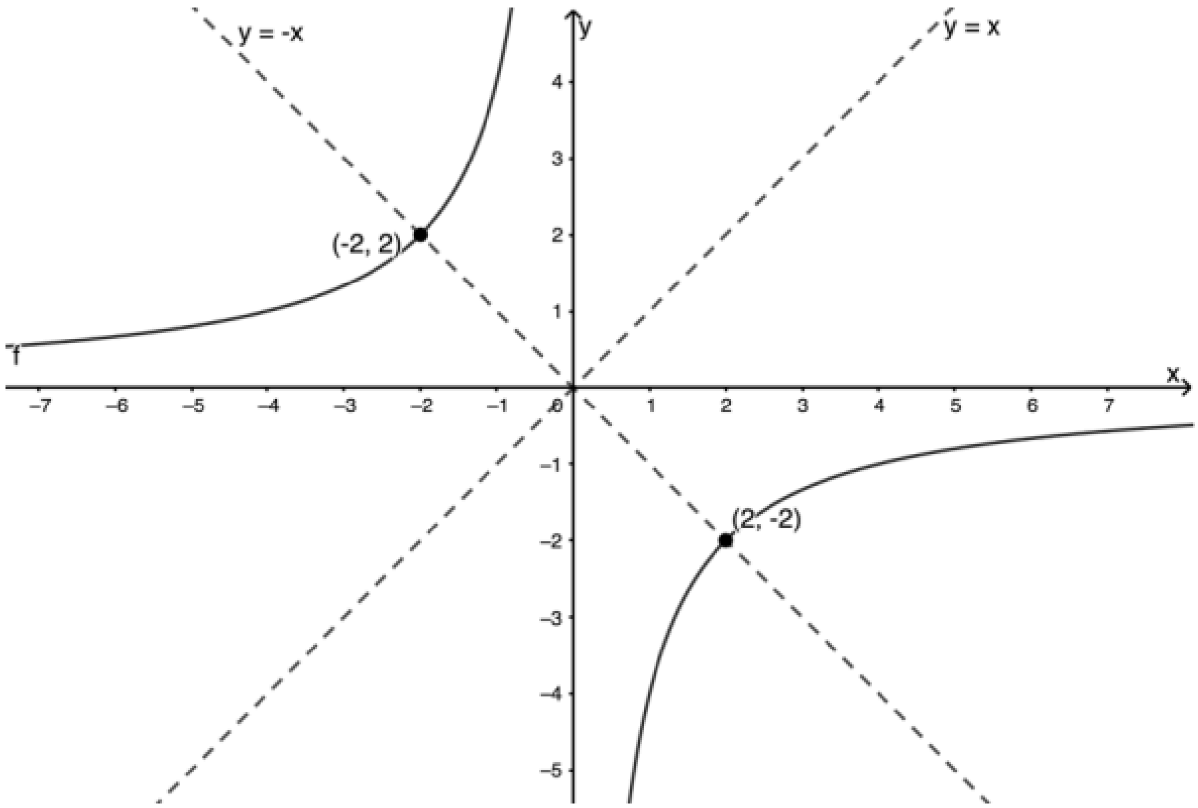
Unit 3: Assessment

1. $y = -\frac{3}{x} + 2$
x-intercept (let $y = 0$):

$$-\frac{3}{x}+2=0$$
$$\therefore \frac{3}{x}=2$$
$$\therefore 2x=3$$
$$\therefore x=\frac{3}{2}$$
$\therefore$ x-intercept is the point $(\frac{3}{2},0)$

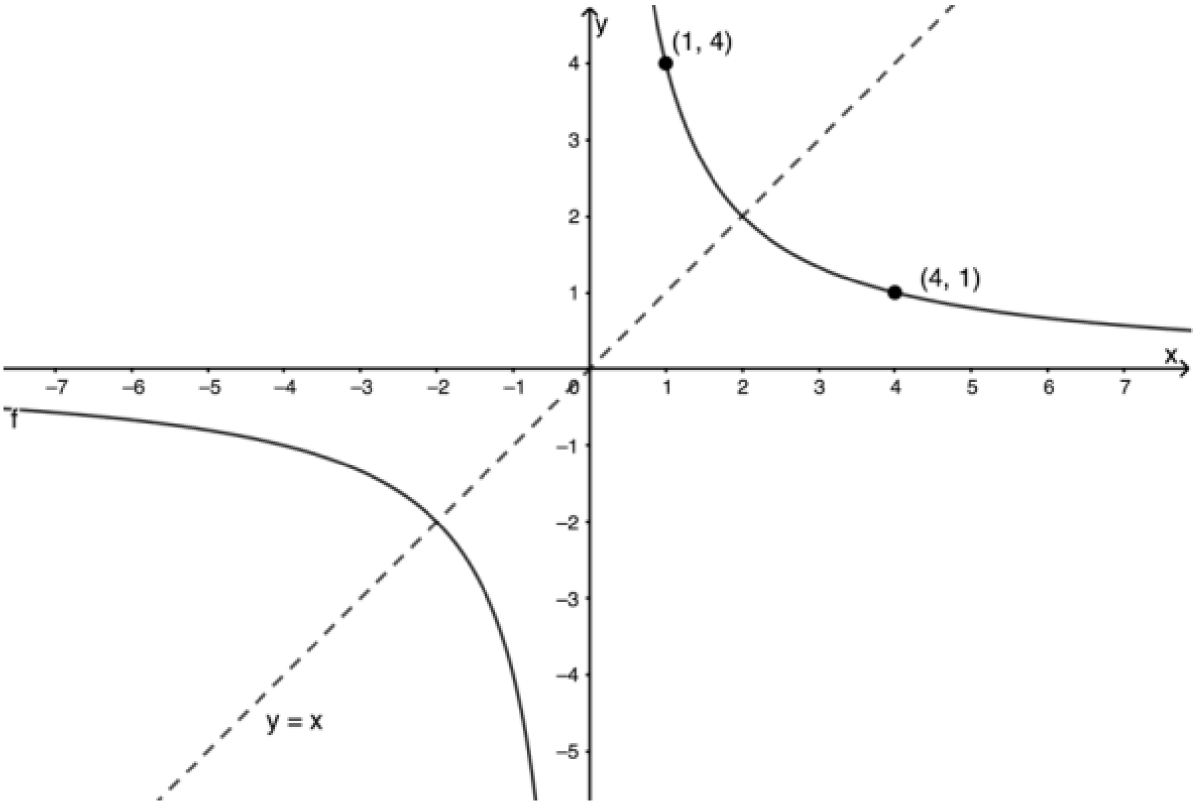
2.

a. $xy=-4$. Therefore, $y=-\frac{4}{x}$. $q=0$ so the horizontal asymptote is $y=0$ and the graph passes through the points $(-2,-2)$ and $(2,2)$.



b. The horizontal asymptote is $y=0$.
The vertical asymptote is $x=0$.

c. The point $(4,1)$ is symmetrical to the point $(1,4)$ about the line $y=x$.



d. We know the equation is $xy = 4$. However $\frac{1}{2} \times -8 \neq 4$. Therefore the point does not lie on the graph.

3.

a.

$3x(y+3) = 6$. Write the equation in standard form.

$3x(y-3) = 6$

$\therefore y + 3 = \frac{6}{3x}$

$\therefore y = \frac{2}{x} - 3$

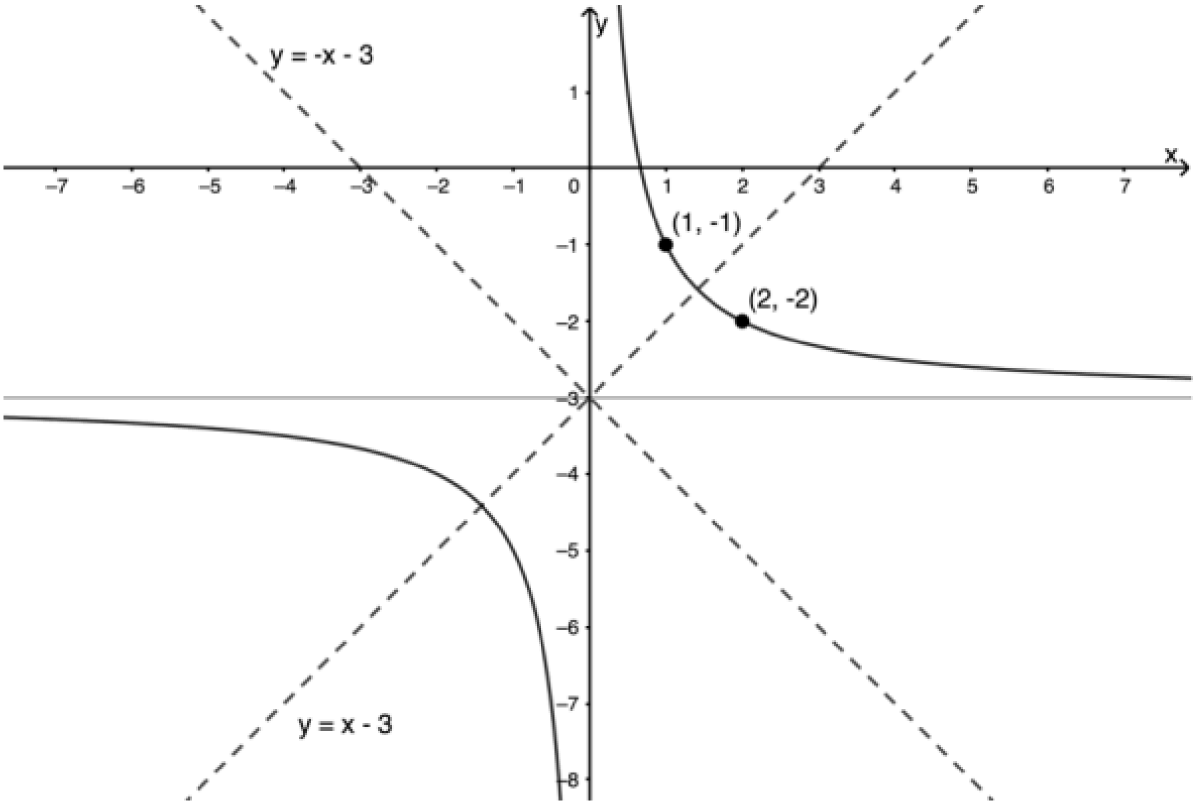
The vertical asymptote is $x = 0$.
The horizontal asymptote is $y = -3$.

b. The axes of symmetry are $y = x - 3$ and $y = -x - 3$.

c. Domain: $\{x \mid x \in \mathbb{R}, x \neq 0\}$

d. Range: $\{f(x) \mid f(x) \in \mathbb{R}, f(x) \neq -3$

e.



4.

a. $g(x)$ is the same shape as $f(x)$ but has been shifted 2 units up.

b. $t(x)$ is the same shape as $s(x)$ but exists in quadrants 2 and 4 rather than quadrants 1 and 3.

c. $b(x)$ is the same shape as $a(x)$ but has been shifted 2 units up and exists in quadrants 2 and 4 rather than quadrants 1 and 3.

d. $r(x)$ is a narrower graph than $q(x)$

5. The axis of symmetry is the line $y = x - 2$. Therefore, $q = -2$. So, the equation is $y = \frac{a}{x} - 2$.

But $(\frac{3}{2}, 0)$ is the x-intercept. Substitute this point into the equation and solve for a.

$$\begin{aligned} y &= \frac{a}{x} - 2 \\ \therefore 0 &= \frac{a}{\frac{3}{2}} - 2 \\ \therefore 2 &= \frac{a}{\frac{3}{2}} \\ \therefore \frac{4}{3} &= a \qquad \text{Multiply through by the inverse of the denominator} \end{aligned}$$

The equation is $y = \frac{4}{3x} - 2$.

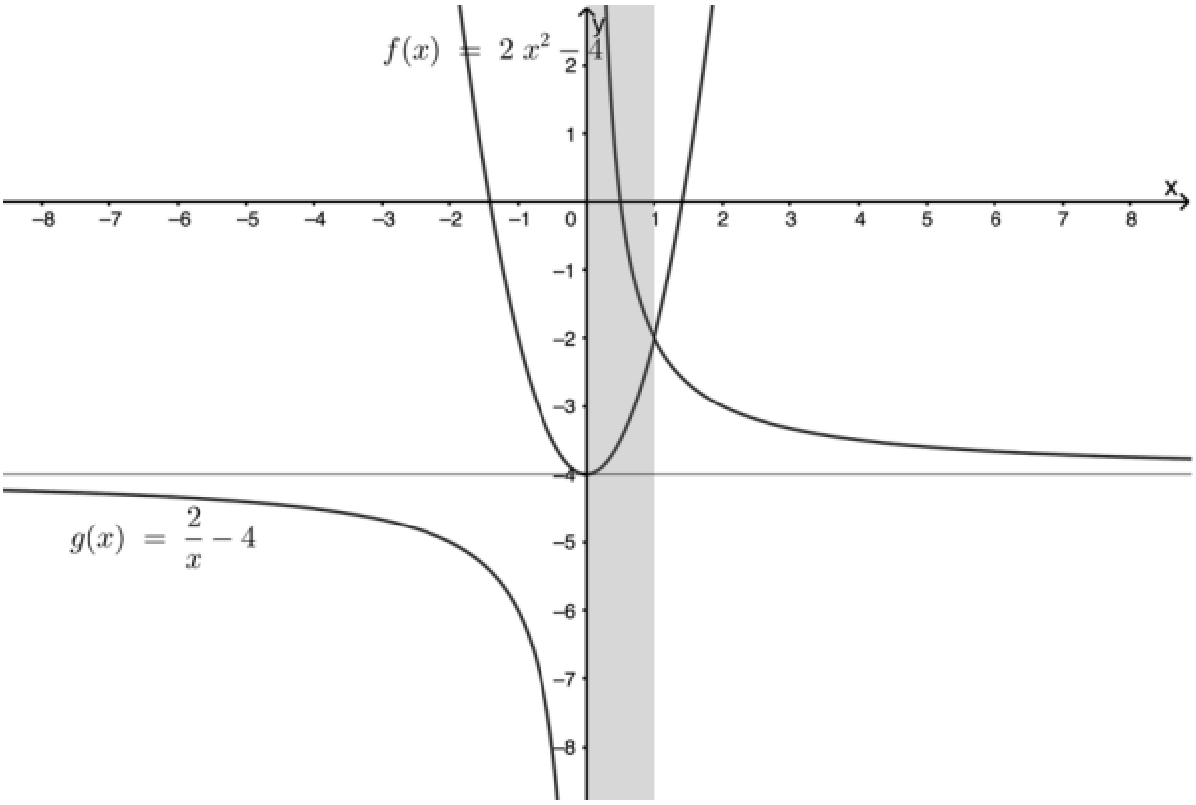
6.

a. $f(x) = 2x^2 - 4$ and $g(x) = \frac{2}{x} - 4$. The graphs intersect when $f(x) = g(x)$.

$$f(x) = g(x)$$
$$\therefore 2x^2 - 4 = \frac{2}{x} - 4$$
$$\therefore 2x^2 = \frac{2}{x}$$
$$\therefore 2x^3 = 2$$
$$\therefore x^3 = 1 \quad \text{take the cube root of both sides of the equation}$$
$$\therefore x = 1 \quad \text{The only number which multiplied by itself three times gives 1 is 1}$$

Now we can substitute $x = 1$ into either function. $g(1) = \frac{2}{1} - 4 = -2$. The graphs intersect at the point $(1, -2)$.

b. By looking at the graph, we can see that $g(x) > f(x)$ when $0 < x < 1$.



Back to Unit 3: Assessment

Media Attributions

Unit 4: Exponential functions

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Sketch and find the equation of the graph $y = a.b^x + q, b > 0$.
- Investigate and generalise the impact of a and q on $y = a.b^x + q, b > 0$.

What you should know

Before you start this unit, make sure you can:

- Determine the domain and range of a function by looking at its graph. Go over Units 1, 2 and 3 in this subject outcome if you need help with this.
- Manipulate and simplify algebraic expressions. Go over Subject outcome 2.2, Unit 1: Simplifying algebraic expressions if you need more help with the basics.
- Solve exponential equations. Go over Subject outcome 2.3, Unit 2: Solve exponential and literal equations if you need more help with the basics.
- Plot points on the Cartesian plane. If you do not know how to plot points onto the Cartesian plane, then you should do Subject outcome 3.3, Unit 1: Plotting points on the Cartesian plane before continuing with this unit.
- Solve linear inequalities. Go over Subject outcome 2.3, Unit 3: Solve algebraic inequalities if you need more help with this.
- Solve simultaneous equations. Go over Subject outcome 2.3, Unit 4: Solve simultaneous equations if you need more help with this.
- Work with the exponential function. This is briefly introduced in Subject outcome 1.2, Unit 2: Introduction to exponents. You should complete this unit before continuing.

Introduction

You have probably heard the term "exponential growth" before. However, the word 'exponential' is often used incorrectly to refer to something that happens very quickly. In Mathematics, 'exponential' has a very specific meaning and exponential functions are specific kinds of functions.

The good news is that these functions behave quite similarly to the other functions we have explored so far. Therefore, much of what we know about linear, quadratic and hyperbolic functions can be applied to exponential functions as well.

There are many processes in nature and finance where the relationship between the inputs and outputs is exponential. One example is how bacteria reproduce. Bacteria reproduce by simply splitting into two new bacteria. No sexual reproduction takes place.

So, if one bacterium splits into two and then each of these splits into two and then each of these splits into two, etc., then pretty soon we have a whole lot of bacteria! How many bacteria? Let's find out.

The exponential function

Activity 4.1: Investigate the exponential function

Time required: 30 minutes

What you need:

- a pen or pencil
- a calculator
- blank paper or a notebook

What to do:

For this activity, we will assume that bacteria multiply once every day. When they multiply, each one splits into two new bacteria. If we start with one bacterium, the progression for the first three days will look like Figure 1.

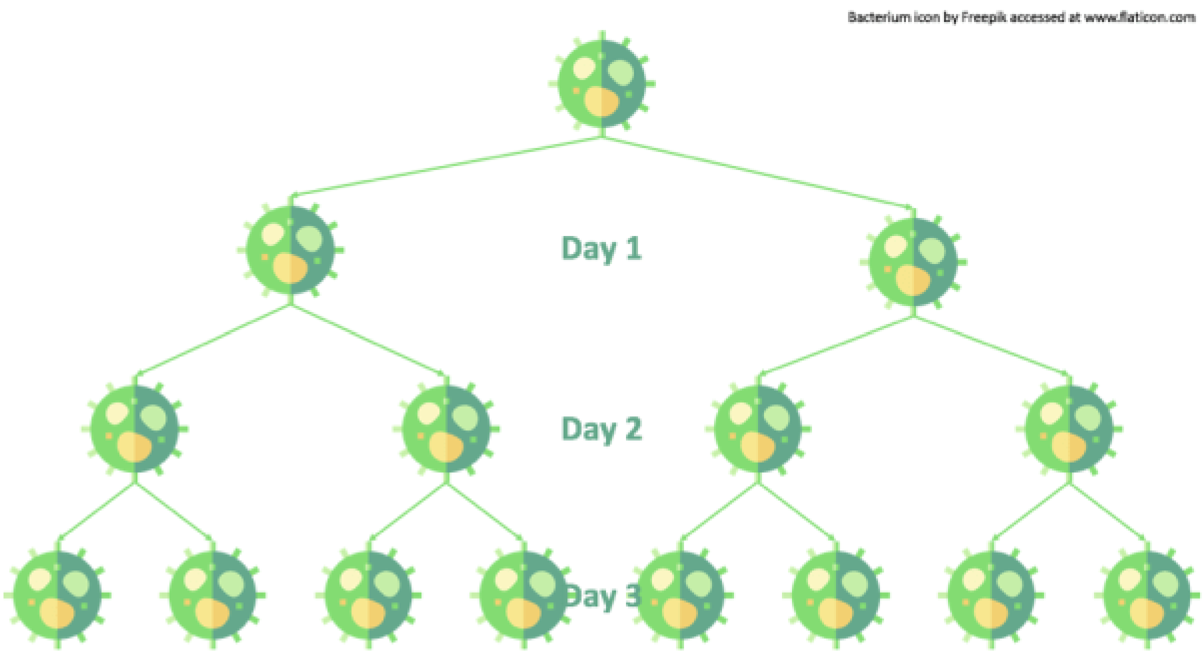


Figure 1: Growth in the number of bacteria

1. Use and extend Figure 1 to create a table of values of the number of bacteria after each day for eight days.
2. Plot these points and join them with a smooth curve. To help you plot the graph use a scale of $1, 2, 3$... on the x-axis and $10, 20, 30$... on the y-axis.
3. Write an expression that describes the relationship between the inputs (the number of days) and the outputs (the number of bacteria). Hint: the outputs increase in powers of 2.
4. How many bacteria are there after two weeks?
5. Where will the graph cut the y-axis? Add this point to your plot.

6. What happens when $x < 0$? Try a few values of x to find out and add these to your sketch.
7. Does the graph have a horizontal asymptote? If so, what is this asymptote?
8. Does the graph have a vertical asymptote? If so, what is this asymptote?
9. What is the domain and range of this graph?
10. Is the relationship between x and y a function? Why?

What did you find?

1. Here is the completed table of values.

Days (x)	1	2	3	4	5	6	7	8
Number of bacteria (y)	2	4	8	16	32	64	128	256

2. If we plot these points and join them with a smooth curve, we get the graph in Figure 2.

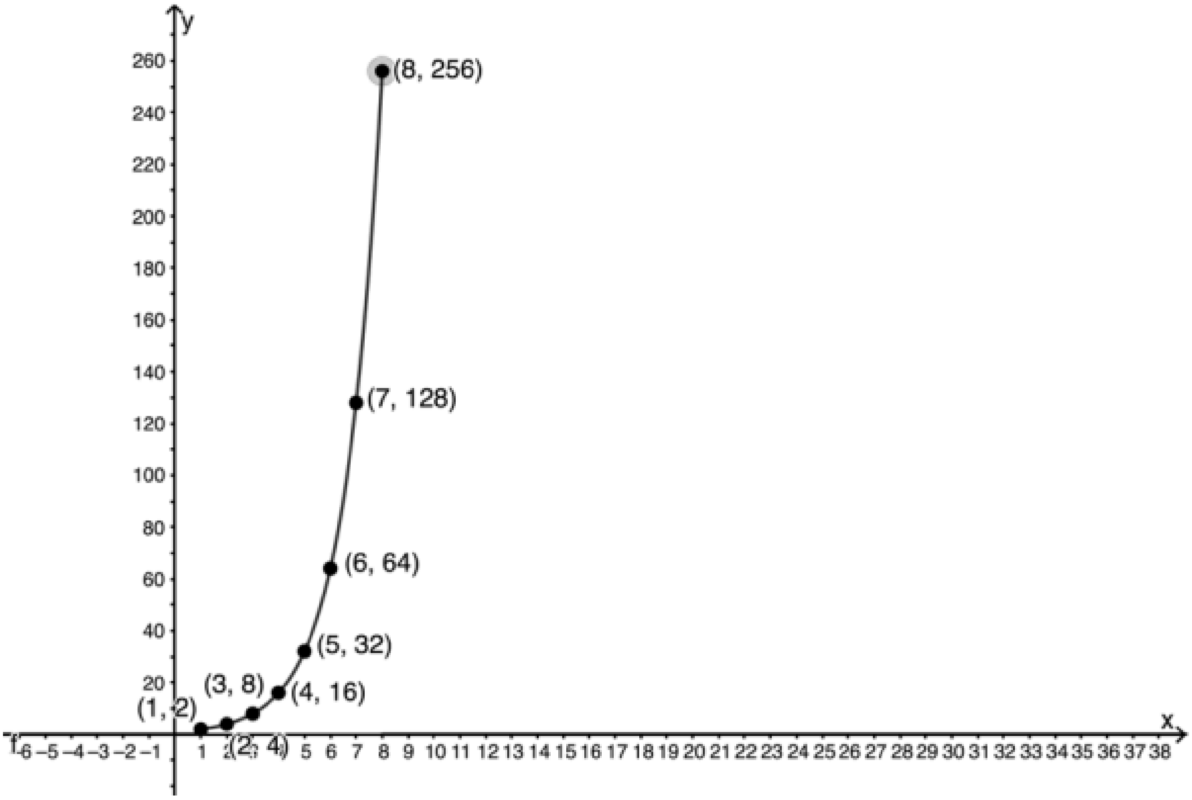


Figure 2: Plot of the number of bacteria after 8 days

3. We can see that the y-values increase by powers of 2. Here is the same table of values but with the y-values expressed as powers of 2.

Days (x)	1	2	3	4	5	6	7	8
Number of bacteria (y)	2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8

We can see that the exponent in each case matches the x-value. Therefore, we can say that the equation of the graph is $y = 2^x$.

4. 2 weeks is 14 days. If we use our equation, we get $y = 2^{14} = 16\ 384$. That is a lot of bacteria!
5. The y-intercept is when $x = 0$. Therefore, $y = 2^0 = 1$. So the y-intercept is the point $(0, 1)$.
6. You can try any values but let's use -1, -2 and -3.
$y = 2^{-1} = \frac{1}{2}$
$y = 2^{-2} = \frac{1}{4}$
$y = 2^{-3} = \frac{1}{8}$
If we add these to our sketch, we get the graph in Figure 3. We have zoomed into this part of the graph to make things clearer.

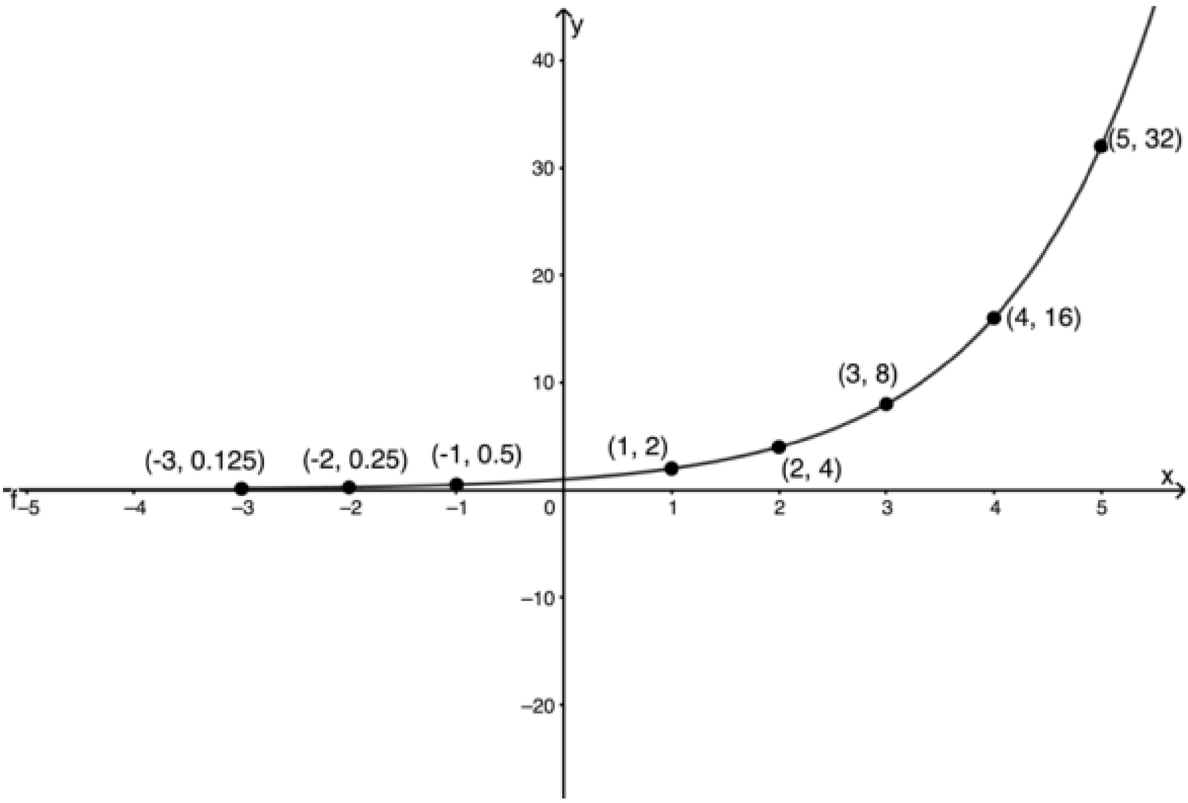


Figure 3: Plot of $y = 2^x$ for $-1 \leq x \leq 5$

7. As the value of x decreases, the value of y will get closer and closer to zero but will never equal zero and will never cross the x-axis to become negative. Therefore the x-axis (the line $y = 0$) is a horizontal asymptote.
8. As x increases, y will get bigger and bigger but there are no values that x gets close to but never reaches. Therefore there is no vertical asymptote.
9. x can be any real number. Therefore the domain is $x \in \mathbb{R}$. We have already seen that the line $y = 0$ is a horizontal asymptote and that y is never negative. Therefore the range is $\{y \mid y \in \mathbb{R}, y > 0\}$.
10. Using the vertical line test (see Figure 4), we can see that for every input value we only ever get one output value. Therefore $y = 2^x$ is a function.

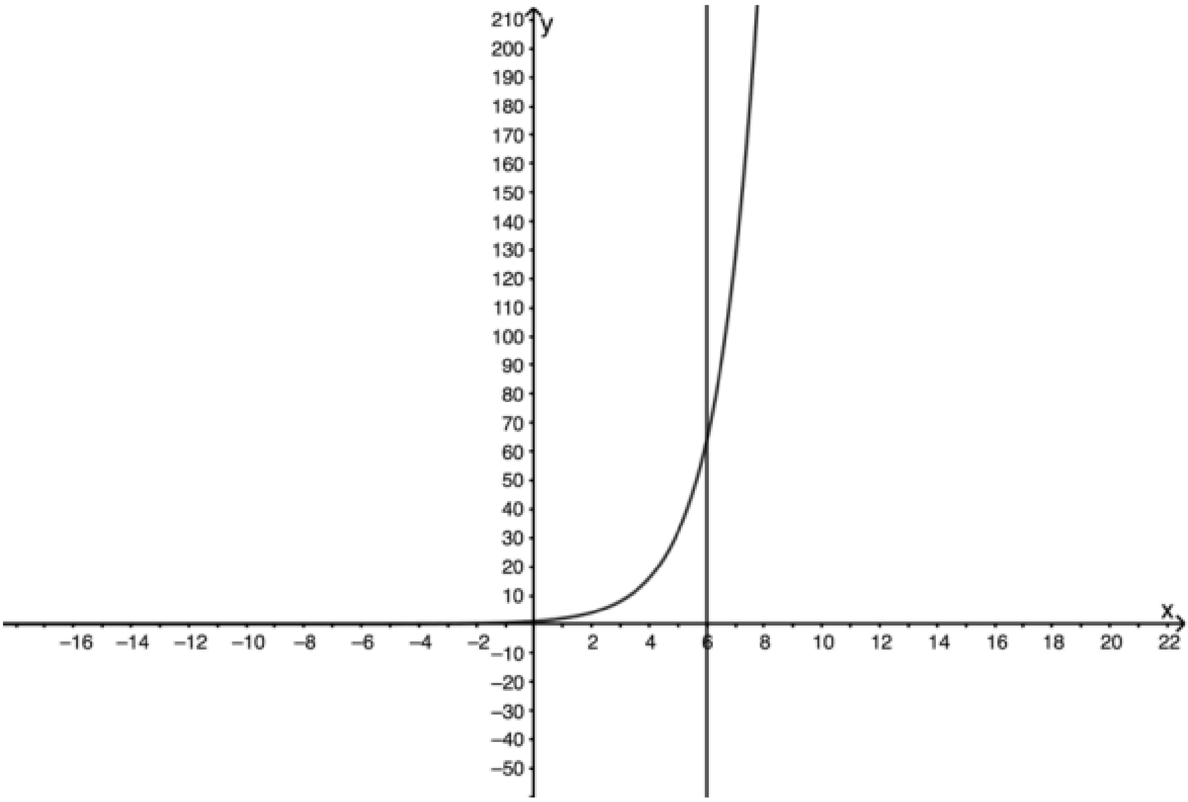


Figure 4: Vertical line test to show that $y = 2^x$ is a function

The effect of b on $y = b^x$

From Activity 4.1 we learnt that equations like $y = 2^x$ (general form $y = b^x$) are functions, and that they have a horizontal asymptote and a y-intercept. But what happens if we have the function $g(x) = 3^x$ or $h(x) = \left(\frac{1}{3}\right)^x$? Can $b > 0$?

Activity 4.2: Investigate the effect of *b* on *y*= *x*^{2}

Time required: 30 minutes

What you need:

- a pen or pencil
- a calculator
- blank paper or a notebook
- an internet connection

What to do:

Part A

1. Given $f(x) = 2^x$, $g(x) = 3^x$, $h(x) = \left(\frac{1}{3}\right)^x$ and $y = (-2)^x$, create and complete the following table of values. **Note:** You are given $y = (-2)^x$. The negative sign is included in the base. If you were given $y = -2^x$, you would assume that only 2 is in the base and the negative sign (or -1) is out front and is not raised to the exponent.

x	-2	-1	0	1	2
$f(x)$					
$g(x)$					
$h(x)$					
$y = (-2)^x$					

2. Sketch the graphs on the same set of axes.
3. What is the difference between $f(x)$ and $g(x)$?
4. What is the difference between $f(x)$ and $h(x)$?
5. What are the y-intercepts of $f(x)$, $g(x)$ and $h(x)$?
6. Why do the graphs all have the same y-intercept?
7. Can you plot $y = -2^x$. If not, why not?

Part B

When you have access to the Internet, visit this exponential function interactive simulation.

Here you will find an exponential function of the form $y = b^x$ with a slider to change the values of b.

1. What happens when b changes from being greater than one to less than one?
2. What happens when $b = 1$? Why is this?
3. What happens when $b < 0$? Why is this?

What did you find?

Part A

1. Here is the completed table of values.

(x)	-2	-1	0	1	2
$(f(x))$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$(g(x))$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
$(h(x))$	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$
$y=(-2)^x$	$\frac{1}{4}$	$-\frac{1}{2}$	1	-2	4

2. The graphs of each of these functions is shown in Figure 5. The different points generated by $y=-2^x$ are also shown.

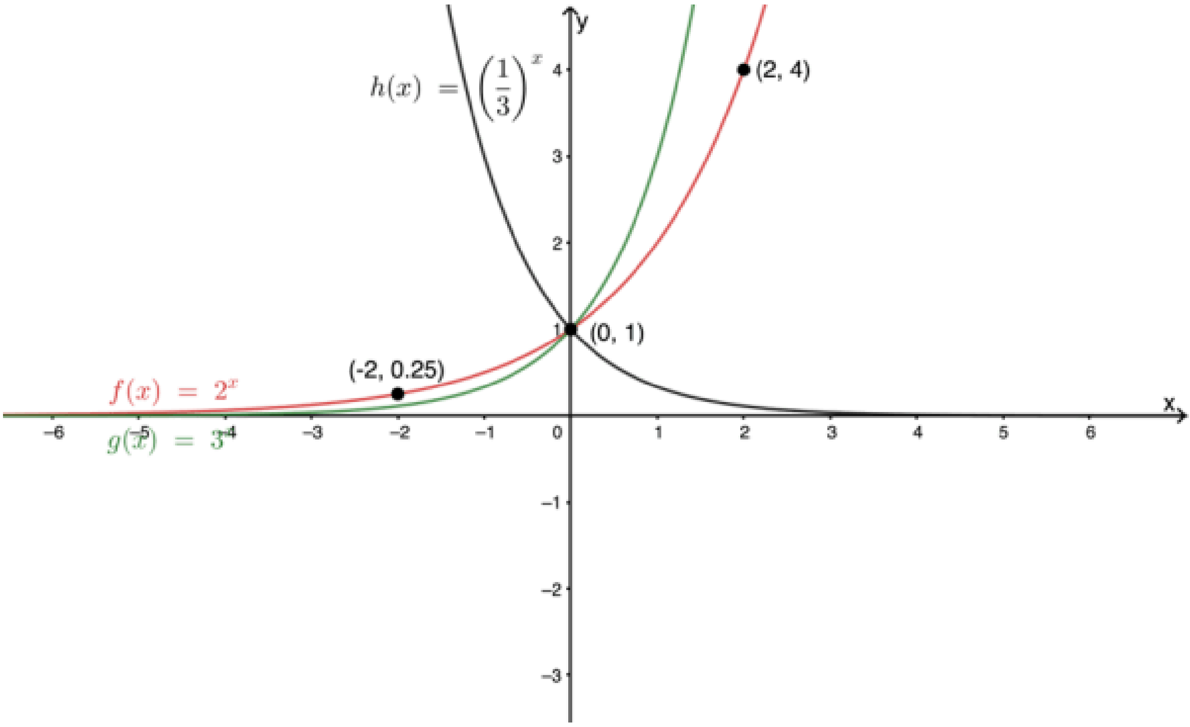


Figure 5: Sketches of $f(x)=2^x$, $g(x)=3^x$ *and* $h(x)=\left(\frac{1}{3}\right)^x$

3. The graph of $g(x)$ is steeper than $f(x)$. The y-values increase faster. This makes sense because for $g(x)$ we have powers of 3 not 2.
4. While $f(x)$ increases for positive x values and approaches the asymptote for negative x values, $h(x)$ does the opposite. If you look carefully you can see that $g(x)=3^x$ and $h(x)=\left(\frac{1}{3}\right)^x$ are mirror images of each other. They are **symmetrical about the y-axis.**
5. (and 6). The y-intercept of all three functions is $(0,1)$. This is because any base to the power of zero is one.

7. We cannot plot $y = -2^x$ because the points jump around. They don't all lie on a smooth curve. This makes sense because if we raise a negative number to an even power, the answer will be positive. But if we raise it to an odd power, the answer will be negative.

Part B

1. As soon as b changes from greater than one to less than one, the graph bends in the opposite direction. When $b > 0$, the graph increases as x get more positive. When $0 < b < 1$, the graph increases as x gets more negative.
2. When $b = 1$, the graph is a horizontal line. This is because one raised to any power is always equal to one. So no matter what the x value is, the function value will always be one.
3. When $b < 0$, the input/output points do not all line on the same smooth curve so we cannot draw the graph.

From Activity 4.2 we learnt that as b in $y = b^x$ increases, the graph gets steeper as the function values increase more quickly. When $0 < b < 1$, the graph flips over to be symmetrical about the y-axis. This means that $y = b^x$ and $y = \left(\frac{1}{b}\right)^x$ are symmetrical to each other about the y-axis. They are mirror images of each other if we treat the y-axis as the mirror (see Figure 6).

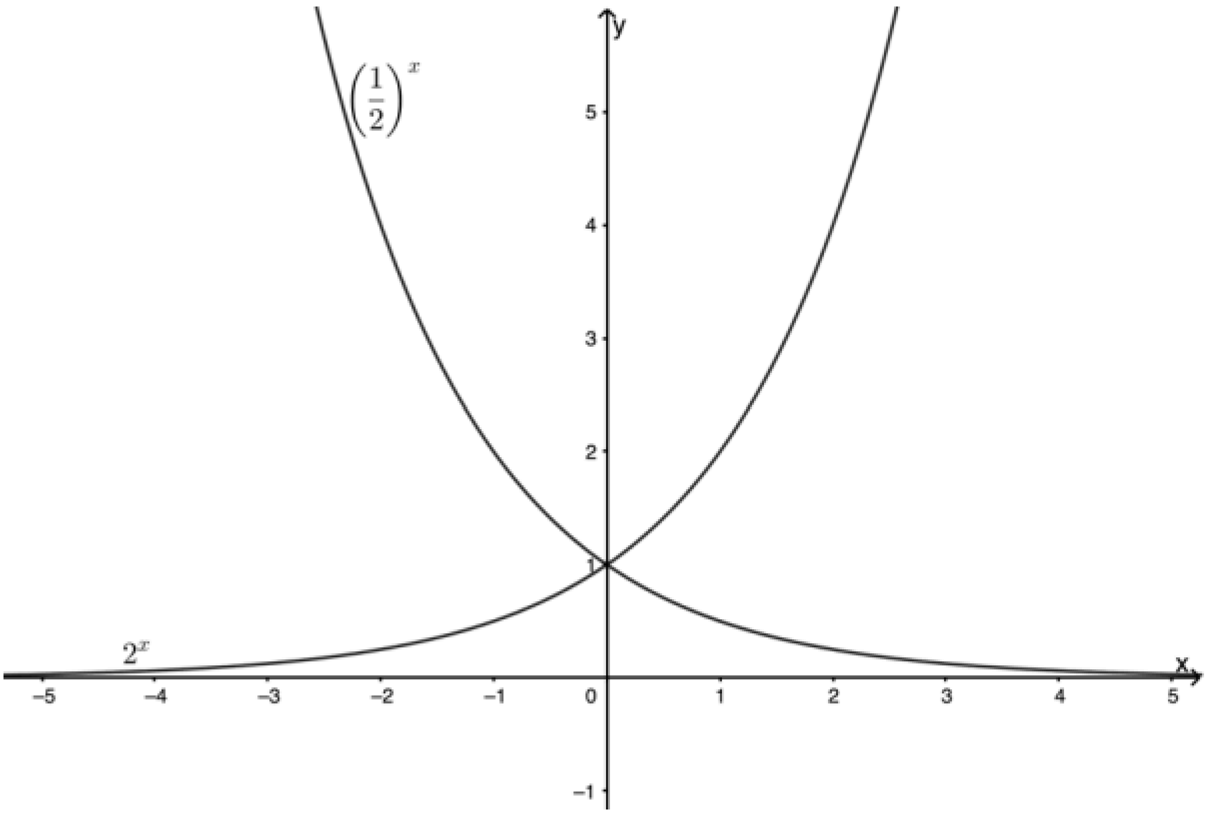


Figure 6: Graph showing that $y = b^x$ and $y = \left(\frac{1}{b}\right)^x$ are symmetrical to each other about the y-axis

We also saw that b cannot be negative. This is why we always write the basic exponential function as $y = b^x, b > 0$.

Note

When you have access to the internet watch the video called “Graphing exponential functions”, to see how to plot an exponential function using a table of values.

Graphing exponential functions (Duration: 05.31)

Example 4.1

Given $j(x) = 4^x$ and $l(x) = \left(\frac{1}{4}\right)^x$.

1. Use a table of values to plot each function on the same set of axes.
2. State the domain and range of each function.
3. About what line are these two graphs symmetrical?

Solutions

1. The graphs are shown in Figure 6.

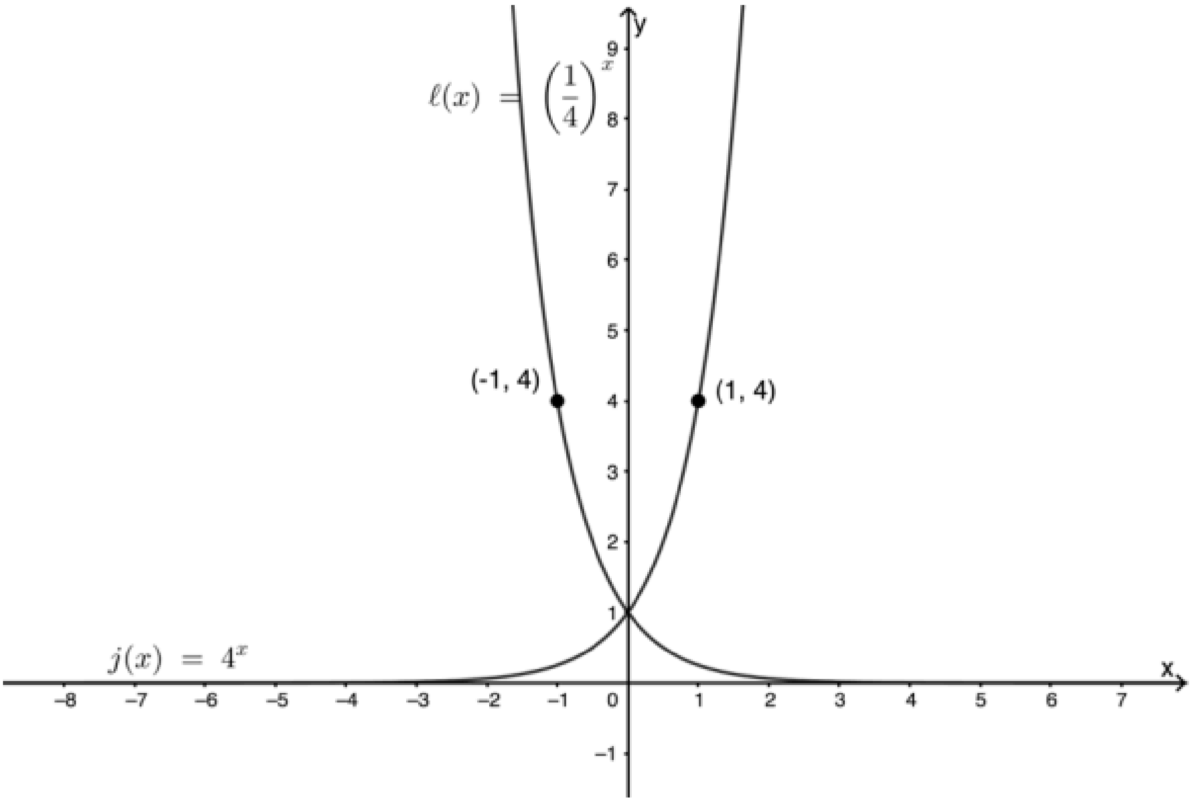


Figure 7: Graph of $j(x) = 4^x$ *and* $l(x) = \left(\frac{1}{4}\right)^x$

2. $j(x)$:
   Domain: $\{x \mid x \in \mathbb{R}\}$
   Range: $\{y \mid y \in \mathbb{R}, y < 0\}$
   $l(x)$:
   Domain: $\{x \mid x \in \mathbb{R}\}$
   Range: $\{y \mid y \in \mathbb{R}, y < 0\}$
3. The graphs are symmetrical about the y-axis, the line $x = 0$.

The effect of q on $y = b^x + q$

So far, we have looked at the graphs of functions like $y = b^x$. But what about graphs of functions like $y = b^x + q$? How do you think the value of q will affect the graph? Study the next example to see.

Example 4.2

1. Using a table of values, sketch the graphs of $d(x) = 2^x + 2$ and $e(x) = 2^x - 1$.

2. What is the y-intercept of each graph?
3. What are the asymptotes of each graph? How do these compare to the values of q in each function?
4. What is the domain and range of each graph?

Solutions

1. You could have used any values you liked. Here is a table of values that can be used to sketch each function (Figure 8).

x	-2	-1	0	1	2
$d(x)$	$\frac{9}{4}$	$\frac{5}{2}$	3	4	6
$e(x)$	$-\frac{3}{4}$	$-\frac{1}{2}$	0	1	3

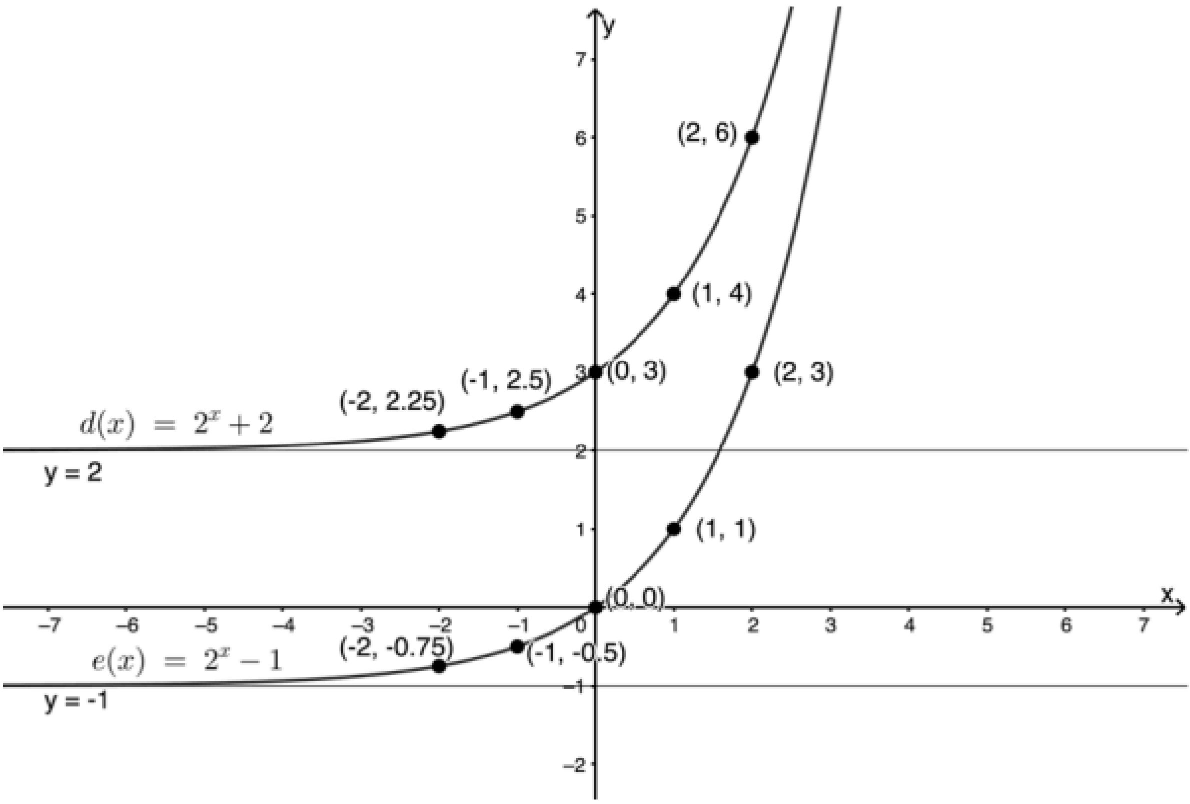


Figure 8: Graph of $d(x) = 2^x + 2$ *and* $e(x) = 2^x - 1$

2. The y-intercept of $d(x)$ is $(0, 3)$ and the y-intercept of $e(x)$ is $(0, 0)$.
3. The horizontal asymptote of $d(x)$ is $y = 2$. The horizontal asymptote of $e(x)$ is $y = -1$. In each case, the horizontal asymptote is the same as the value of q.
4. $d(x)$:
   Domain: $\{x : x \in \mathbb{R}\}$
   Range: $\{y : y \in \mathbb{R}, y > 2\}$

$e(x)$:
Domain: $\{x : x \in \mathbb{R}\}$
Range: $\{y : y \in \mathbb{R}, y > -1\}$

In Example 4.2 we saw that changing the value of q shifts the whole graph vertically up or down. This is the same effect changing the value of q has on the linear function ($y = ax + q$), the quadratic function ($y = ax^2 + q$) and the hyperbolic function ($y = \frac{a}{x} + q$).

In the exponential function, the horizontal asymptote is the line $y = q$. This is the same as for the hyperbolic function.

The effect of a on $y = a.b^x + q$

Finally, we need to have a look at exponential functions of the form $y = a.b^x + q$. The functions $j(x) = 2^x$, $k(x) = 2.2^x$, and $l(x) = -2.2^x$ are shown in Figure 9.

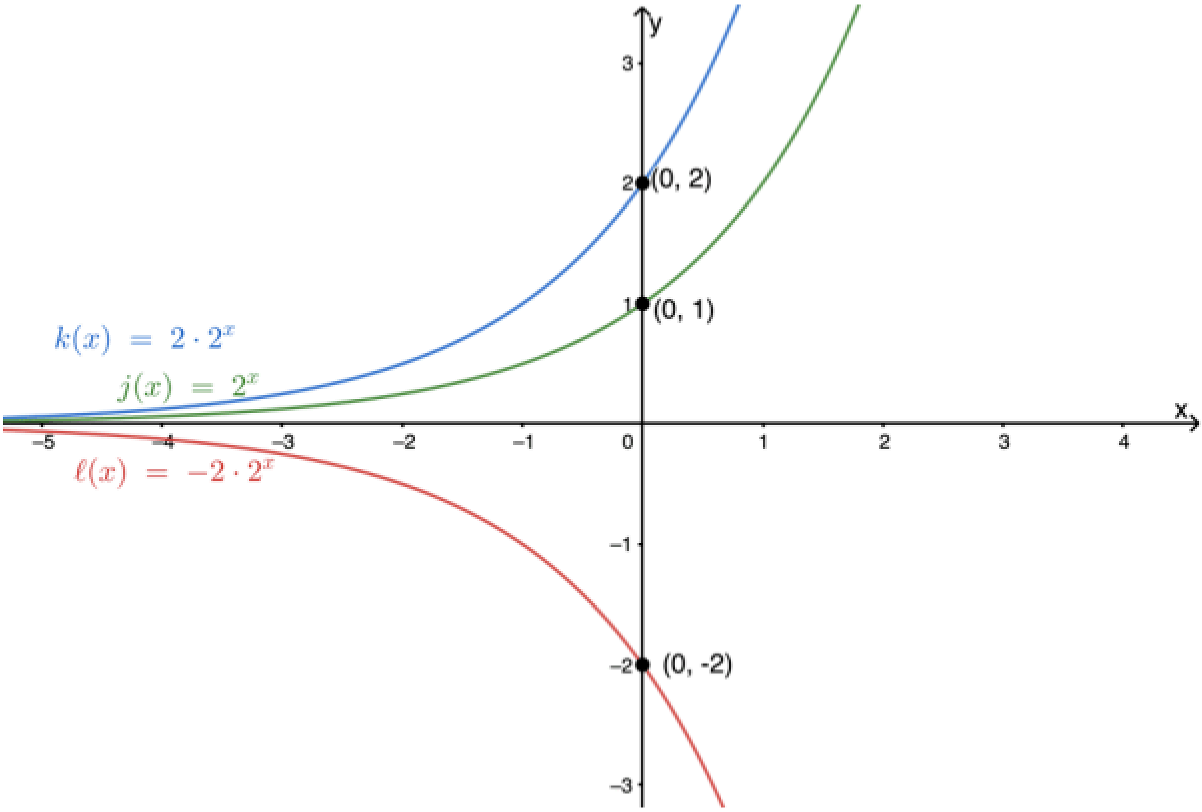


Figure 9: Graphs of $j(x) = 2^x$, $k(x) = 2.2^x$, and $l(x) = -2.2^x$

Firstly, what is the difference between $j(x) = 2^x$, $k(x) = 2.2^x$? Can you see that the graph of $k(x)$ is stretched up so that it intersects the y-axis at $(0, 2)$ instead of $(0, 1)$? Every point on the graph of $j(x) = 2^x$ has been multiplied by two to make the graph of $k(x) = 2.2^x$.

Secondly, we can also see that the graph of $l(x) = -2.2^x$ is the same shape as $k(x) = 2.2^x$ except that it has been flipped over horizontally. Instead of going to positive infinity as x gets bigger, the graph goes to negative infinity. Each function value is multiplied by negative two instead of positive two. The function

$l(x) = -2.2^x$ still has a horizontal asymptote at $y = 0$ (remember that the value of q in the function is zero) but this asymptote now represents a **maximum** value for the graph rather than a **minimum** value. The graphs of $k(x) = 2.2^x$ and $l(x) = -2.2^x$ are mirror images of each other if the mirror is the x-axis. In general, we say that $y = a.b^x + q$ and $y = -a.b^x q$ are **symmetrical** about the x-axis.

The domains of each of these functions are all still $\{x \mid x \in \mathbb{R}\}$. However, the ranges are different and depend on whether $a > 0$ or $a < 0$.

When $a > 0$, the range of the function is $\{y \mid y \in \mathbb{R}, y > q\}$.
When $a < 0$, the range of the function is $\{y \mid y \in \mathbb{R}, y < q\}$.

Figures 10 and 11 show a summary of what we know about the exponential function so far.

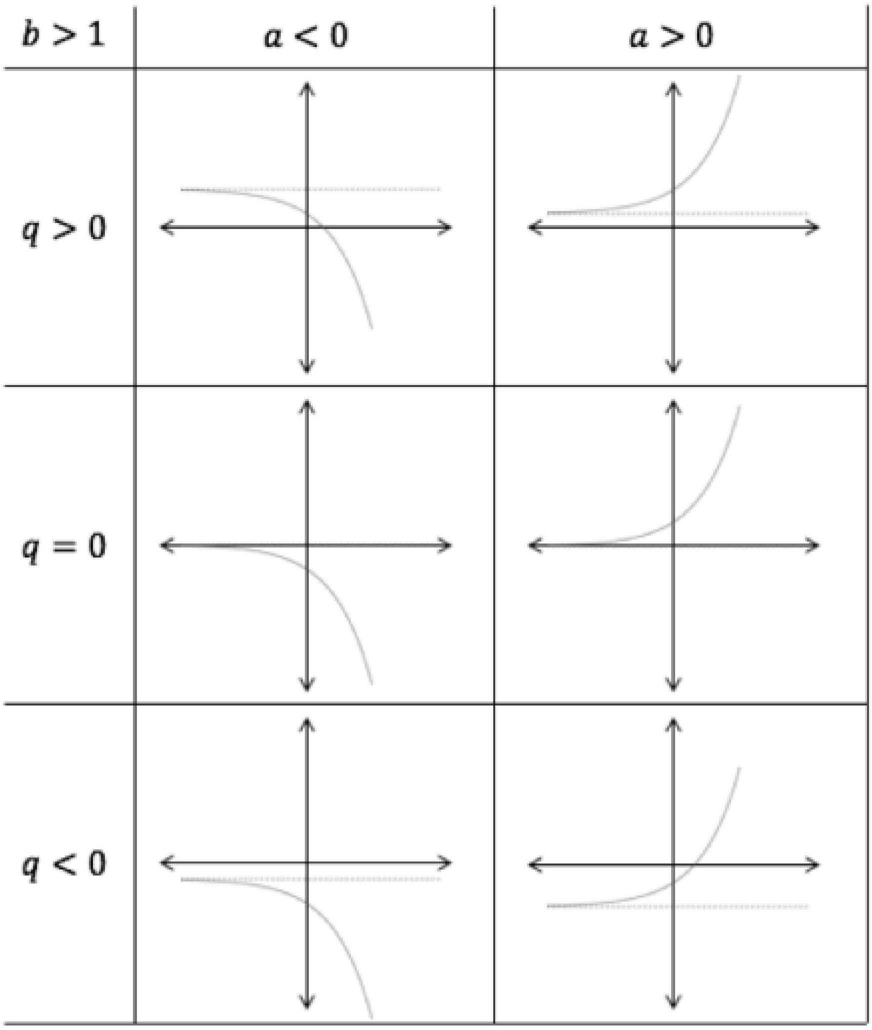


Figure 10: The effects of a and q on the exponential graph when $b > 1$

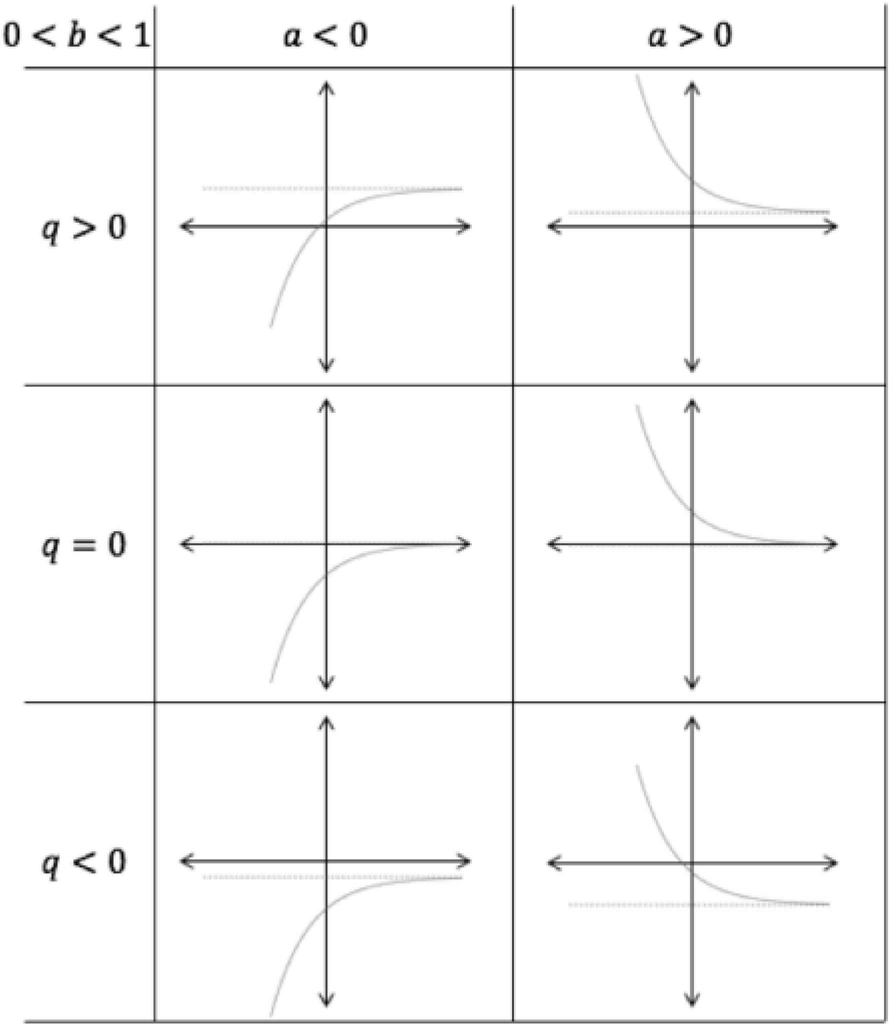


Figure 11: The effects of a and q on the exponential graph when $0 < b < 1$

Before looking at the next example, get online and visit this interactive simulation.

Here you will find an exponential function of the form $y = a.b^x + q, b < 0$ with sliders to change the values of a, b and q. Spend some time playing with the graph to make sure that you understand how changing the values of a, b and q affects the shape and position of the exponential graph of the form $y = a.b^x + q, b < 0$.

Example 4.3

1. Sketch the graph of $t(x) = 3.2^2 + 2$. Mark the intercepts and asymptote.

2. Find the domain and range of $v(x) = -5.3^x - 1$.

Solutions

1. To sketch an exponential function, we need to know the sign of b, the sign of a, the y-intercept, the x-intercept (if one exists) and the horizontal asymptote.

   In $t(x) = 3.2^2 + 2$, we can see that $b > 1$ and $a > 0$. This means that the graph will grow to positive infinity as x gets larger. $q = 2$, which means that the graph will be shifted two units up.

   y-intercept (let $x = 0$):
   $y = 3.2^0 + 2 = 3.1 + 2 = 5$
   The y-intercept is $(0, 5)$.

   x-intercept (let $y = 0$):
   $0 = 3.2^x + 2$
   $\therefore 3.2^x = -2$
   $\therefore 2^x = -\frac{2}{3}$
   Therefore, there is no real solution and hence, no x-intercept.

   The horizontal axis is the line $y = 2$. We plot the y-intercept and the asymptote and draw the graph as shown in Figure 12.

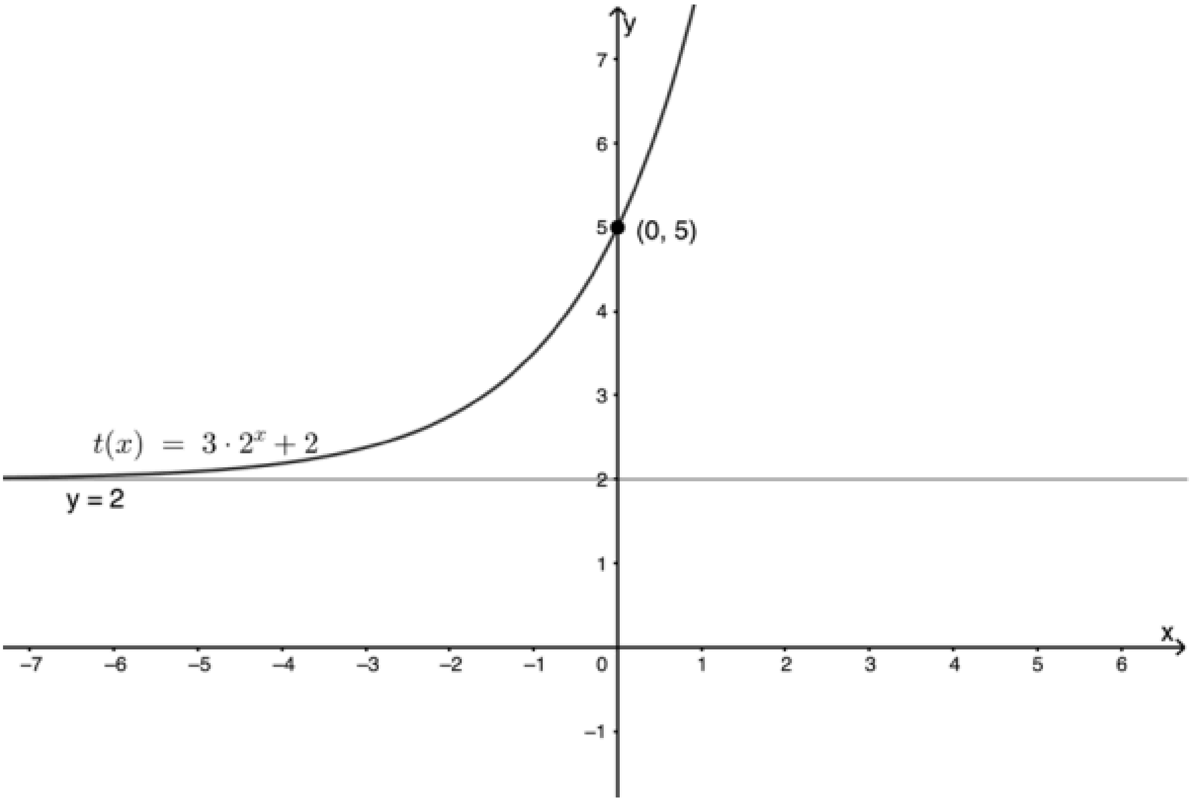


Figure 12: Graph of $t(x) = 3.2^x + 2$

2. We need to find the domain and range of $v(x) = -5.3^x - 1$. The domain is all real values. So the

domain is $\{x \mid x \in \mathbb{R}\}$.

We can see that $a < 0$ and $b > 1$. This means that the graph will decrease infinitely as x gets larger and the horizontal asymptote (the line $y = -1$) is an upper limit on the range. The range is $y < q$ and so is $\{v(x) \mid v(x) \in \mathbb{R}, v(x) < -1\}$.

Alternatively, you could use inequalities to determine the range.
$-5.3^x < 0$
$\therefore -5.3^x - 1 < -1$
Therefore the range will be all real values less than negative one: $\{v(x) \mid v(x) \in \mathbb{R}, v(x) < -1\}$

Exercise 4.1

1. Given $f(x) = -\frac{3}{4}.4^x + 3$:
   a. Calculate the y-intercept.
   b. Calculate the x-intercept.
   c. Calculate the equation of the horizontal asymptote.
   d. Calculate the domain and range of $f(x)$.
   e. Make a sketch of $f(x)$ indicating the intercepts and equation of the asymptote.
2. Sketch the graph of $y = -2 \times 4^x + 4$ indicating the intercepts and asymptote.
3. Make a neat sketch of $g(x) = -3.\left(\frac{1}{3}\right)^x + 3$ showing the intercepts and horizontal asymptote.

The full solutions are at the end of the unit.

Find the equations of exponential functions

Example 4.4

1. Figure 13 shows the graph of an exponential function of the form $y = a \times \left(\frac{1}{2}\right)^x + q$ which cuts the y-axis at $(0, 2)$ and the x-axis at $(-1, 0)$. Find the values of a and q and hence, the equation of the function.

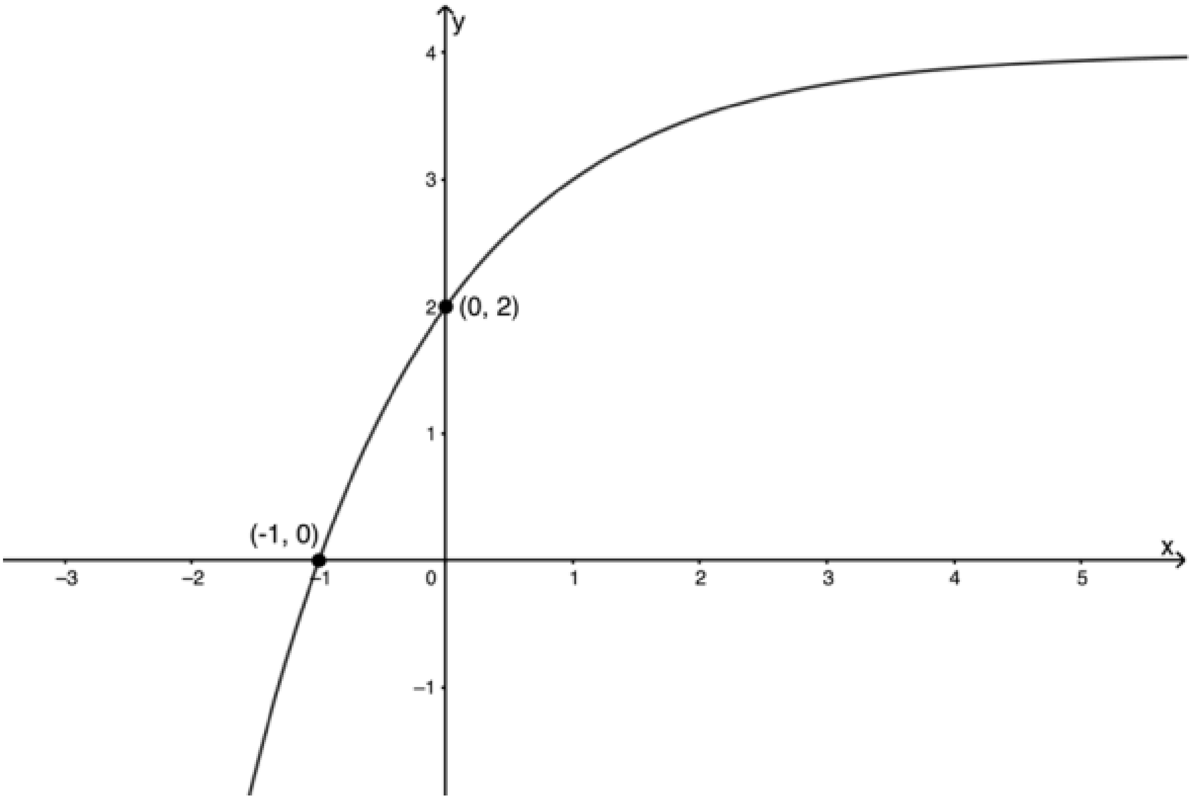


Figure 13: Graph of $y = a \times \left(\frac{1}{2}\right)^x + q$

2. An exponential function with an asymptote of $y = 0$ passes through $\mathrm{A}(0, 1)$ and $\mathrm{B}(2, 9)$. What is the equation of the function?

Solutions

1. We are told that the function is of the form $y = a \times \left(\frac{1}{2}\right)^x + q$. Therefore, we already know that $b = \frac{1}{2}$. With the two intercepts, we can set up a system of two simultaneous equations as follows:

   Using $(-1, 0)$:

   $$0 = a\left(\frac{1}{2}\right)^{-1} + q$$

   $$\therefore 2a + q = 0$$
   $$\therefore 2a = -q \quad \text{Equation 1}$$

   Using $(0, 2)$:

   $$2 = a \times \left(\frac{1}{2}\right)^0 + q$$

   $$\therefore 2 = a + q$$
   $$\therefore a = 2 - q \quad \text{Equation 2}$$

   Substitute equation 2 into equation 1:

$$2(2-q) = -q$$
$$\therefore 4 - 2q = -q$$
$$\therefore q = 4$$

Substitute $q = 4$ into equation 2:

$$a = 2 - 4$$
$$\therefore a = -2$$

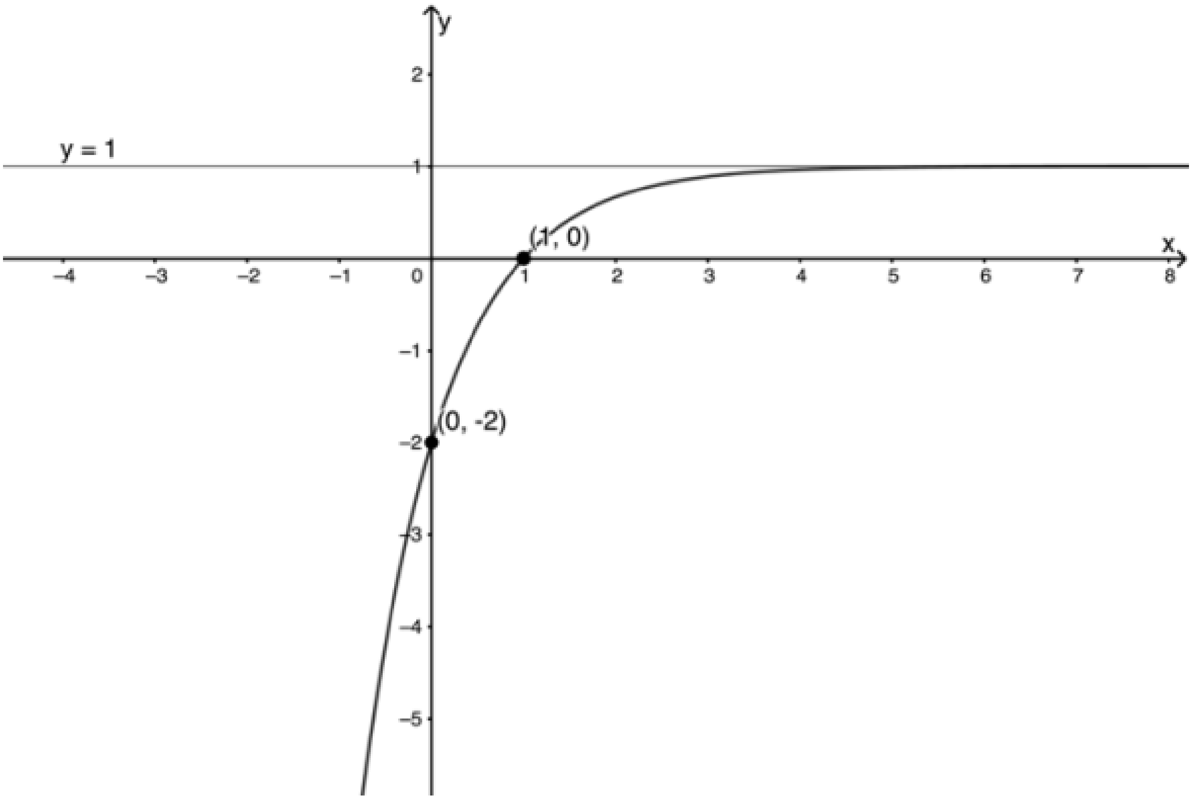
Therefore, the equation of the function is $y = -2 \times \left(\frac{1}{2}\right)^x + 4$.

2. We are told that the asymptote of the exponential function is the line $y = 0$. Therefore, $q = 0$ and the function is of the form $y = a.b^x$. Using $\mathrm{A}(0, 1)$, we get $1 = a.b^0 = a$. Now we can use $\mathrm{B}(2, 9)$ to find the value of b.

   $9 = b^2$ But we know that $b \geq 0$. Therefore, we need to exclude the answer $b = -3$.
   $\therefore b = \pm 3$
   Therefore, the equation of the function is $y = 3^x$.

Exercise 4.2

1. What is the equation of the exponential function of the form $y = a.b^x + q$ represented in the graph?



2. An exponential function of the form $y = a.b^x + q$ has a range of $\{y \mid y \in \mathbb{R}, y < -2\}$ and a y-inter-

cept of $(0, -\frac{5}{2})$. If it passes through the point $(2, -10)$, what is the equation of the function?

The full solutions are at the end of the unit.

Summary

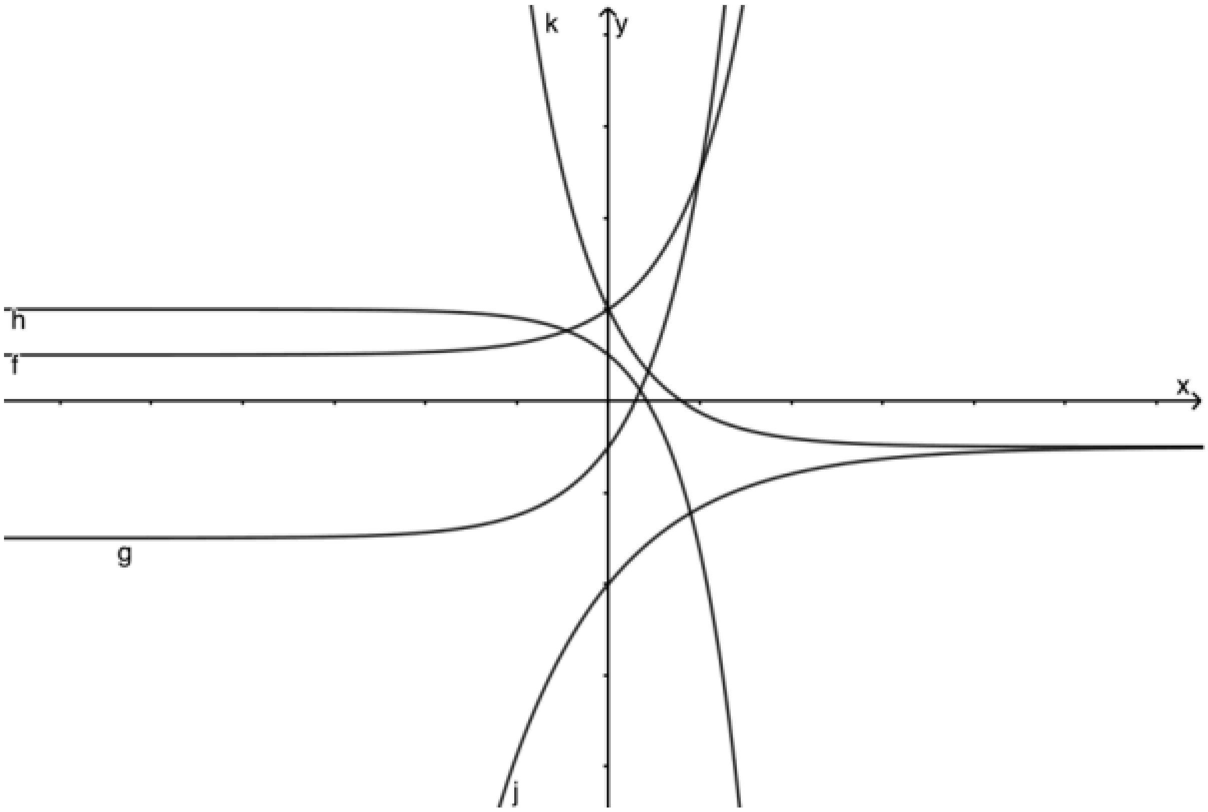
In this unit you have learnt the following:

- How to sketch exponential functions of the form $y = a.b^x + q, b > 0$.
- How to find the equation of graphs of the form $y = a.b^x + q, b > 0$.
- The effect of a, b and q on the shape and position of $y = a.b^x + q, b > 0$.
- How to find the asymptote of an exponential function of the form $y = a.b^x + q, b > 0$.
- How to determine the domain and range of an exponential function of the form $y = a.b^x + q, b > 0$.

Unit 4: Assessment

Suggested time to complete: 60 minutes

1. Pictured below are the following functions:
$y = -2.3^x + 2$
$y = 3\left(\frac{1}{2}\right)^x - 1$
$y = 2^x + 1$
$y = 2.2^x - 3$
$y = -3\left(\frac{2}{3}\right)^x - 1$



Match each one to its graph.

2. Given the functions $q(x) = 3^x + 1$ and $r(x) = \left(\frac{1}{3}\right)^x + 1$.

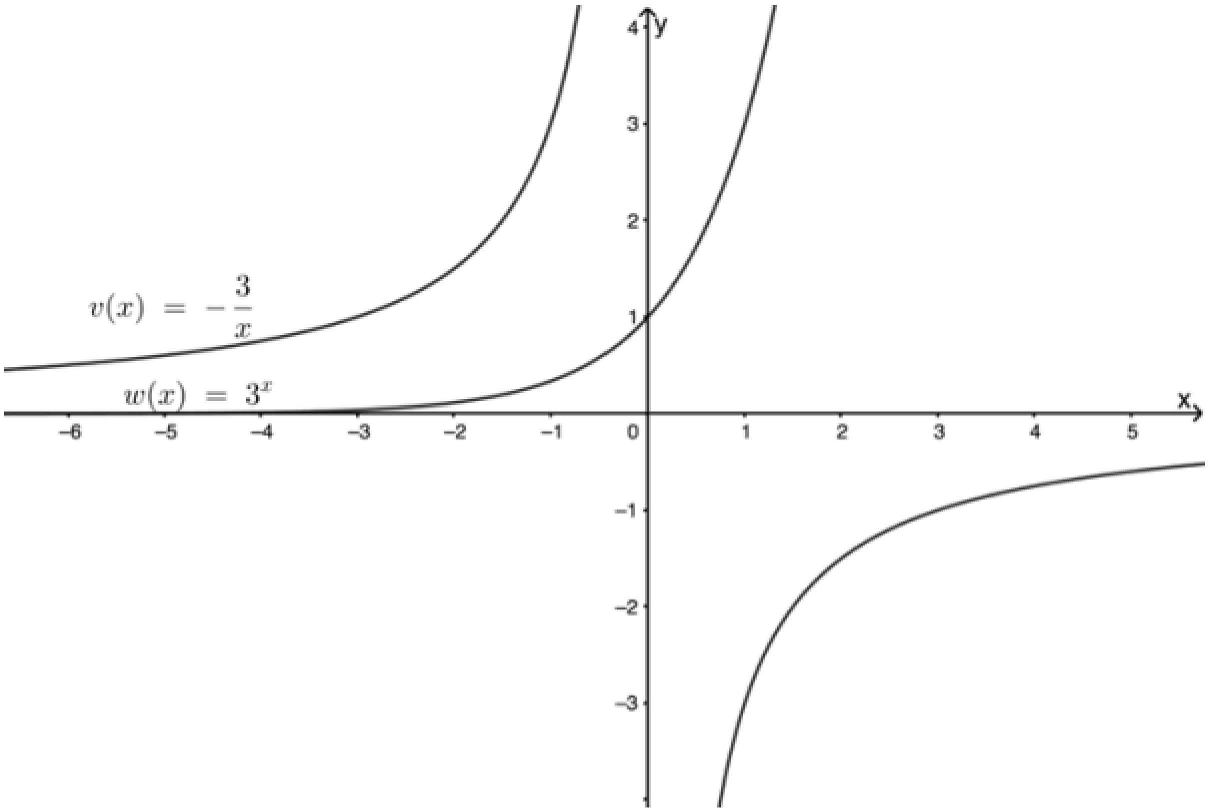
   a. Draw the graphs on the same set of axes.

   b. Is the line $y = 1$ an asymptote or axis of symmetry of each graph?

   c. Solve the equation $3^x + 1 = \left(\frac{1}{3}\right)^x + 1$ graphically.

   d. About what line are these two functions symmetrical?

3. Sketched below are the functions $v(x) = -\frac{3}{x}$ and $w(x) = 3^x$.

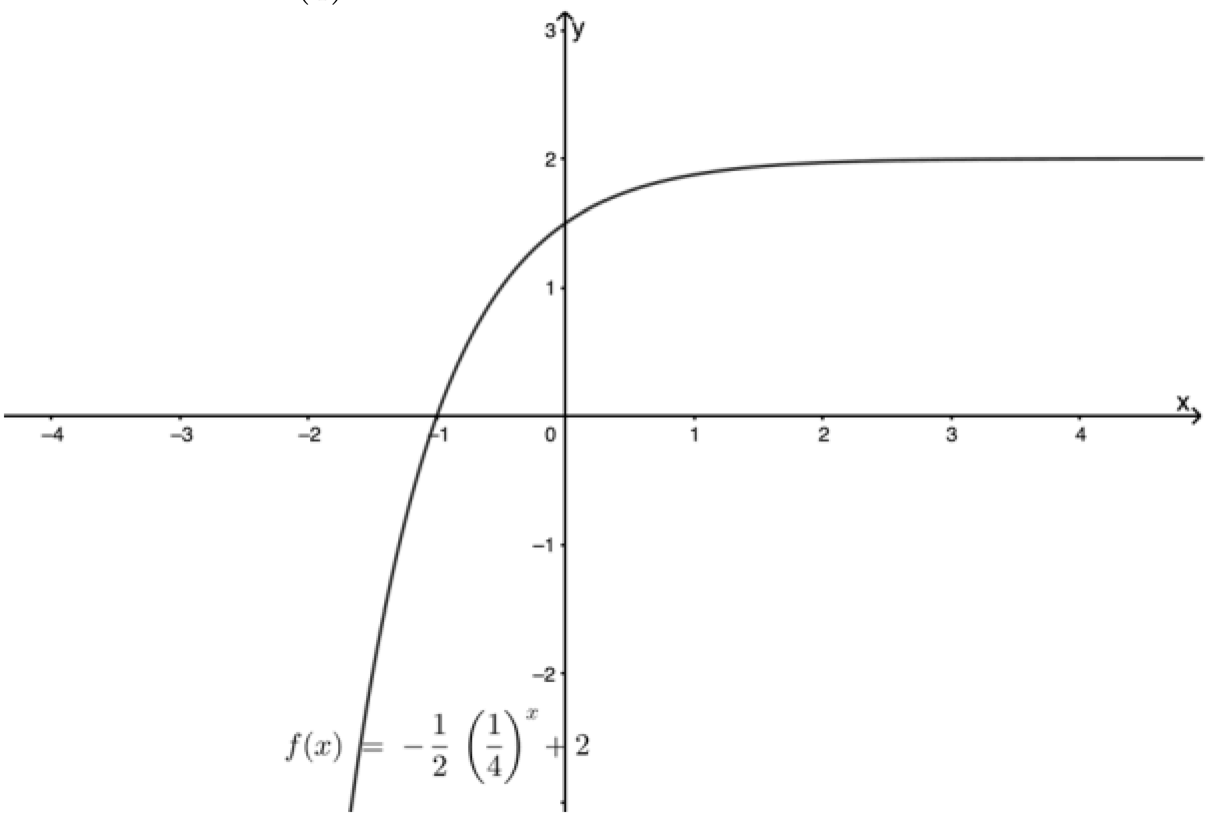


a. What feature(s) or characteristic(s) do the graphs share?

b. Evaluate $w(1) - v(1)$.

c. Describe what the expression $w(1) - v(1)$ means graphically.

4. The function $f(x) = -\frac{1}{2}\left(\frac{1}{4}\right)^x + 2$ is shown in the diagram.



a. What is the domain and range of $f(x)$?

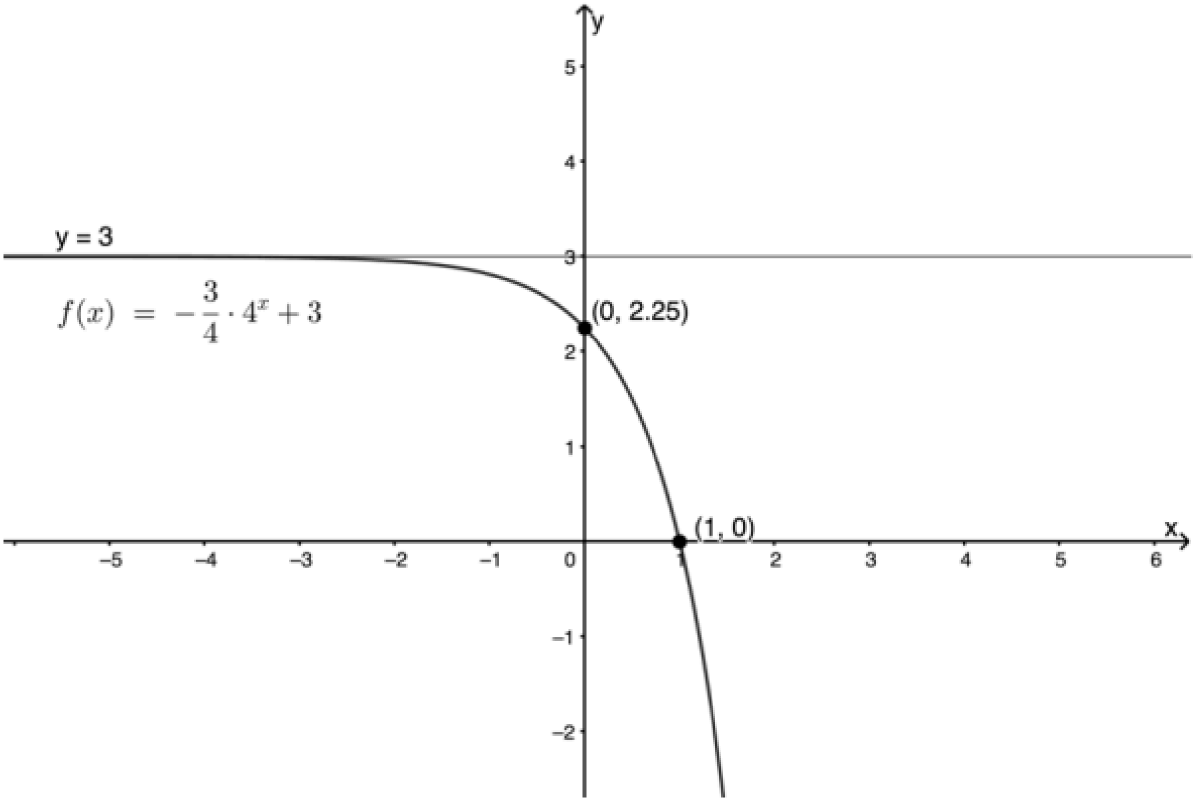
b. What is the equation of the function $g(x)$, $f(x)$ shifted 3 units down?
c. What is the equation of $h(x)$, the reflection of $g(x)$ about the y-axis?

The full solutions are at the end of the unit.

Unit 4: Solutions

Exercise 4.1

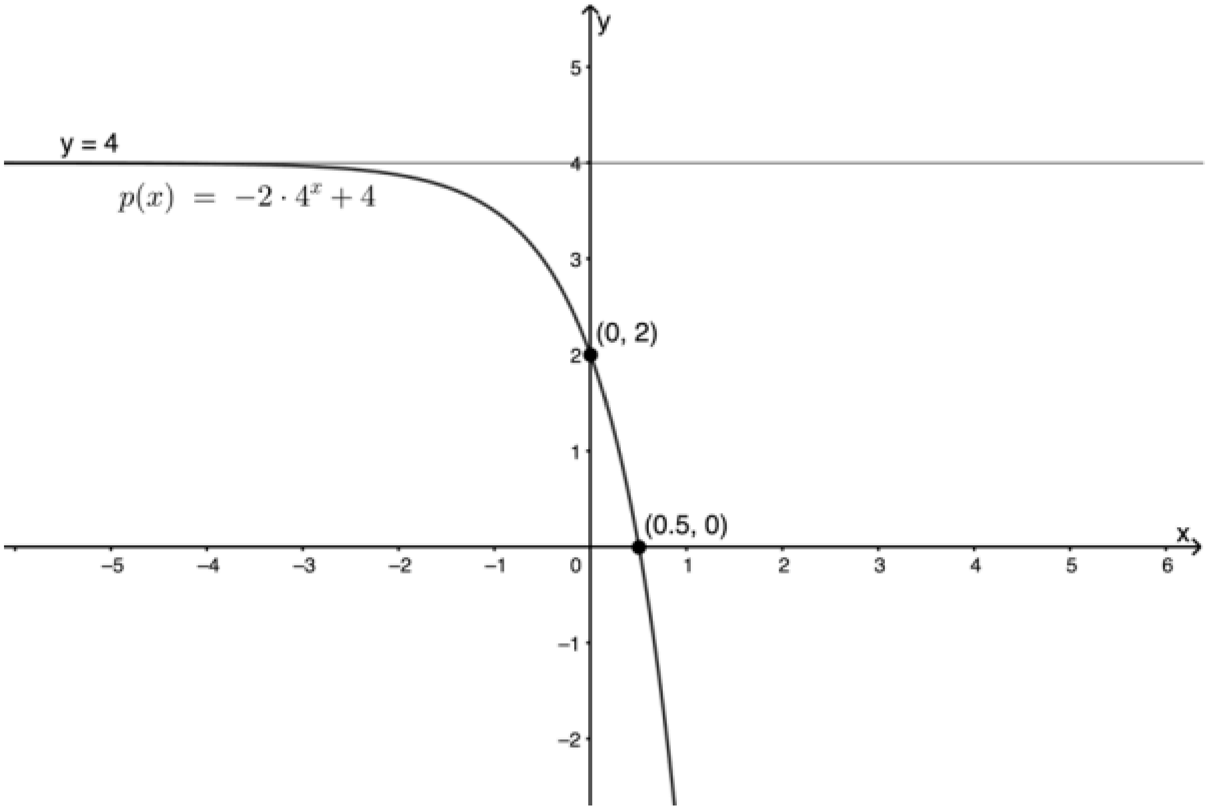
1.
  a. y-intercept (let $x = 0$):
  $$f(0) = -\frac{3}{4}.4^0 + 3$$
  $$\therefore f(0) = -\frac{3}{4} + 3$$
  $$\therefore f(0) = \frac{9}{4}$$
  Therefore the y-intercept is $(0, \frac{9}{4})$.
  b. x-intercept (let $y = 0$):
  $$0 = -\frac{3}{4}4^x + 3$$
  $$\therefore \frac{3}{4}4^x = 3$$
  $$\therefore 4^x = 4$$
  $$\therefore x = 1$$
  Therefore the x-intercept is $(1, 0)$.
  c. $q = 3$ therefore the horizontal asymptote is $y = 3$.
  d. Domain: $\{x \mid x \in \mathbb{R}\}$
  Range:
  $a < 0$ therefore
  $$-\frac{3}{4}4^x < 0$$
  $$\therefore -\frac{3}{4}4^x + 3 < 3$$
  So the range is $\{y \mid y \in \mathbb{R}, y < 3\}$
  e.



2. y-intercept (let $x = 0$):
   $y = -2 \times 4^0 + 4$
   $\therefore y = 2$
   The y-intercept is $(0, 2)$.

   x-intercept (let $y = 0$):
   $0 = -2 \times 4^x + 4$
   $\therefore 2 \times 4^x = 4$
   $\therefore 4^x = 2$
   $\therefore x = \frac{1}{2}$
   The x-intercept is $(\frac{1}{2}, 0)$.

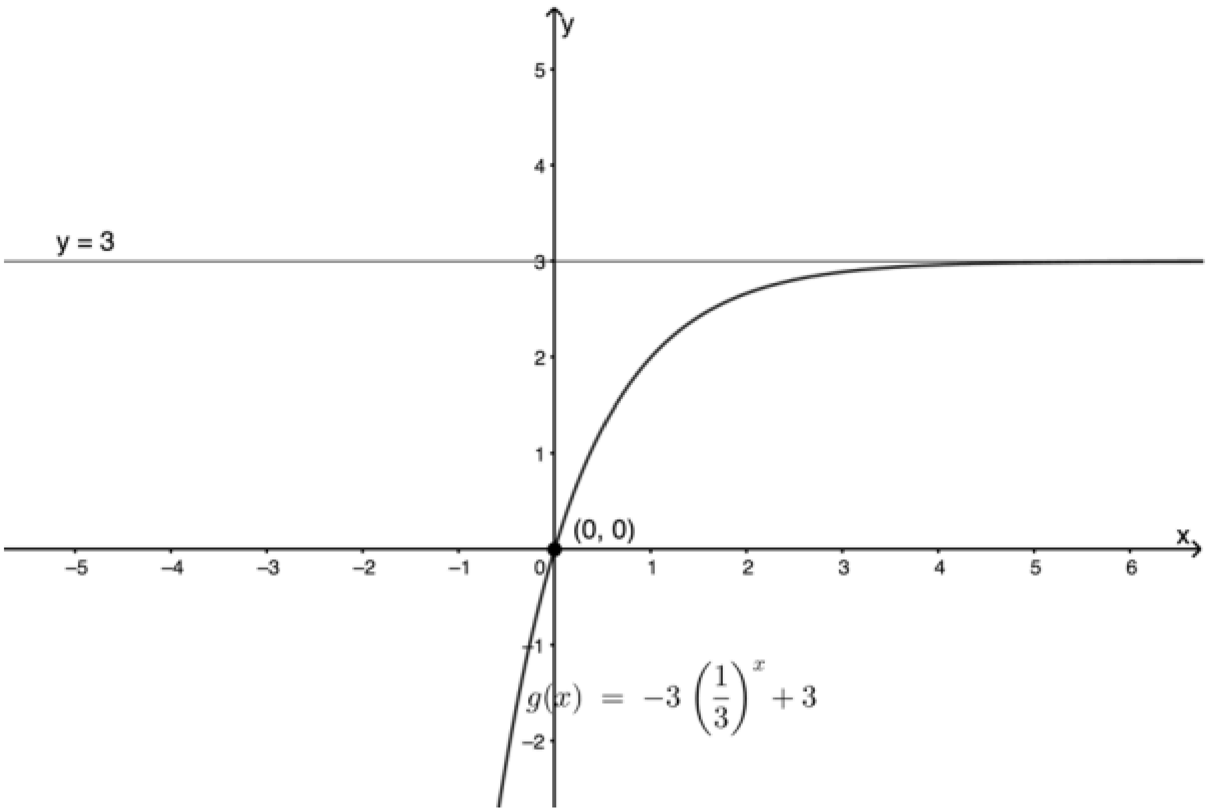
   The asymptote is $y = 4$.



3. y-intercept (let $x = 0$):
   $g(x) = -3.\left(\frac{1}{3}\right)^0 + 3$ The y-intercept is $(0, 0)$. Therefore the graph passes through the origin and the
   $\therefore g(x) = 0$
   x- and y-intercepts are not different points.

   The asymptote is $y = 3$.

   $a < 0$ and $0 < b < 1$. Therefore the graph goes to negative infinity as x gets more negative.



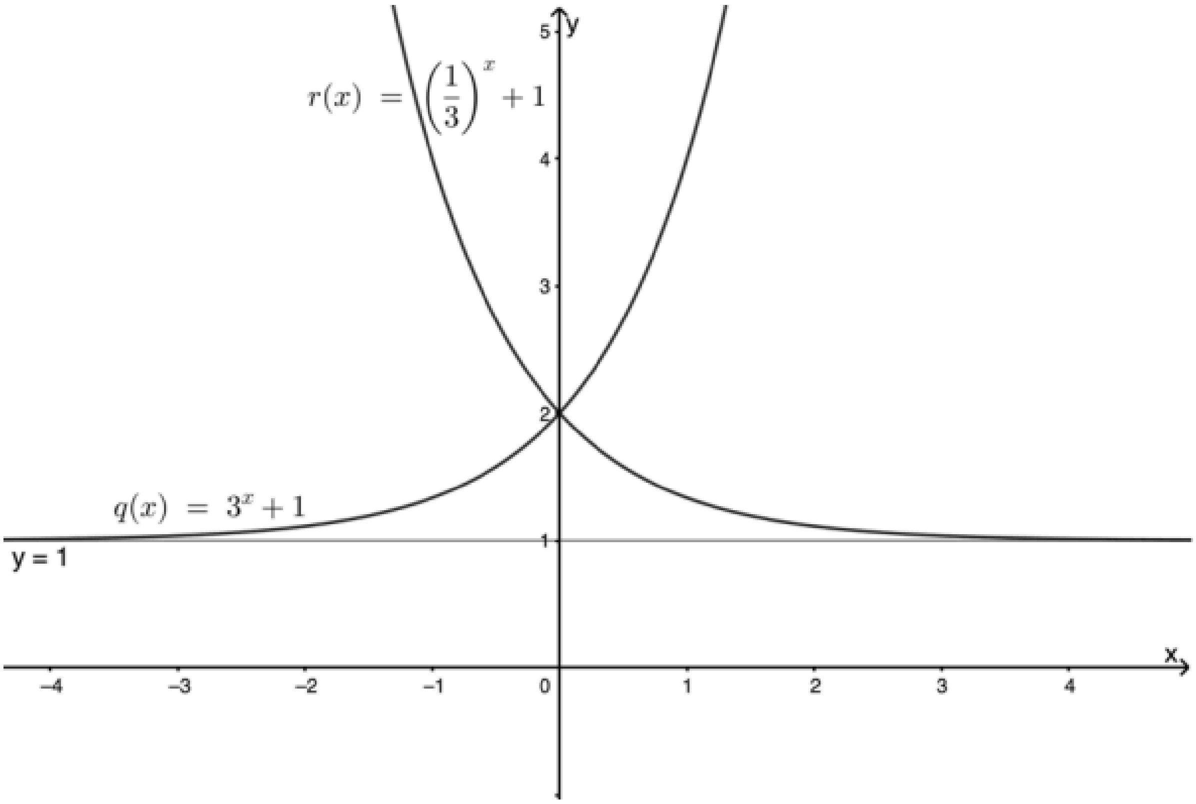
Back to Exercise 4.1

Exercise 4.2

1. From the graph, the horizontal asymptote is $y = 1$. Therefore $q = 1$. Therefore the function is of the form $y = a.b^x + 1$.Substitute $(0, -2)$:
   $-2 = a.b^0 + 1$
   $\therefore -2 = a + 1$
   $\therefore a = -3$
   Substitute $(1, 0)$:
   $0 = -3 \times b^1 + 1$
   $\therefore 3b = 1$
   $\therefore b = \frac{1}{3}$
   Therefore the equation of the function is $y = -3\left(\frac{1}{3}\right)^x + 1$.
2. We are told that the range of the function is $y < -2$. Therefore $y = -2$ is the asymptote and $q = -2$.We are told that the y-intercept is $(0, -\frac{5}{2})$ which is below the asymptote. Therefore the graph curves down and so:Substitute $(0, -\frac{5}{2})$ into $y = a.b^x - 2$:
   $-\frac{5}{2} = a.b^0 - 2$
   $\therefore a = -\frac{1}{2}$
   Now substitute $(2, -10)$ into $y = -\frac{1}{2}.b^x - 2$:
   $-10 = -\frac{1}{2}b^2 - 2$
   $\therefore -8 = -\frac{1}{2}b^2$
   $\therefore b^2 = 16$
   $\therefore b = \pm 4$
   But for an exponential function we know that $b > 0$. Therefore $b = 4$ and the equation of the function is $y = -\frac{1}{2}4^x - 2$.

Back to Exercise 4.2

Unit 4: Assessment

1. $h(x) = -2.3^x + 2$
   $k(x) = 3\left(\frac{1}{2}\right)^x - 1$
   $f(x) = 2^x + 1$
   $g(x) = 2.2^x - 3$
   $j(x) = -3\left(\frac{2}{3}\right)^x - 1$
2.
   a.


   b. The line $y = x$ is an asymptote of each graph.
   c. The solution to $3^x + 1 = \left(\frac{1}{3}\right)^x + 1$ is the point where the graphs intersect each other. From the graph, this is the point $(0, 2)$.
   d. The functions are symmetrical about the y-axis.
3.
   a. Both graphs have the line $y = 0$ as a horizontal asymptote.
   b.
$$\begin{aligned} w(1) - v(1) &= 3^1 - \left(-\frac{3}{1}\right) \\ &= 3 + 3 \\ &= 6 \end{aligned}$$
   c. $w(1)$ and $v(1)$ are the y-values of the functions when $x = 1$. Therefore $w(1) - v(1)$ is the difference in or distance between these y-values. Therefore the graphs are six units apart vertically when $x = 1$.
4.

a. Domain: $\{x \mid x \in \mathbb{R}\}$
Range:
$q = 2$ and $a < 0$
Range is $\{f(x) \mid f(x) \in \mathbb{R}, f(x) < 2\}$

b.

$$
\begin{aligned}
g(x) &= f(x) - 3 \\
\therefore g(x) &= -\frac{1}{2}\left(\frac{1}{4}\right)^x + 2 - 3 \\
\therefore g(x) &= -\frac{1}{2}\left(\frac{1}{4}\right)^x - 1
\end{aligned}
$$

c. We know that $g(x) = -\frac{1}{2}\left(\frac{1}{4}\right)^x - 1$. We also know that $y = a.b^x + q$ and $y = a.\left(\frac{1}{b}\right)^x + q$ are symmetrical about the y-axis. Therefore, if $b = \frac{1}{4}$ in $g(x)$, the value of b for the graph symmetrical about the y-axis to $g(x)$ will be 4. Therefore $h(x) = -\frac{1}{2} \times 4^x - 1$.

Back to Unit 4: Assessment

Media Attributions

Unit 5: Trigonometric functions

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Identify the following characteristics of trigonometric functions:
    - periodicity
    - amplitude.
- Sketch the graph $y = a\sin x + q$, $y = a\cos x + q$ and $y = a\tan x + q$.
- Identify the asymptotes of $y = a\tan x + q$.
- Investigate and generalise the impact of a and q on $y = asinx + q$, $y = a\cos x + q$ and $y = a\tan x + q$.

What you should know

Before you start this unit, make sure you can

- Use the trigonometric ratios $\sin\theta$, $\cos\theta$ and $\tan\theta$. Go over Subject outcome 3.6, Unit 1: Trigonometric ratios if you need more help with this.
- Determine the domain and range of a function by looking at its graph. Go over Units 1 to 4 in this subject outcome if you need help with this.
- Determine the gradient of a straight line or linear function. Go over Unit 1 in this subject outcome if you need help with this.
- Explain what an asymptote of a function is. Go over Unit 3 in this subject outcome if you need help with this.
- Manipulate and simplify algebraic expressions. Go over Subject outcome 2.2, Unit 1: Simplifying algebraic expressions if you need more help with the basics.
- Plot points on the Cartesian plane. If you do not know how to plot points onto the Cartesian plane, then you should do Subject outcome 3.3, Unit 1: Plotting points on the Cartesian plane before continuing with this unit.

Introduction

All modern music is produced with the aid of computers at some stage in the recording and mastering process. Computers are used to manipulate, transform and blend the sounds made by instruments of various kinds. These computer programmes use trigonometric functions to help them model and change sounds.

When sound travels through the air, the energy of the sound creates pockets of air molecules at slightly different pressures (see Figure 1). In some of these areas the air is slightly **compressed** and at a higher pressure than normal. In others, it is **decompressed** and at a lower pressure.

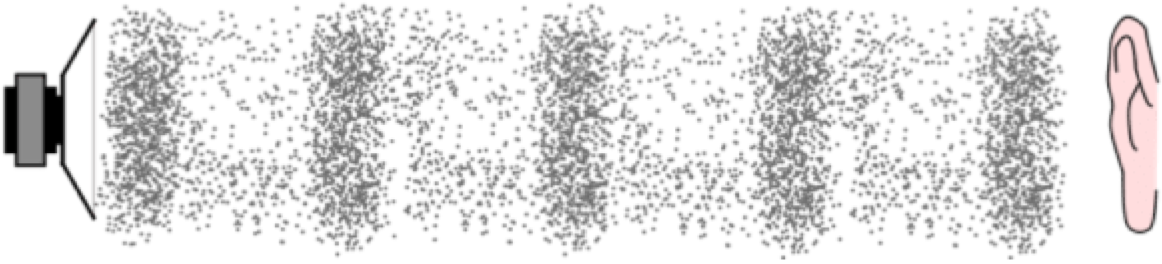


Figure 1: Sound waves

These sound waves can be thought of in the same way as waves travelling across a pond, where the areas of compression correspond to the crests or peaks and the areas of decompression correspond to the troughs or valleys (see Figure 2).

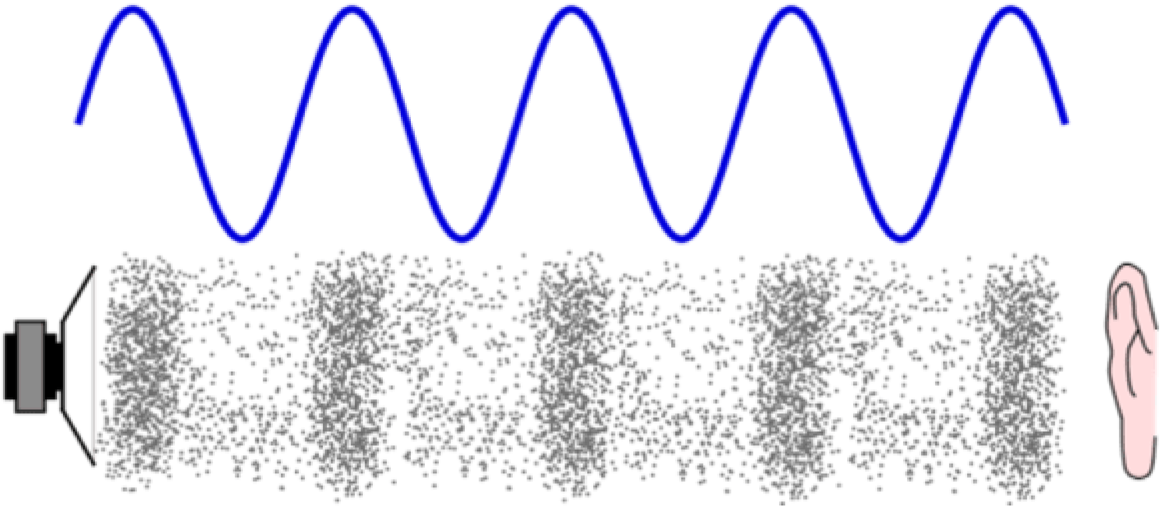
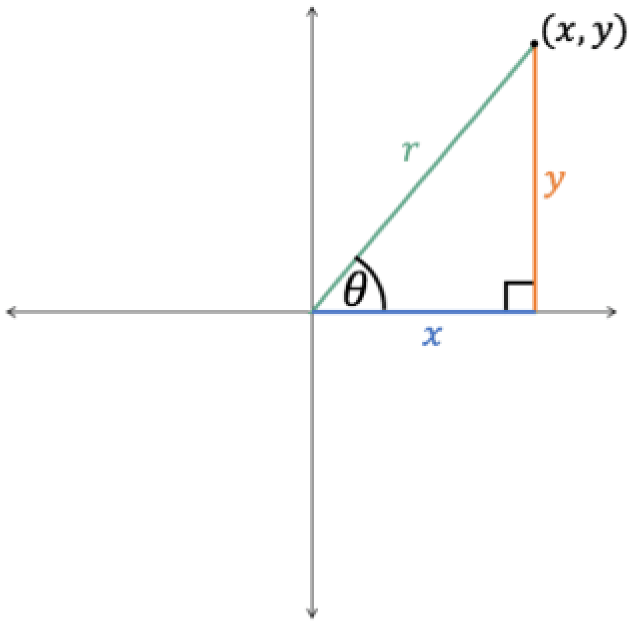


Figure 2: Sound waves modelled as a sine wave

These waves can be modelled mathematically using a trigonometric function called sine. If we can model the wave using a function, it means we can manipulate it in all sorts of ways. Computers, being particularly good at maths, can make very complicated changes to very complex waves quickly and easily.

This is just one example of where trigonometric functions are used in real life. Another example is an engineer or architect calculating forces that are at an angle that will be experienced by physical structures like buildings or bridges.

In this unit, we will explore the trigonometric functions called sine $(\sin)$, cosine $(\cos)$ and tangent $(\tan)$. All three functions are based on how the ratios of the length of sides in a right-angled triangle change as the angle inside the triangle changes. Figure 3 shows a summary.



$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{r}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x}$$

Figure 3: Summary of the three basic trigonometric ratios

- $\sin\theta$ is the ratio of the length of the side **opposite** the angle θ and the **hypotenuse** (the side opposite the right angle). Another way of saying this is that it is the value of the y-coordinate, of the point made by the radius that is r units long and that is θ° away from the positive x-axis, divided by r.
- $\cos\theta$ is the ratio of the length of the side **adjacent** (next to) the angle θ and the **hypotenuse**. Another way of saying this is that it is the value of the x-coordinate, of the point made by the radius that is r units long and that is θ° away from the positive x-axis, divided by r.
- $\tan\theta$ is the ratio of the length of the side **opposite** the angle θ and the length of the side **adjacent** to the angle. Another way of saying this is that it is the value of the y-coordinate, of the point made by the radius that is r units long and that is θ° away from the positive x-axis, divided by the x-coordinate of this point.

Note

If you need more help understanding these three basic trigonometric ratios, watch the introductory video called “Basic trigonometry” or complete Subject outcome 3.6, Unit 1: Trigonometric ratios before continuing with this unit.

Basic trigonometry (Duration: 9.16)

The sine and cosine functions

In this unit, we will deal with the sine and cosine functions together because they are very similar. As we will see, the cosine function is really just the sine function shifted horizontally by 90°.

Activity 5.1: Investigate the sine and cosine functions

Time required: 60 minutes

What you need:

- a pen or pencil
- a calculator
- blank paper or a notebook
- a ruler
- a protractor
- an internet connection (highly recommended)

What to do:

Part A

1. On a piece of paper, draw a large Cartesian plane. Each axis should be at least 15 cm long.
2. Now put your protractor on the origin and measure an angle of 30° from the positive x-axis and mark this point. Draw a line 10 cm long from the origin going through the point you marked off. We will call this line the radius.
3. Draw a perpendicular line from the end of the radius to the x-axis to create a right-angled triangle with 30° at the origin.
4. Create a table like this on another piece of paper.

θ	30°	45°	60°	90°	120°	135°	150°	180°	210°	240°	270°	300°	330°	360°
y														
$\frac{y}{r}$														

5. Measure the length of the side opposite the 30° angle. This gives you the y-coordinate of the point made by the end of the radius. Write this in the table under the 30°.
6. Now measure an angle of 45° from the positive x-axis, draw another radius 10 cm long, drop a perpendicular line, measure the length of this perpendicular line to determine the y-coordinate of the point at the end of the radius and fill it into your table.
7. Complete the rest of the table in the same way. Just remember that when the y-coordinate at the end of your radius is below the x-axis, you need to show that this is a negative value in your table.
8. Once the second row of your table is complete, calculate the values in the third row. Remember that in every case $r = 10$.
9. Now draw a Cartesian plane like the one in Figure 4 and plot the points from your table on it.
10. Finally, join these points with as smooth a 'wavy' curve as you can.

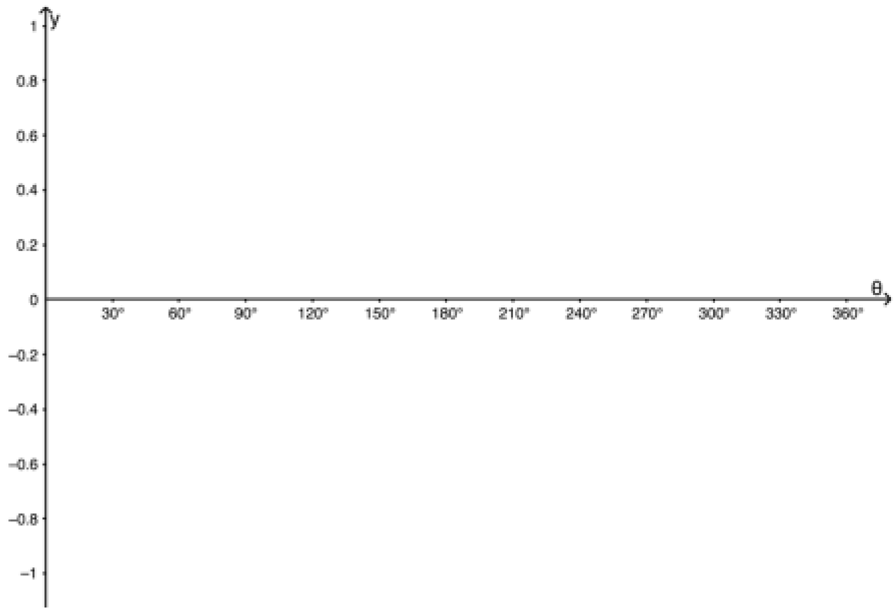


Figure 4: Cartesian plane template

You have just drawn the sine function $f(\theta) = \sin\theta$ for $0^{\circ} \leq \theta \leq 360^{\circ}$. You plotted the value of the ratio $\frac{y}{r}$ as θ increased from 0° to 360°.

11. Now answer these questions about the graph you have just drawn.
    1. What is the maximum value the function reaches? What is the minimum value?
    2. What are the turning points of the graph?
    3. What do you think happens to the function if we continue past 360° or to the left of 0°?
    4. What is the domain and range of the function $f(\theta) = \sin\theta$?

Part B

1. Make a copy of this table on a piece of paper

θ	30°	45°	60°	90°	120°	135°	150°	180°	210°	240°	270°	300°	330°	360°
x														
$\frac{x}{r}$														

2. Go back to your first Cartesian plane and now measure the length of the line adjacent to the angle θ in each case. Remember, the adjacent line is the line next to the angle that is **not** the radius. Fill these values into the second row of your table.
3. Calculate the values in the third row of the table.
4. Now draw a Cartesian plane like the one in Figure 4 and plot the points from your table on it.
5. Finally, join these points with as smooth a 'wavy' curve as you can. You have just drawn the cosine function $f(\theta) = \cos\theta$ for $0^{\circ} \leq \theta \leq 360^{\circ}$. You plotted the value of the ratio $\frac{x}{r}$ as θ increased from 0° to 360°.

6. Now answer these question about the graph you have just drawn.
   a. What is the maximum value the function reaches? What is the minimum value?
   b. What are the turning points of the graph?
   c. What do you think happens to the function if we continue past 360° or to the left of 0°?
   d. What is the domain and range of the function $f(\theta) = \cos\theta$?

Part C

1. When you have an internet connection, visit the sine function interactive simulation.
   Here you will see a unit circle similar to the one you created in Part A and a slider with which you can change the value of the angle θ. Drag the slider and notice how the function value of the graph of $f(\theta) = \sin\theta$ always tracks the length of the side **opposite** the angle.
2. Now visit the cosine function interactive simulation.
   Here you will find a unit circle similar to the one you used in Part B and a slider with which you can change the value of the angle θ. Drag the slider and notice how the function value of the graph of $f(\theta) = \cos\theta$ always tracks the length of the side **adjacent** to the angle.

What did you find?

Part A

Figure 5 shows what the various triangles you drew should look like. The figure indicates the lengths of the sides opposite the angle as well as the x- and y-coordinates of the points formed by the radius and the angle. We call a circle like this a unit circle. In our case, $10\ \text{cm} = 1\ \text{unit}$.

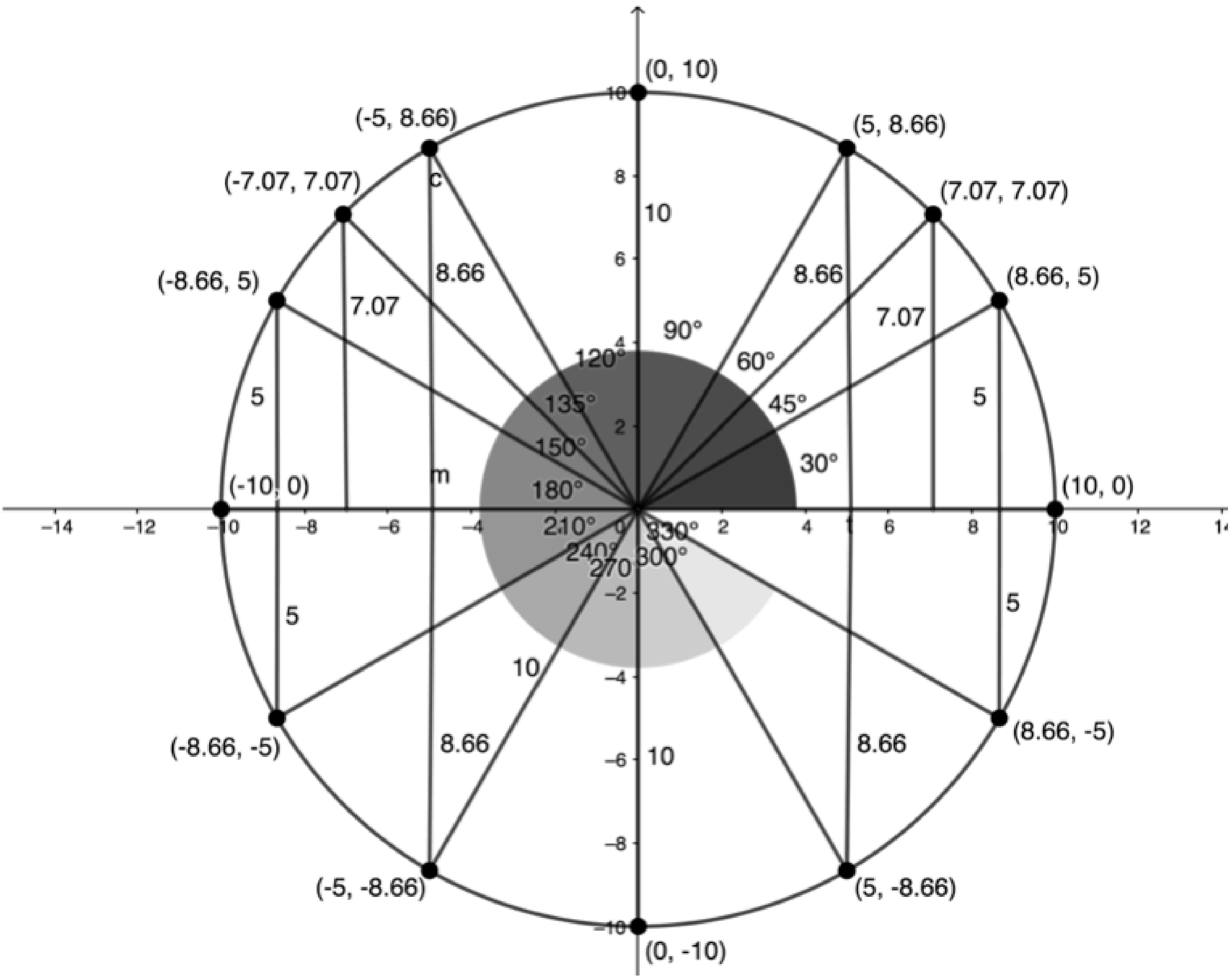


Figure 5: A unit circle

Table 1 shows what your completed table should look like.

Table 1: Plotting $\sin\theta$

θ	30°	45°	60°	90°	120°	135°	150°
y	5	7.07	8.66	10	8.66	7.07	5
$\frac{y}{r}$	$\frac{1}{2}$	0.707	0.866	1	0.866	0.707	$\frac{1}{2}$

Table 1 continued: Plotting $\sin\theta$

θ	180°	210°	240°	270°	300°	330°	360°
y	0	-5	-8.66	-10	-8.66	-5	0
$\frac{y}{r}$	0	$-\frac{1}{2}$	-0.866	-1	-0.866	$-\frac{1}{2}$	0

When you plot the coordinates, formed by the values of θ and $\frac{y}{r}$, you get the graph shown in Figure 6.

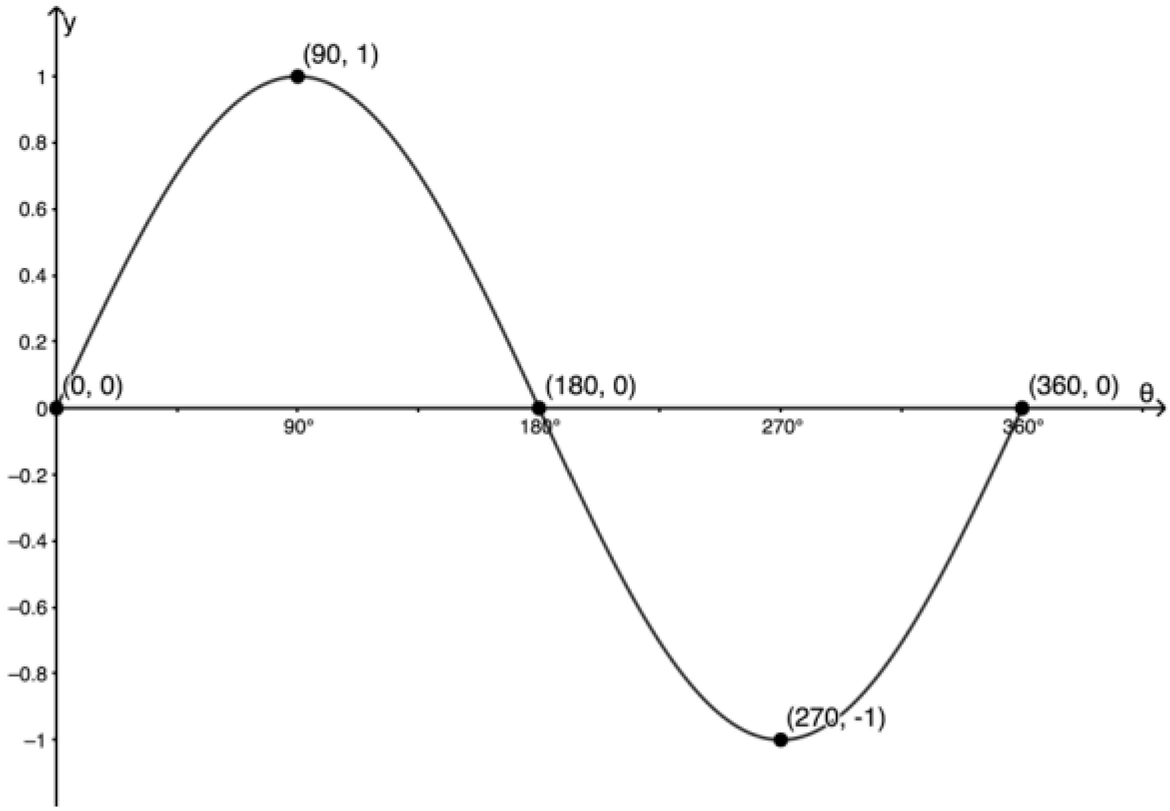


Figure 6: Graph of $f(\theta) = \sin\theta$ *for* $0^\circ \leq \theta \leq 360^\circ$

11. This is what you should notice about the graph.
    a. The maximum value the graph reaches is 1. Its minimum value is -1.
    b. The graph has a maximum turning point at $(90^\circ, 1)$ and a minimum turning point at $(270^\circ, -1)$.
    c. If we continue past 360° or 0° the graph will repeat itself. This is because if we keep measuring angles on our unit circle, we will keep generating the same function values. An angle of 390° will give a triangle on the unit circle exactly the same as 30°. If we measured in the other direction, an angle of -30° for example, it would be the same as an angle of 330°.
    d. There are no values that θ cannot take. Therefore, the domain of the function is $\{\theta \mid \theta \in \mathbb{R}\}$. The range of the function is all values between -1 and 1. Therefore the range is $\{f(\theta) \mid f(\theta) \in \mathbb{R}, \text{-}1 \leq y \leq 1\}$.

Part B

Table 2 shows the completed table of values of the length of the side **adjacent** to the angle in each case.

Table 2: Plotting $\cos\theta$

θ	30°	45°	60°	90°	120°	135°	150°
x	8.66	7.07	5	0	8.66	7.07	8.66
$\frac{x}{r}$	0.866	0.707	0.5	0	-0.866	-0.707	-0.866

Table 2 continued: Plotting $\cos\theta$

θ	180°	210°	240°	270°	300°	330°	360°
x	-10	-8.66	5	0	5	-8.66	10
$\frac{x}{r}$	-1	-0.866	-0.5	0	0.5	0.866	1

When you plot the coordinates, formed by the values of θ and $\frac{x}{r}$, you get the graph shown in Figure 7.

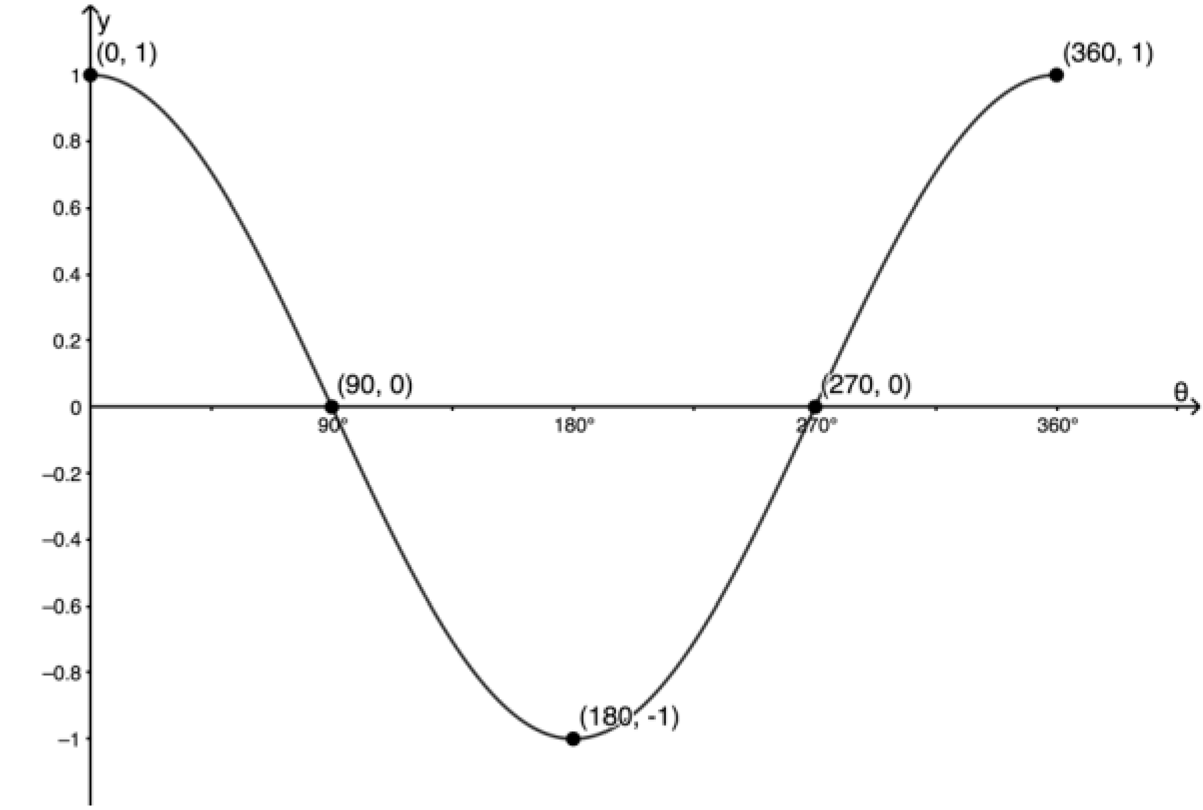


Figure 7: Graph of $f(\theta) = \cos\theta$ *for* $0^\circ \leq \theta \leq 360^\circ$

6. This is what you should notice about the graph.
   a. The maximum value the graph reaches is 1. Its minimum value is -1.
   b. The graph has maximum turning points at $(0^\circ, 1)$ and $(360^\circ, 1)$, and a minimum turning point at $(180^\circ, -1)$.
   c. If we continue past 360° or 0° the graph will repeat itself. This is because if we keep measuring angles on our unit circle, we will keep generating the same function values.
   d. There are no values that θ cannot take. Therefore, the domain of the function is $\{\theta \mid \theta \in \mathbb{R}\}$. The range of the function is all values between -1 and 1. Therefore the range is $\{f(\theta) \mid f(\theta) \in \mathbb{R}, -1 \leq y \leq 1\}$.

Part C

1. Th sine function interactive simulation constructed the graph of the function $f(\theta) = \sin\theta$ for $0^{\circ} \leq \theta \leq 360^{\circ}$.
2. Th cosine function interactive simulation constructed the graph of the function $f(\theta) = \cos\theta$ $0^{\circ} \leq \theta \leq 360^{\circ}$.

Amplitude and period

In Activity 5.1 we discovered the general shape of the functions of $y = \sin x$ and $y = \cos x$. We also saw that the graphs have maximum values of 1 (a maximum turning point) and minimum values of -1 (a minimum turning point) and that they 'bounce' between these limits. This is represented by the range $\{f(x) \mid f(x) \in \mathbb{R}, -1 \leq y \leq 1\}$.

We say that each of these functions has an **amplitude** of 1. Amplitude is a measure of the maximum distance of the graph from the 'middle' or 'point of rest' of the graph. In this case, the point of rest is the x-axis.

Look at Figure 8 which shows the graphs of $y = \sin x$ and $y = \cos x$ for $-720^{\circ} \leq x \leq 720^{\circ}$. Here we can clearly see how the graphs repeat themselves again and again. How many degrees does it take for each graph to repeat itself?

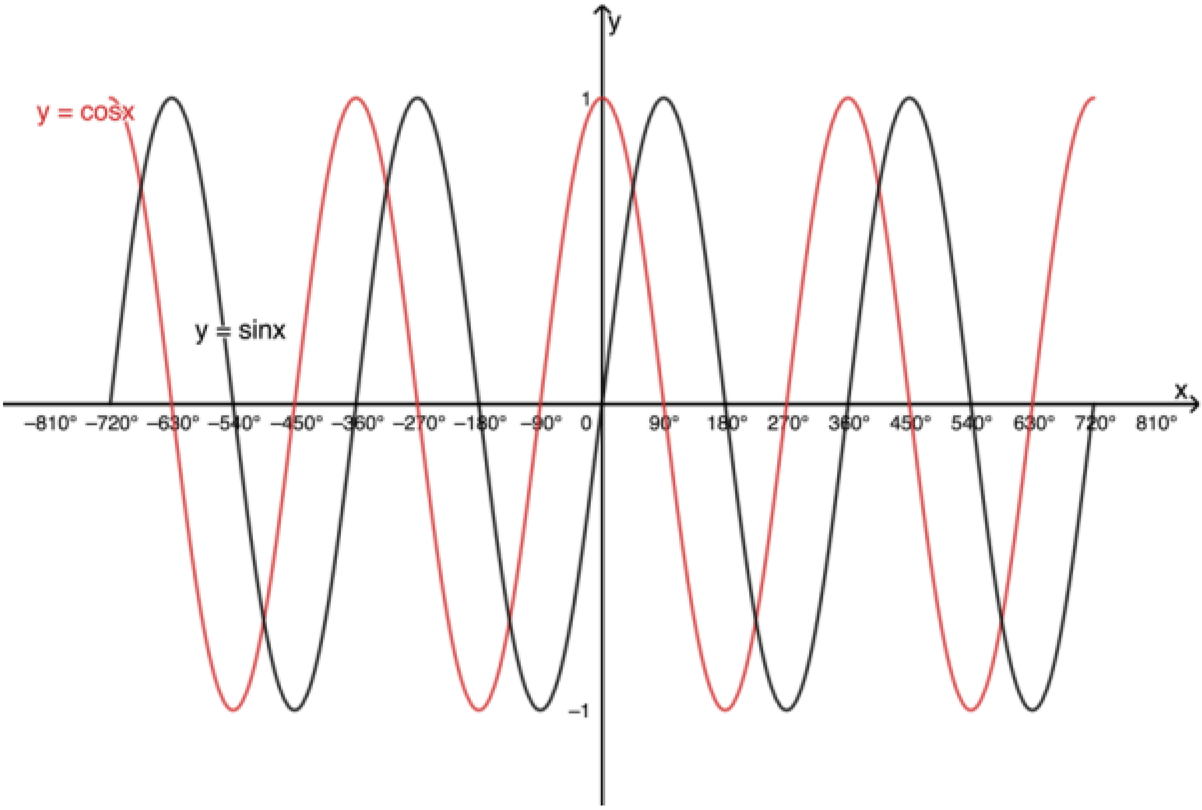


Figure 8: Graphs of $y = \sin x$ and $y = \cos x$ for $-720^{\circ} \leq x \leq 720^{\circ}$

If you look at Figure 8, you can see that each graph repeats itself every 360°. We say that each graph has a **period** of 360°.

Can you also see that both functions are exactly the same shape? The graph of $y = \cos x$ is really just the graph of $y = \sin x$ shifted 90° to the left. You will see in Level 3 why this is the case.

The effects of a and q on the graphs of y=a.sin x+q and y=a.cos x+q

Before working through Activity 5.2, what do you think the effect of changing the value of q will be on the functions of $y = \sin x + q$ and $y = \cos x + q$? Think about what changing q does to other functions we have studied.

The effect of q on the graphs of $y = a \sin x + q$ and $y = a \cos x + q$

Work through this activity to understand the effect of *q* on the sine and cosine functions.

Activity 5.2: Investigate the effect of *q* on the sine and cosine functions

Time required: 15 minutes

What you need:

- an internet connection

What to do:

When you have an internet connection, visit this interactive simulation.

Here you will find graphs of $f(x) = \sin x + q$ and $g(x) = \cos x + q$ with a slider to change the value of q. Change the value of q and see the effect on each graph. You can show and hide each function by clicking the coloured circle next to it.

Now answer these questions:

1. What effect does changing q have of each function?
2. Does changing q change the maxima and minima of the functions? If so, how?
3. Does changing q affect the amplitude of the functions?
4. Does changing q affect the period of the functions?

What did you find?

1. By changing the value of q the functions are moved vertically up and down. Increasing q, shifts the functions up by q units. Decreasing q shifts the functions down by q units.
2. Changing q does change the maximum and minimum turning points of the functions. The normal maxima and minima of the functions are 1 and -1. The new values are $1+q$ and $-1+q$.
3. The amplitude of the functions does not change. The distance from the middle of the graph to the maximum or minimum turning point is still 1. However, the position of the 'middle' of the graphs does change. This is now the straight line $y=q$.
4. Both graphs have the same period of 360°.

Example 5.1

Given $y=\sin x-1$.

1. What is the period of the function?
2. What is the amplitude of the function?
3. At what y-coordinates will the maximum and minimum turning points be?
4. State the domain and range.
5. What are the intercepts with the axes?
6. Make a neat sketch of the function for $0^{\circ} \leq x \leq 360^{\circ}$.

Solutions

We need to first consider what the basic sine function looks like (see Figure 9).

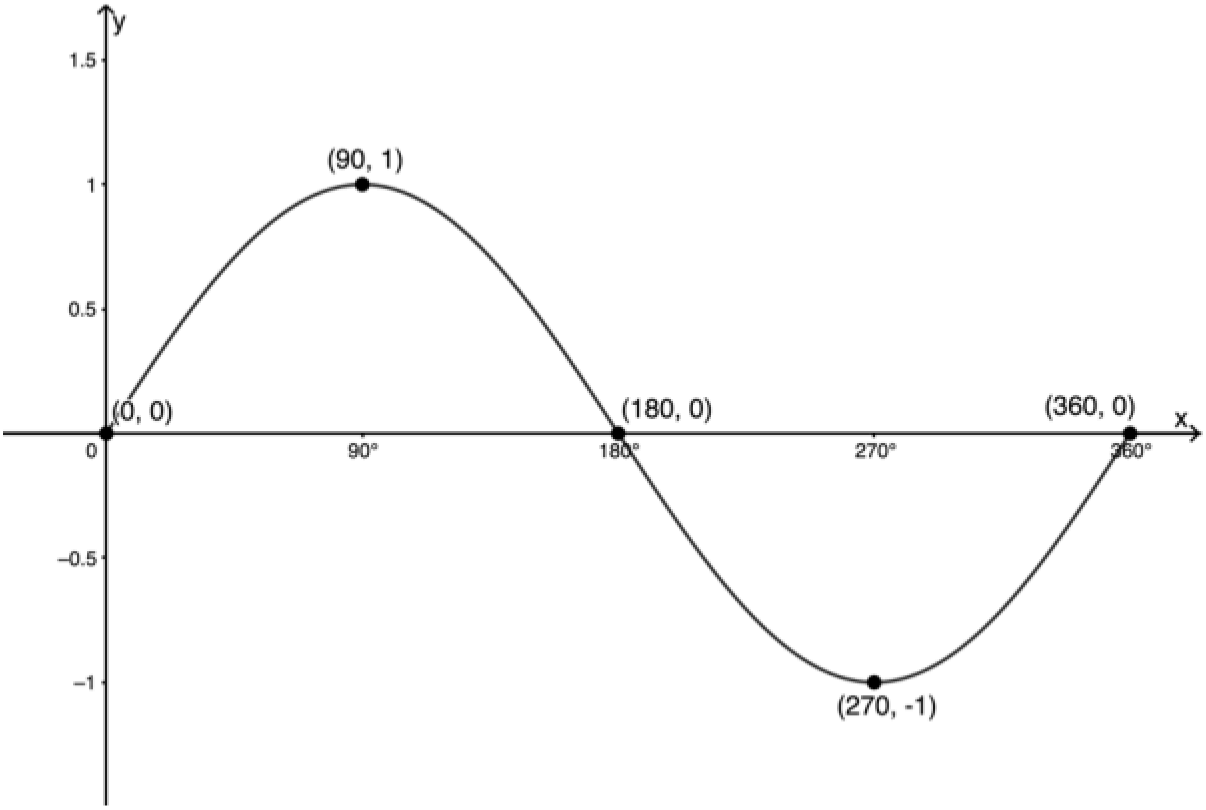


Figure 9: Graph of $y = \sin x$ *for* $0^{\circ} \leq x \leq 360^{\circ}$

We can see that the graph passes through the origin to a maximum at $(90^{\circ}, 1)$, back through the x-axis at $(180^{\circ}, 0)$ to a minimum at $(270^{\circ}, -1)$ and then back to the x-axis at $(360^{\circ}, 0)$. The graph has a period of 360° and an amplitude of 1. The 'middle' of the graph is the x-axis (the line $y = 0$).

The function we need to sketch is $y = \sin x - 1$. Here $q = -1$ and the whole graph in Figure 9 will be moved one unit down.

1. The period of the function will remain unchanged at 360°.
2. The amplitude of the function will remain unchanged at 1. However, the 'middle' of the graph will now be the line $y = -1$.
3. Because the whole graph has been moved down one unit, the maximum turning points will have y-coordinates of $1 - 1 = 0$ and the minimum turning points will have y-coordinates of $-1 - 1 = -2$.
4. Domain: $\{x \mid x \in \mathbb{R}\}$
   Range: $\{y \mid y \in \mathbb{R}, -2 \leq y \leq 0\}$
5. We find the x- and y-intercepts by letting $y = 0$ and $x = 0$.
   x-intercept (let $y = 0$):
   $$0 = \sin x - 1$$
   $$\therefore \sin x = 1$$
   $$\therefore x = 90^{\circ}$$
   The point $(90^{\circ}, 0)$ will be the x-intercept.

   y-intercept (let $x = 0$):

$y = \sin 0^{\circ} - 1$
$\therefore y = 0 - 1 = -1$
The point $(0, -1)$ will be the y-intercept.

6. The graph of $y = \sin x - 1$ is shown in Figure 10. The light dotted line represents the 'middle' of the function and the dark dotted line the function $y = \sin x$.

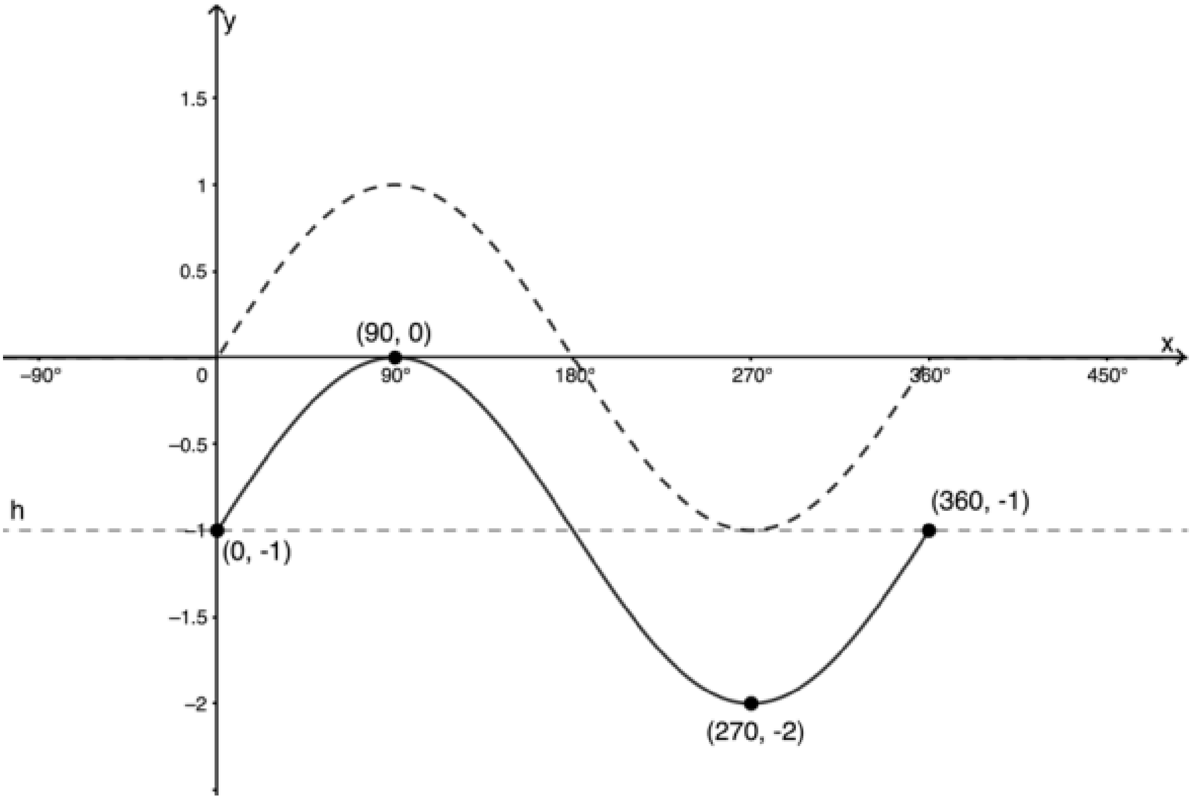


Figure 10: Graph of $y = \sin x - 1$ *for* $0^{\circ} \leq x \leq 360^{\circ}$

Note: You could also have sketched the function using a table of values or your calculator. When using a table of values, it is often only necessary to use these angles: $0^{\circ}; 90^{\circ}; 180^{\circ}; 270^{\circ}; 360^{\circ}$.

Note

In Example 5.1 it was necessary to calculate x in $\sin x = 1$. You can do this on your calculator by using the inverse sine button, usually marked as $\sin^{-1}$. For this calculation, key in $1 \rightarrow \text{2nd func} \rightarrow \sin \rightarrow \; =$ (on most Sharp calculators) or $\text{SHIFT} \rightarrow \sin \rightarrow 1 \rightarrow =$ (on most Casio calculators).

It was also necessary to calculate y in $y = \sin 0^{\circ} - 1$. You can do this on your calculator by using the sine button, marked as $\sin$. For this calculation, key in $0 \rightarrow \sin \rightarrow -1 \rightarrow \; =$.

You can also use your calculator to generate sets of ordered pairs for any function. When you have an internet connection, watch the video called "CASIO FX 991ES PLUS – Calculator skills – plotting a graph" to learn how to do this.

CASIO FX 991ES PLUS – Calculator skills – plotting a graph (Duration: 2.59)

Exercise 5.1

Make a neat sketch of the function $f(x) = \sin x + 1$ for the interval $0^{\circ} \leq x \leq 360^{\circ}$ showing the intercepts with the axes and turning points.

Make a neat sketch of the function $g(\theta) = \cos\theta - 3$ for the interval $0^{\circ} \leq \theta \leq 540^{\circ}$ showing the intercepts with the axes and turning points.

The full solutions are at the end of the unit.

We have established that the effect of q on $y = \sin x + q$ and $y = \cos x + q$ is to move the graph vertically up or down by q units. There is no change in the overall shape of the graph.

The effect of a on $y = \sin x + q$ and $y = \cos x + q$

Based on your knowledge of the effect of a on the other functions we have studied, what do you think the effect of a will be on $y = a\sin x + q$ and $y = a\cos x + q$?

Consider the following table of values (Table 3) for the function $y = \sin x$.

Table 3

θ	0°	90°	180°	270°	360°
$f(x) = \sin x$	0	1	0	-1	0

What values will go into the second row if instead we calculated the values for $g(x) = 2\sin x$? What values would go into the second row if we calculated the values for $h(x) = -3\sin x$. Before you read on, make a copy of Table 4 on a piece of paper and try to complete it on your own.

Table 4

θ	0°	90°	180°	270°	360°
$f(x) = \sin x$	0	1	0	-1	0
$g(x) = 2\sin x$					
$h(x) = -3\sin x$					

It should be clear that for $g(x)$ we simply double the values obtained for $f(x)$, and that for $h(x)$ we multiply each of these values by -3 (see Table 5).

Table 5

θ	0°	90°	180°	270°	360°
$f(x) = \sin x$	0	1	0	-1	0
$g(x) = 2\sin x$	0	2	0	-2	0
$h(x) = -3\sin x$	0	-3	0	3	0

This means that the y-coordinate of every point on $g(x) = 2\sin x$ is going to be double the y-coordinate of every point on $f(x) = \sin x$. The y-coordinate of every point on $h(x) = -3\sin x$ is going to be three times the y-coordinate of every point on $f(x) = \sin x$ and on the opposite side of the x-axis.

Figure 11 shows the graphs of these three functions. From it, we can see how $g(x) = 2\sin x$ is the same basic shape as $f(x) = \sin x$, but that the graph has been vertically stretched so that the maxima and minima are now 2 and -2 instead of 1 and -1. This means that the range of $g(x) = 2\sin x$ has been extended to be $\{g(x) \mid g(x) \in \mathbb{R}, -2 \leq g(x) \leq 2\}$. The amplitude of $g(x)$ is 2.

The domain and period of $g(x)$ are the same as for $f(x)$.

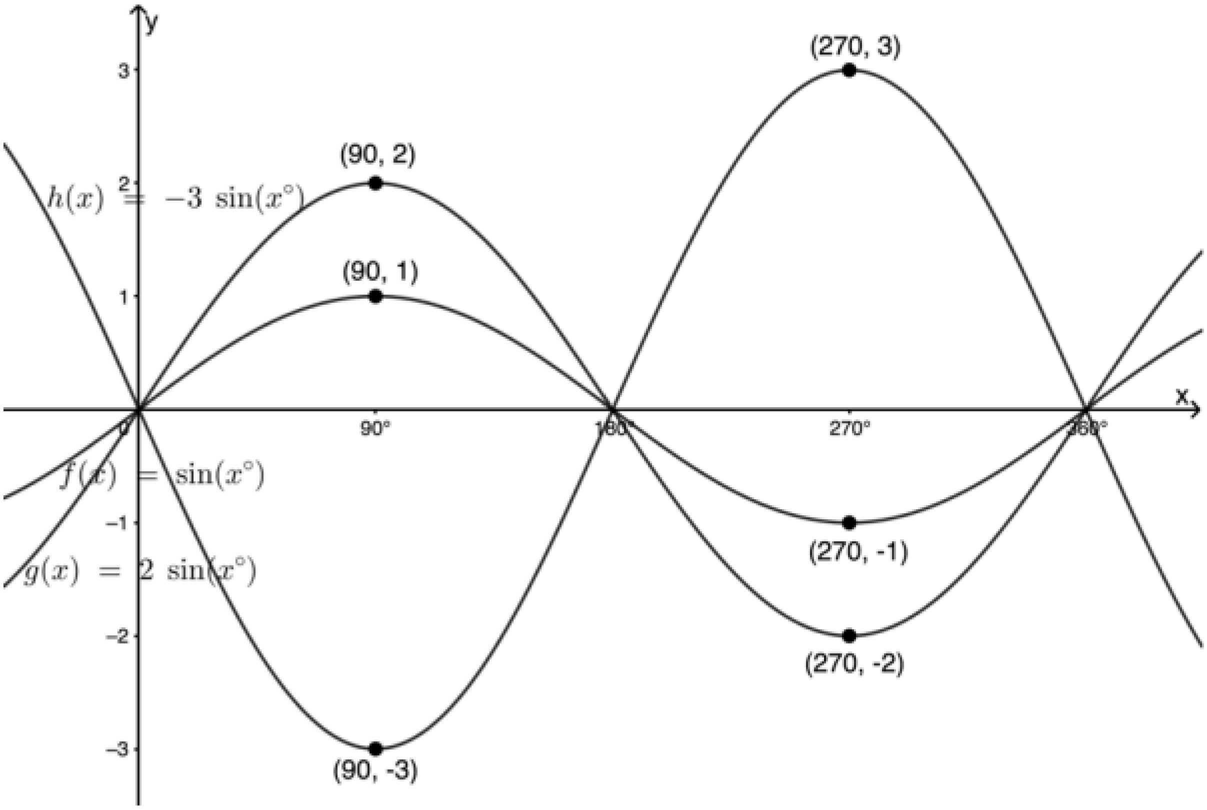


Figure 11: Graphs of $f(x) = \sin x$, $g(x) = 2\sin x$ and $h(x) = -3\sin x$

We can also see how $h(x) = -3\sin x$ is the same basic shape as $f(x)$, but that the graph has been vertically stretched so that the maxima and minima are now 3 and -3 instead of 1 and -1. Because $a < 0$ it also means that the graph has been flipped over the x-axis. What was a maximum value is now a minimum.

This means that the range of $h(x) = -3\sin x$ has been extended to $\{h(x) \mid h(x) \in \mathbb{R}, -3 \leq h(x) \leq 3\}$. The amplitude of $h(x)$ is 3.

The domain and period of $h(x)$ are the same as for $f(x)$.

Naturally, the effect of a on the function $y = a\cos x$ will be the same. The graph will be stretched vertically and the new maxima and minima will be multiples of a. Also, if $a < 0$, the graph will be flipped over the x-axis and what was a maximum value will become a minimum value.

Before you look at the next example, visit this interactive simulation.

Here you will find graphs of $f(x) = a\sin x + q$ and $g(x) = a\cos x + q$ with sliders to change the values of a and q. Change the values to see the effect on each graph. You can show and hide each function by clicking the coloured circle next to it.

Example 5.2

Given $s(x) = 3\cos x - 1$.

1. What is the period of the function?
2. What is the amplitude of the function?
3. Find the y-coordinates of the maximum and minimum turning points?
4. State the domain and range.
5. What are the intercepts with the axes (to one decimal place)?
6. Make a neat sketch of the function for $0^\circ \leq x \leq 360^\circ$, marking the turning points and intercepts.

Solutions

1. The period is 360°.
2. The absolute value of a is 3 therefore, we can say the amplitude of the function is 3.
3. $q = -1$. Therefore, the whole graph will be shifted one unit down. The 'middle' of the graph will be the line $y = -1$. This means that the maxima will be at $3 - 1 = 2$ and the minima will be at $-3 - 1 = -4$.
4. Domain: $x \in \mathbb{R}$
   Range: $y \in [-4, 2]$
   Note: These shortened representations of the domain and range are perfectly acceptable. The range has been represented in interval notation where the square brackets indicate that the end values are included. Interval notation assumes that all real values between the end values are included.
5. x-intercept (let $y = 0$):
   $$0 = 3\cos x - 1$$
   $$\therefore 3\cos x = 1$$
   $$\therefore \cos x = \frac{1}{3}$$
   $$\therefore x = 70.5^\circ$$

The point $(70.5^\circ, 0)$ will be the x-intercept. y-intercept (let $x = 0$):

$y = 3\cos 0^\circ - 1$

$\therefore y = 3 - 1 = 2$

The point $(0, 2)$ will be the y-intercept.

Figure 12 shows a sketch of the function.

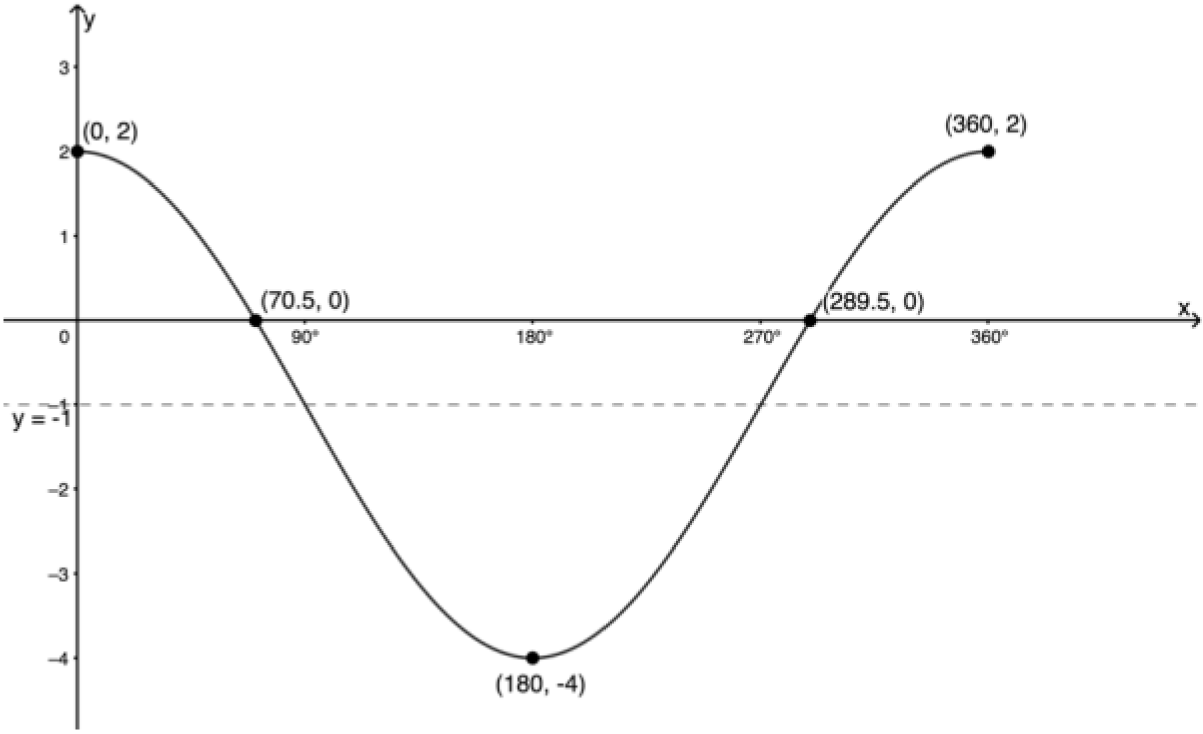


Figure 12: Graph of $s(x) = 3\cos x - 1$ for $0^\circ \leq x \leq 360^\circ$

Note

You will notice in Figure 12 that a second x-intercept of $(289.5^\circ, 0)$ is shown that was not calculated in the example. You will learn how to calculate the general solution for trigonometric equations in Level 3.

Exercise 5.2

1. Sketch $y = -2\sin x - 2$ for $x \in [0^\circ, 180^\circ]$ showing all the intercepts with the axes and the maxima and minima.
2. Sketch $y = -\dfrac{1}{2}\cos x + 1$ for $x \in [0^\circ, 180^\circ]$ showing all the intercepts with the axes and the maxima

and minima.

The full solutions are at the end of the unit.

Figures 13 and 14 show a summary of what we know about the effects of a and q on the functions $y = a\sin x + q$ and $y = a\cos x + q$.

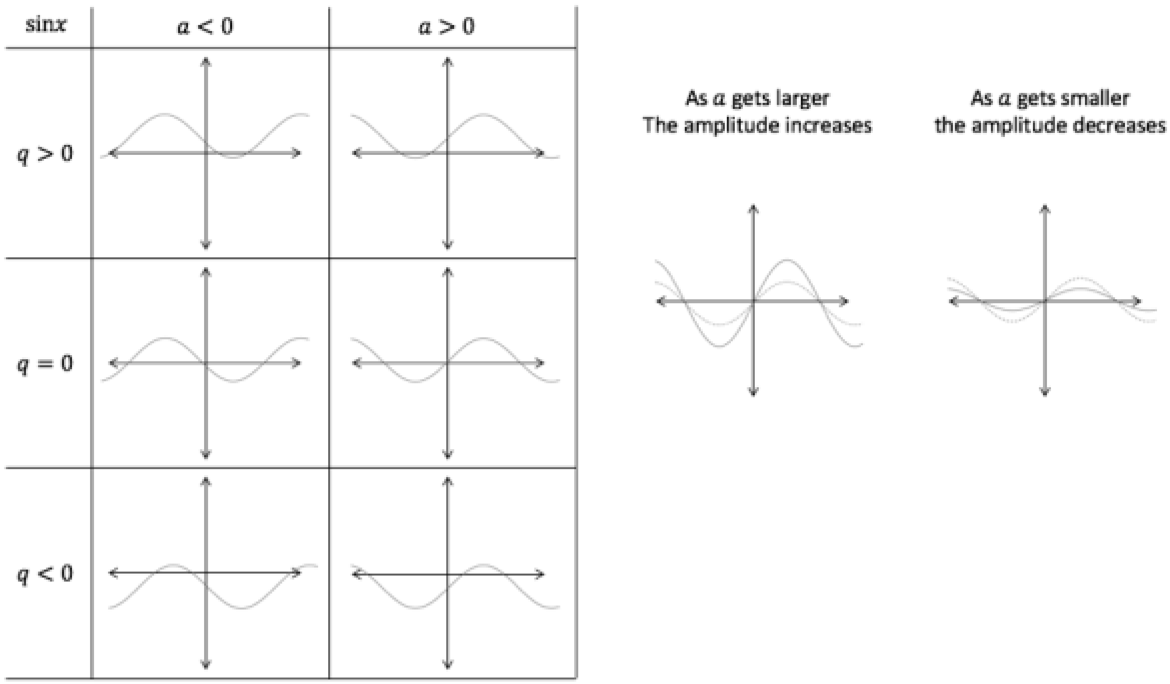


Figure 13: Effects of a and q on the function $y = a\sin x + q$

For $y = a\sin x + q$
Period: 360°
Amplitude: $|a|$ (the absolute value of a)
Domain: $x \in \mathbb{R}$
Range: $y \in [a + q, -a + q]$

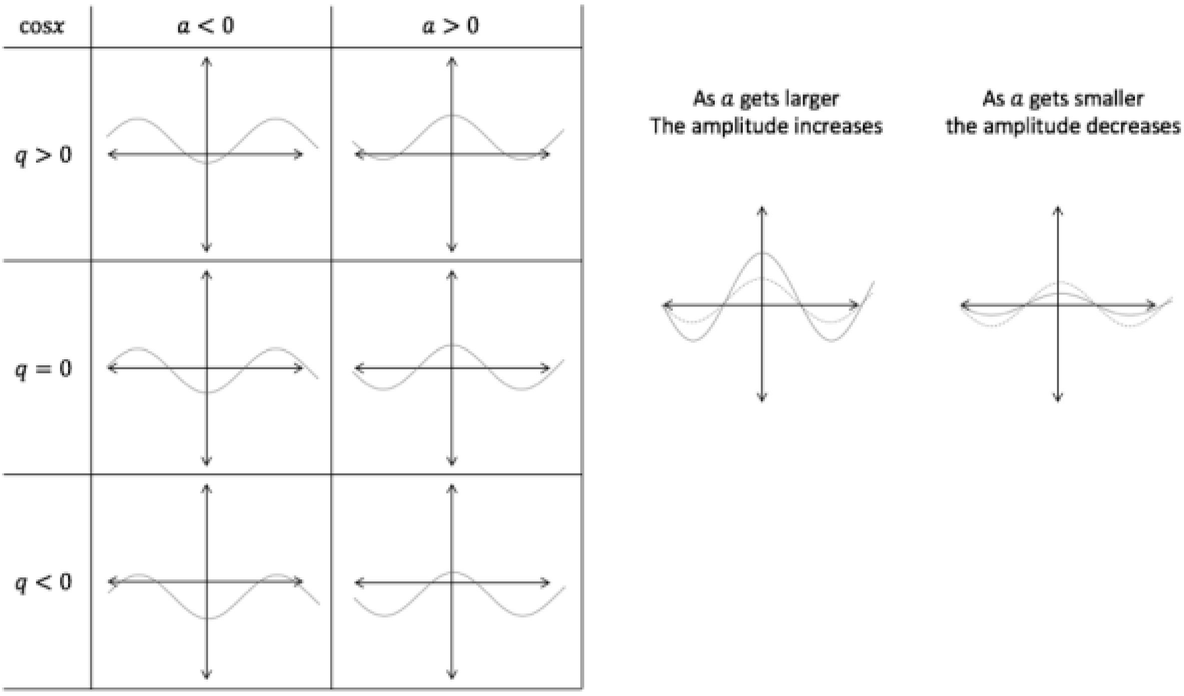


Figure 14: Effects of a and q on the function $y = a\cos x + q$

For $y = a\cos x + q$
Period: 360°
Amplitude: $|a|$ (the absolute value of a)
Domain: $x \in \mathbb{R}$
Range: $y \in [a + q, -a + q]$

The tangent function

We know that the tangent function is the ratio of the length of the side opposite the angle to the length of the side adjacent to the angle. In other words $\tan\theta = \dfrac{y}{x}$. But if we consider the point (x, y) as the point at the end of a radius drawn at an angle of θ from the positive x-axis at the origin in the unit circle, we can see that the tangent function is actually the gradient of this radius (see Figure 15).

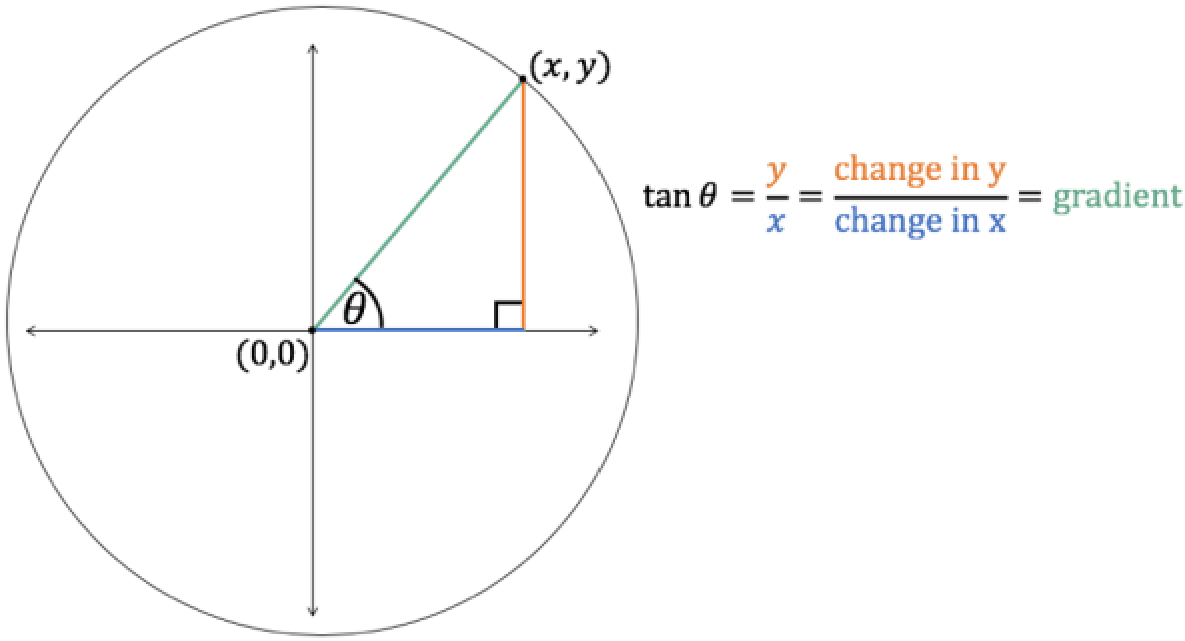


Figure 15: The tangent function as the gradient of the radius in the unit circle

The basic tangent function

Let's think about how the gradient or slope of the radius in Figure 15 changes as we increase the angle from 0° to 360°. If you have an Internet connection, visit the Interactive tangent function simulation.

Drag the green slider to change the angle. Notice how the gradient of the radius in the unit circle corresponds to the function value.

When the angle the radius of the unit circle makes with the x-axis is 0°, the slope of the radius is also zero and, therefore, $\tan 0^{\circ} = 0$ (see Figure 16). As the angle increases, so does the slope and so does the value of $\tan\theta$. When $\theta = 45^{\circ}$, the slope of the radius is 1 and so $\tan 45^{\circ} = 1$ (see Figure 17). However, as θ approaches 90°, the slope of the radius increases and, as it gets more and more vertical, it approaches infinity. Therefore, the graph of the tangent function has an asymptote at $\theta = 90^{\circ}$ because $\tan 90^{\circ}$ is undefined (see Figure 18).

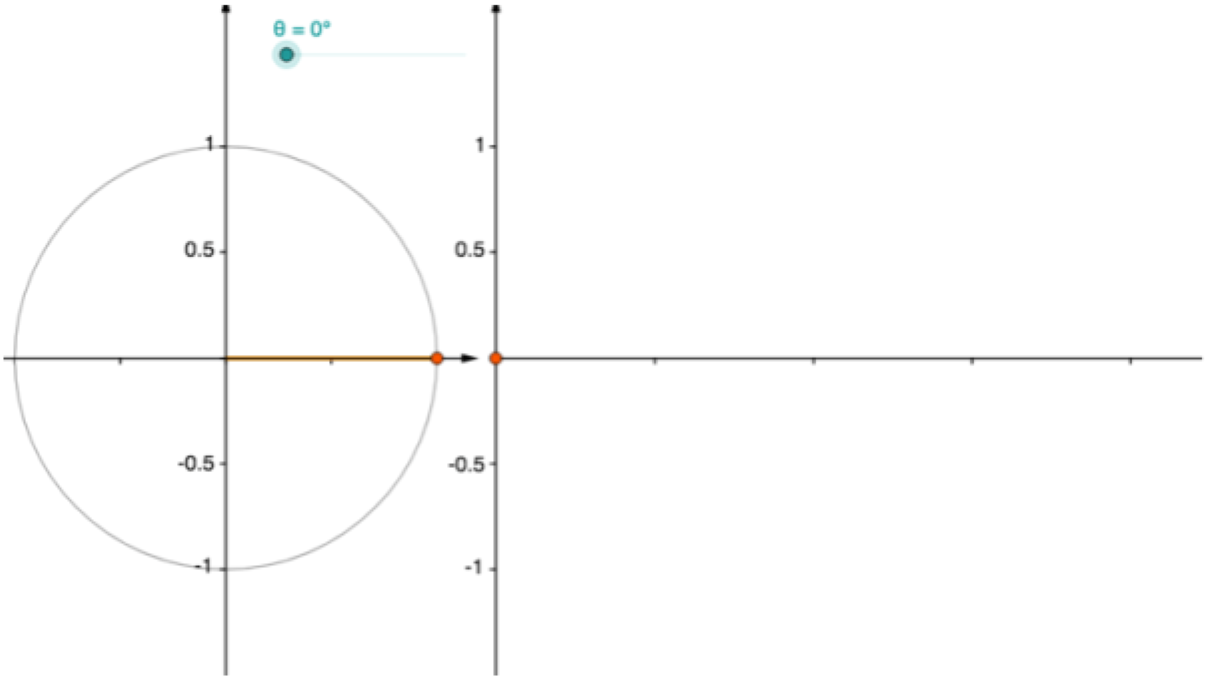


Figure 16: Tangent function value when $\theta = 0^{\circ}$

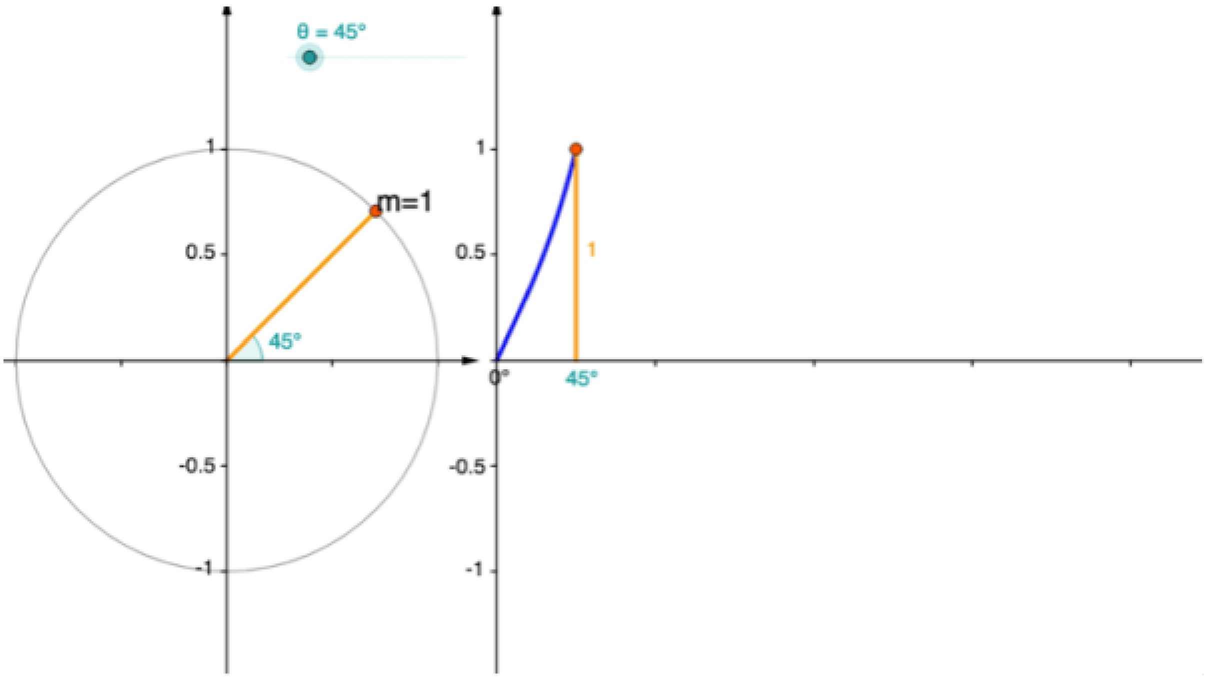


Figure 17: Tangent function value when $\theta = 45^{\circ}$

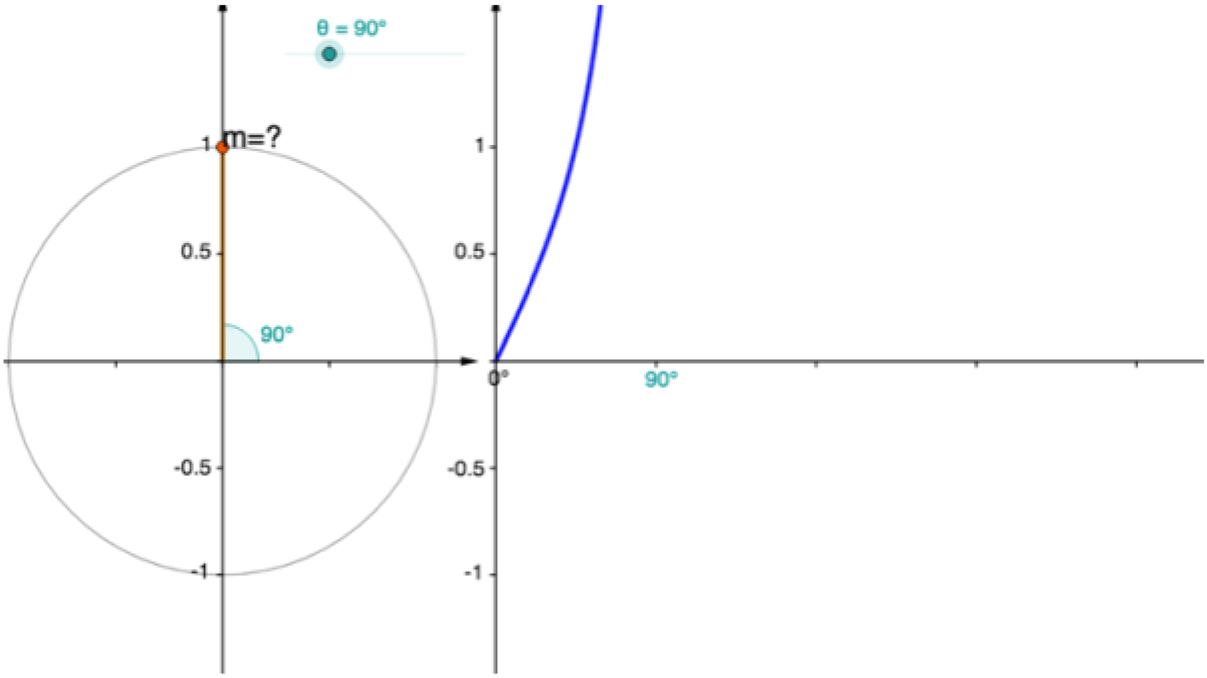


Figure 18: Tangent function value when $\theta = 90^{\circ}$

On the other side of $\theta = 90^{\circ}$, the gradient of the line suddenly switches to being very large negative but gradually decreases until it is -1 when $\theta = 135^{\circ}$ (see Figure 19). When $\theta = 180^{\circ}$, the slope is again zero and so $\tan 180^{\circ} = 0$ (see Figure 20).

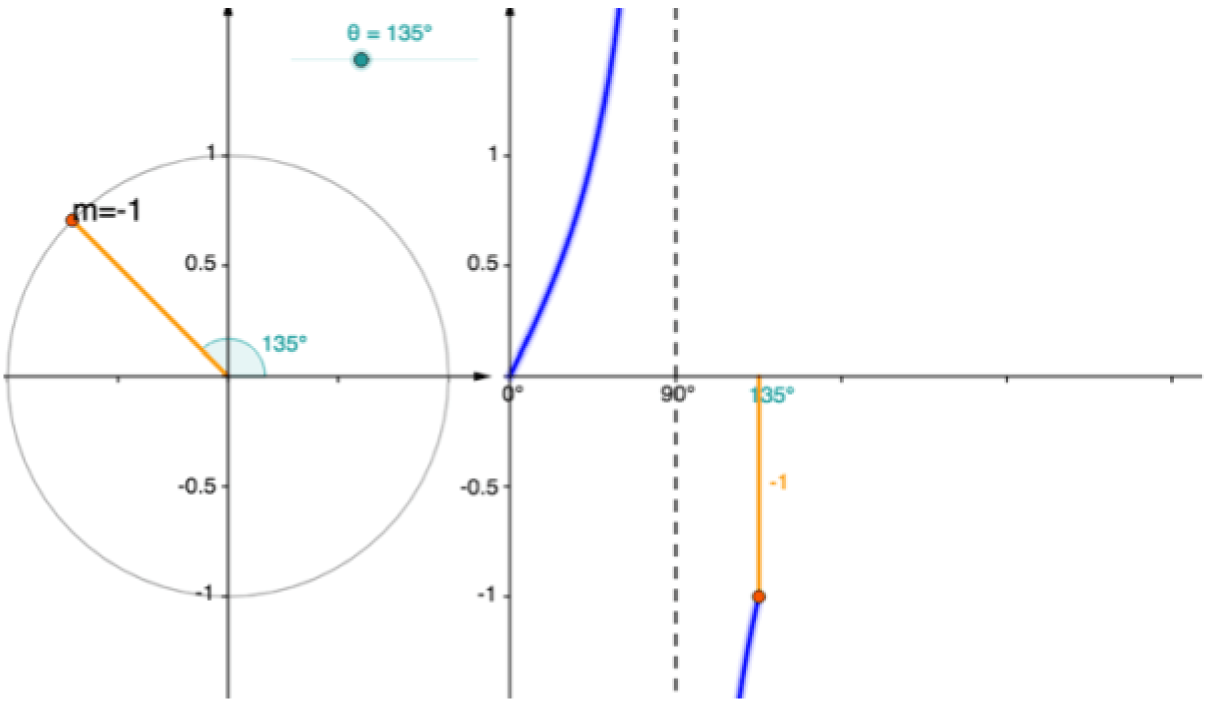


Figure 19: Tangent function value when $\theta = 135^{\circ}$

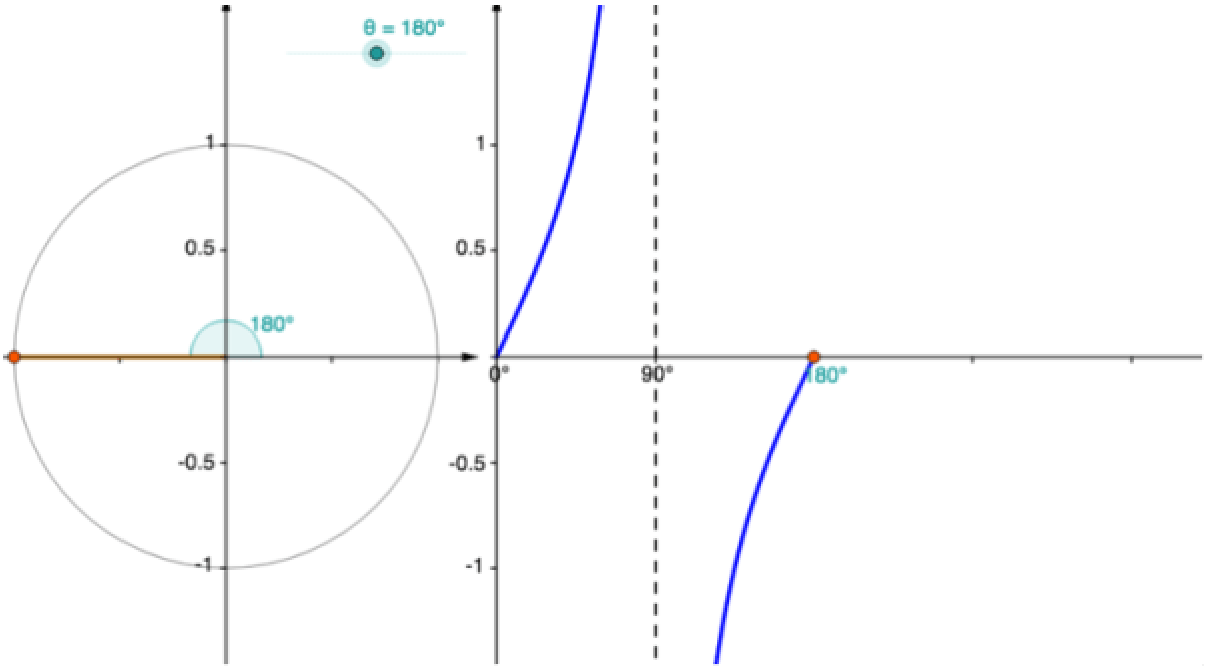


Figure 20: Tangent function value when $\theta = 180^{\circ}$

As θ continues to increase, the gradient becomes positive again and increases to a value of 1 when $\theta = 225^{\circ}$ (see Figure 21). As θ approaches 270°, the slope again approaches infinity and we have another asymptote at 270° (see Figure 22). After this, the slope again switches to being negative and $\tan 315^{\circ} = -1$ and $\tan 360^{\circ} = 0$ (see Figure 23).

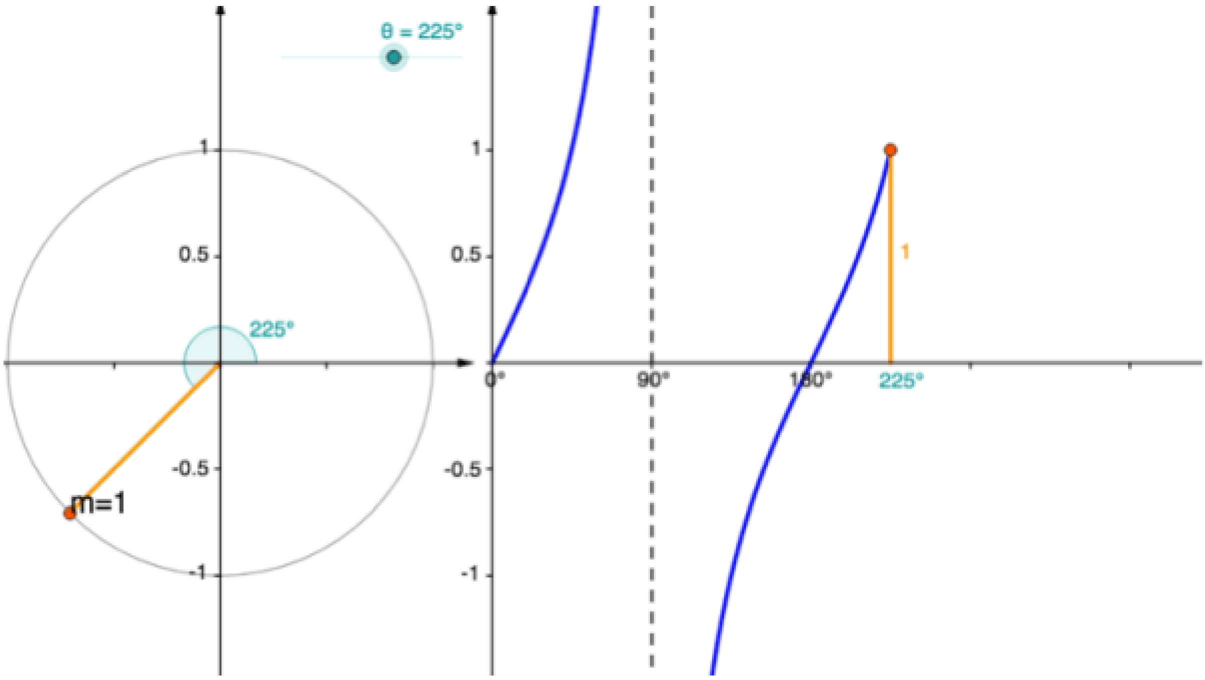


Figure 21: Tangent function value when $\theta = 225^{\circ}$

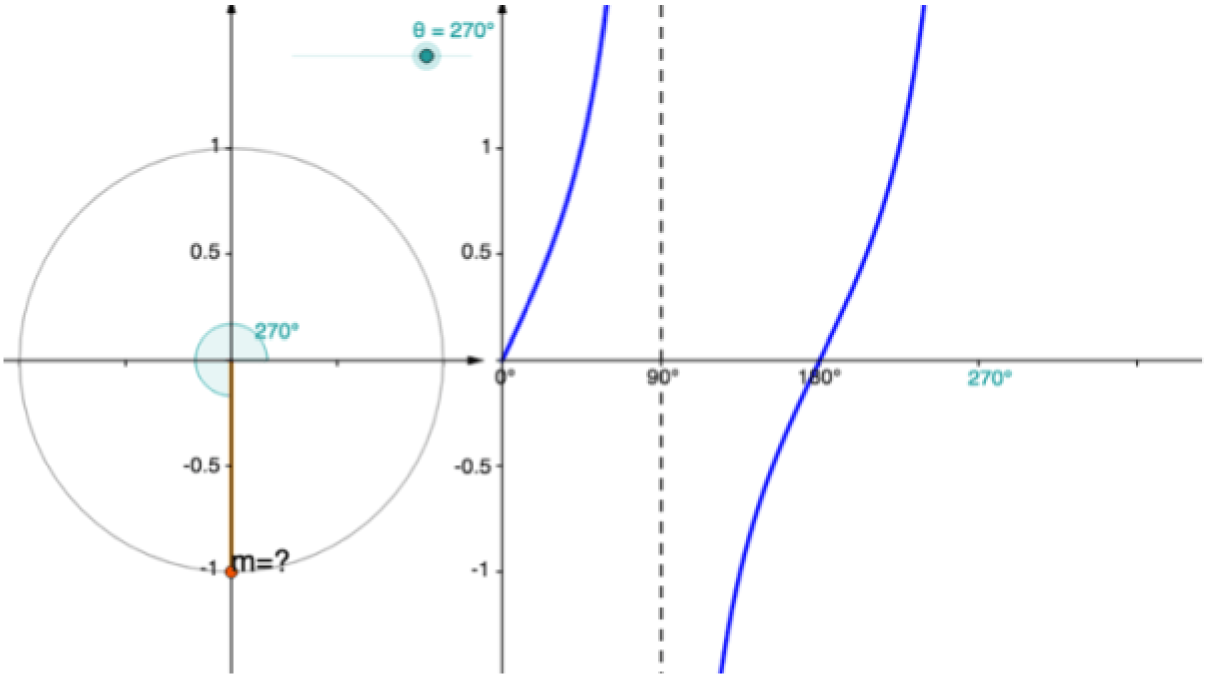


Figure 22: Tangent function value when $\theta = 270^{\circ}$

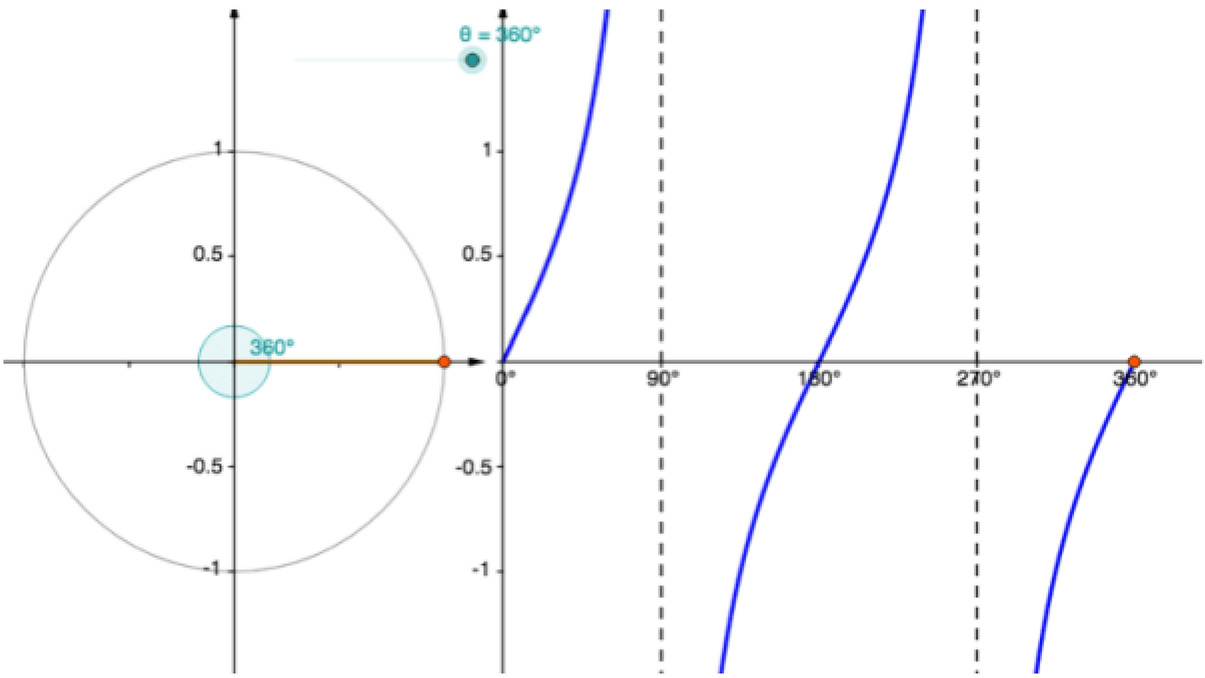


Figure 23: Tangent function value when $\theta = 360^{\circ}$

Domain, range, amplitude and period

In Figure 24 we can see the graph of the function $y = \tan x$ for $0^{\circ} \leq \theta \leq 360^{\circ}$. The period of the function is 180°. It repeats itself every 180°.

The graph in Figure 24 has vertical asymptotes at $\theta = 90^{\circ}$ and $\theta = 270^{\circ}$, the domain of the function is all real values except 90° and 270° in this interval. We write the domain as $\{\theta \mid \theta \in \mathbb{R}, 0^{\circ} \leq \theta \leq 360^{\circ}, \theta \neq 90^{\circ}; 270^{\circ}\}$. In general, we can write the domain as $\{\theta \mid \theta \in \mathbb{R}, \theta \neq (2k-1)90^{\circ}, k \in \mathbb{Z}\}$. This might look long and complicated but all it says is that we have excluded every odd multiple of 90°.

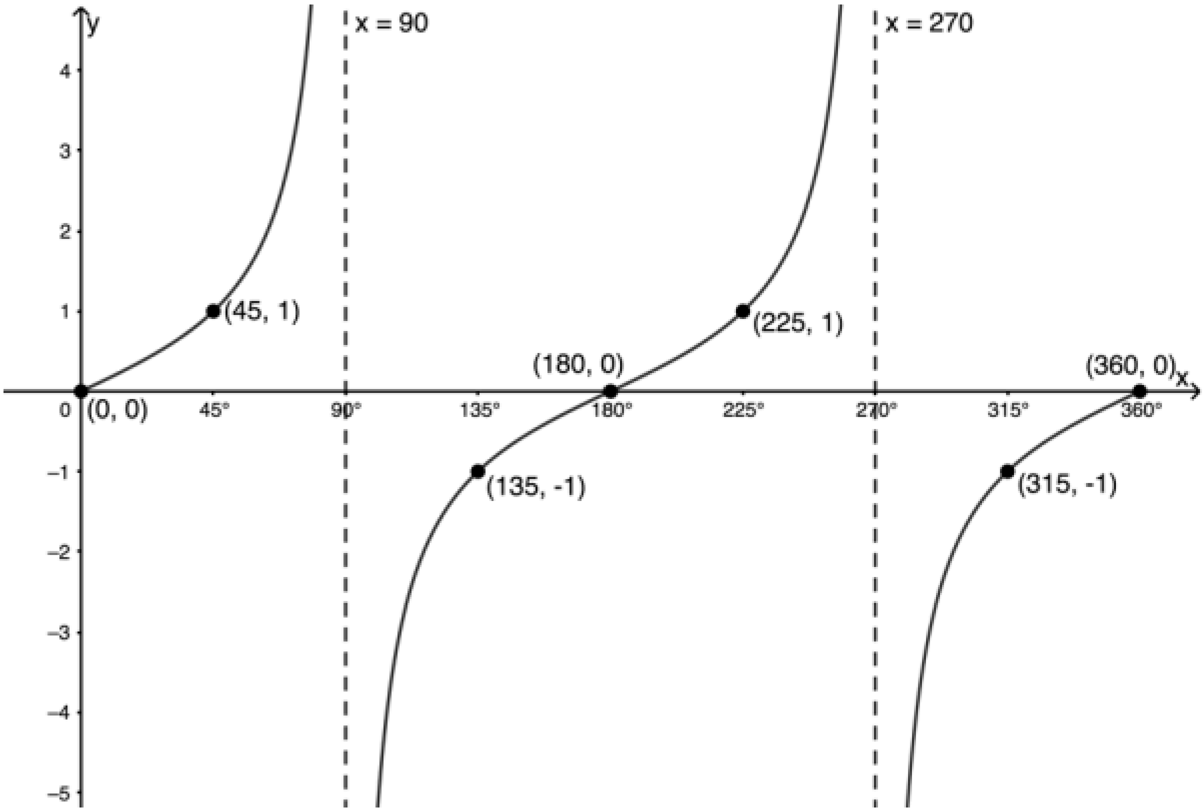


Figure 24: Graph of $y = \tan x$ *for* $0^{\circ} \leq \theta \leq 360^{\circ}$

Because the asymptotes are vertical, they do not impact the values that the function can have, hence the range is $y \in \mathbb{R}$. For this reason, the graph of the tangent function does not have a defined amplitude.

The graph has intercepts with the axes but it also has some other important 'anchor points' that help when sketching the graph as we shall see. These are the points $(45^{\circ}, 1)$, $(135^{\circ}, -1)$,$(225^{\circ}, 1)$ and $(315^{\circ}, -1)$.

The effects of a and q on the graph of y=a.tan x+q

The effects of a and q on the tangent function are quite similar to those on the sine and cosine functions even if the graph looks very different.

Changing q moves the whole graph vertically up or down by q units. Changing a stretches or squashes the graph vertically. If $a < 0$, the graph is flipped over the x-axis.

Activity 5.3: Investigate the effect of *a* and *q* on the tangent function

Time required: 15 minutes

What you need:

- an internet connection

What to do:

When you have an internet connection, visit this interactive simulation.

Here you will find a graph of $f(x) = a \tan x + q$ with sliders to change the value of a and q. Change the value of q only.

1. What affect does changing q have on the function and, in particular, the value of the seven labelled anchor points?
2. Does changing q affect the period of the function?
3. Does changing q affect the asymptotes of the function?

Set q back to zero and change the value of a.

4. What affect does increasing a have on the function and in particular the value of the seven labelled anchor points?
5. What affect does $a < 0$ have on the function and in particular the value of the seven labelled anchor points?
6. Does changing a affect the period of the function?
7. Does changing a affect the asymptotes of the function?

Now change the values of a and q together.

8. What affect will $a = -2$ and $q = -1$ have on the graph and, in particular, the value of the seven anchor points? Make the changes to see if you are correct.

What did you find?

1. If we increase q the graph moves up by q units. If we decrease q, the graph moves down by q units. The y-coordinates of all seven anchor points increase or decrease by q units. The overall shape of the graph stays the same.
2. Changing q does not affect the period of the graph. It remains 180°.
3. Changing q does not affect the asymptotes. These remain at 90° and 270°. This is in keeping with the fact that the period does not change.
4. If we increase the value of a, we stretch the graph out vertically. The intercepts with the x-axis remain as they are but the other four anchor points move further away from the axis by a factor of a units. This makes sense if we think about the fact that every value of $\tan x$ is being multiplied by a to produce the final function value.
5. As soon as $a < 0$ the graph flips over the x-axis. The intercepts with the x-axis remain the same but the anchor points that were above the axis move below the axis and vice versa. Now each value of

$\tan x$ is being multiplied by a negative number so values that were positive become negative and those that were negative become positive.

6. Changing a does not affect the period of the graph. It remains 180°.
7. Changing a does not affect the asymptotes. These remain at 90° and 270°. This is in keeping with the fact that the period does not change.
8. If $a = 2$, the graph will be stretched vertically and flipped over the x-axis. The x-intercepts will not change but the other anchor points will be two units away from the x-axis but on opposite sides to where they were. The point $(45^{\circ}, 1)$, for example, will now be at $(45^{\circ}, -2)$ If $q = -1$ this whole altered graph will move one unit down. This will affect all the anchor points. The point $(45^{\circ}, -2)$, for example, will move to $(45^{\circ}, -3)$.

Example 5.3

Given $t(x) = -2\tan x + 2$.

1. State the period of the function.
2. State the asymptotes of the function.
3. What are the intercepts with the axes?
4. State the domain and range of the function.
5. Make a neat sketch of the function for $0^{\circ} \leq x \leq 180^{\circ}$ showing all the key points.

Solutions

1. Period is 180°.
2. Asymptotes are $x = 90^{\circ}$ and $x = 270^{\circ}$.
3. x-intercepts (let $y = 0$):
$$0 = -2\tan x + 2$$
$$\therefore 2\tan x = 2$$
$$\therefore \tan x = 1$$
$$\therefore x = 45^{\circ}$$
x-intercept is the point $(45^{\circ}, 0)$

   y-intercept (let $x = 0$):
$$t(x) = -2\tan 0^{\circ} + 2$$
$$\therefore t(x) = 2$$
y-intercept is the point $(0^{\circ}, 2)$
4. Domain: $\{x \mid x \in \mathbb{R}, x \neq 90^{\circ}; 270^{\circ}\}$
Range: $t(x) \in \mathbb{R}$
5. To make a sketch, we need to think about how the normal anchor points on the tangent graph have been changed. A useful approach is to use a table of these points for at least one period.

Original point	After a	After q
$(0^{\circ}, 0)$	$(0^{\circ}, 0)$	$(0^{\circ}, 2)$ Note: This is the y-intercept we calculated
$(45^{\circ}, 1)$	$(45^{\circ}, -2)$	$(45^{\circ}, 0)$ Note: This is the x-intercept we calculated
$(135^{\circ}, -1)$	$(135^{\circ}, 2)$	$(135^{\circ}, 4)$
$(180^{\circ}, 0)$	$(180^{\circ}, 0)$	$(180^{\circ}, 2)$

We know that there is an asymptote at $x = 90^{\circ}$ and that the graph is flipped over the x-axis so therefore, it approaches negative infinity from the left of the asymptote and positive infinity from the right (the opposite way around to normal). Figure 25 shows the sketch of $t(x) = -2\tan x + 2$ for $0^{\circ} \leq x \leq 180^{\circ}$.

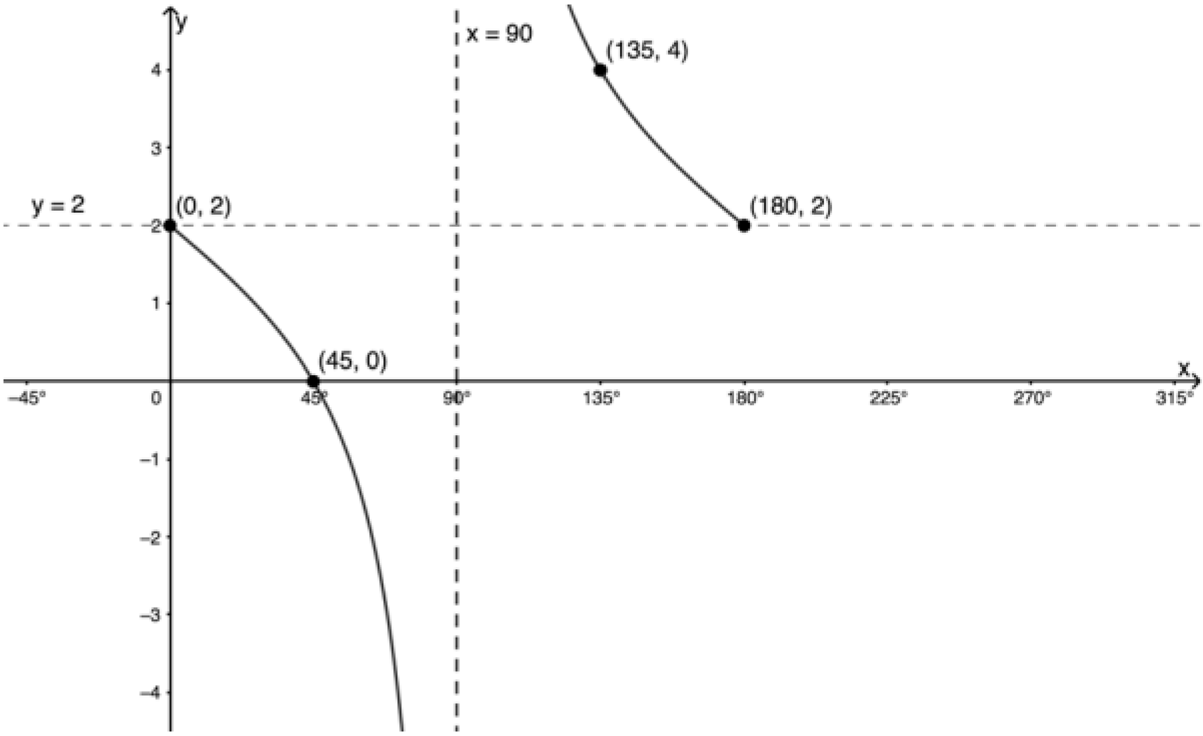


Figure 25: Graph of $t(x) = -2\tan x + 2$ *for* $0^{\circ} \leq x \leq 180^{\circ}$

Exercise 5.3

Make a neat sketch of $v(x) = 3\tan x - 1$ for the interval $0^{\circ} \leq x \leq 180^{\circ}$.

The full solutions are at the end of the unit.

Figure 26 shows a summary of what we know about the effects of a and q on the function $y = a\tan x + q$.

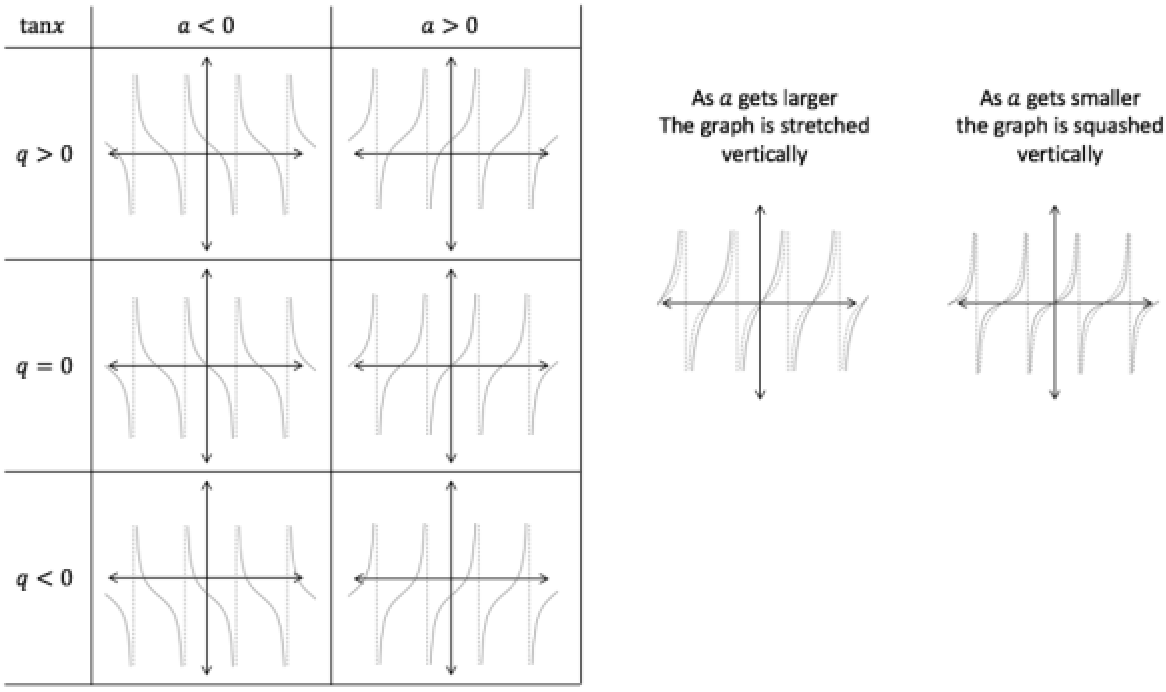


Figure 26: Effects of a and q on the function $y = a\tan x + q$

For $y = a\tan x + q$
Period: 180°
Asymptotes: $(2k-1)90^{\circ}, k \in \mathbb{Z}$
Domain: $x \in \mathbb{R}, x \neq (2k-1)90^{\circ}, k \in \mathbb{Z}$
Range: $y \in \mathbb{R}$

Finding the equations of trigonometric functions

In this section, we will finding the equations of functions of the form $y = a\sin x + q$, $y = a\cos x + q$ and $y = a\tan x + q$.

Example 5.4

1. Figure 27 shows the graph of the function $f(x) = a\sin x + q$ where $\mathrm{A}(90^{\circ}, 2)$ and $\mathrm{B}(180^{\circ}, -1)$. Find the values of a and q.

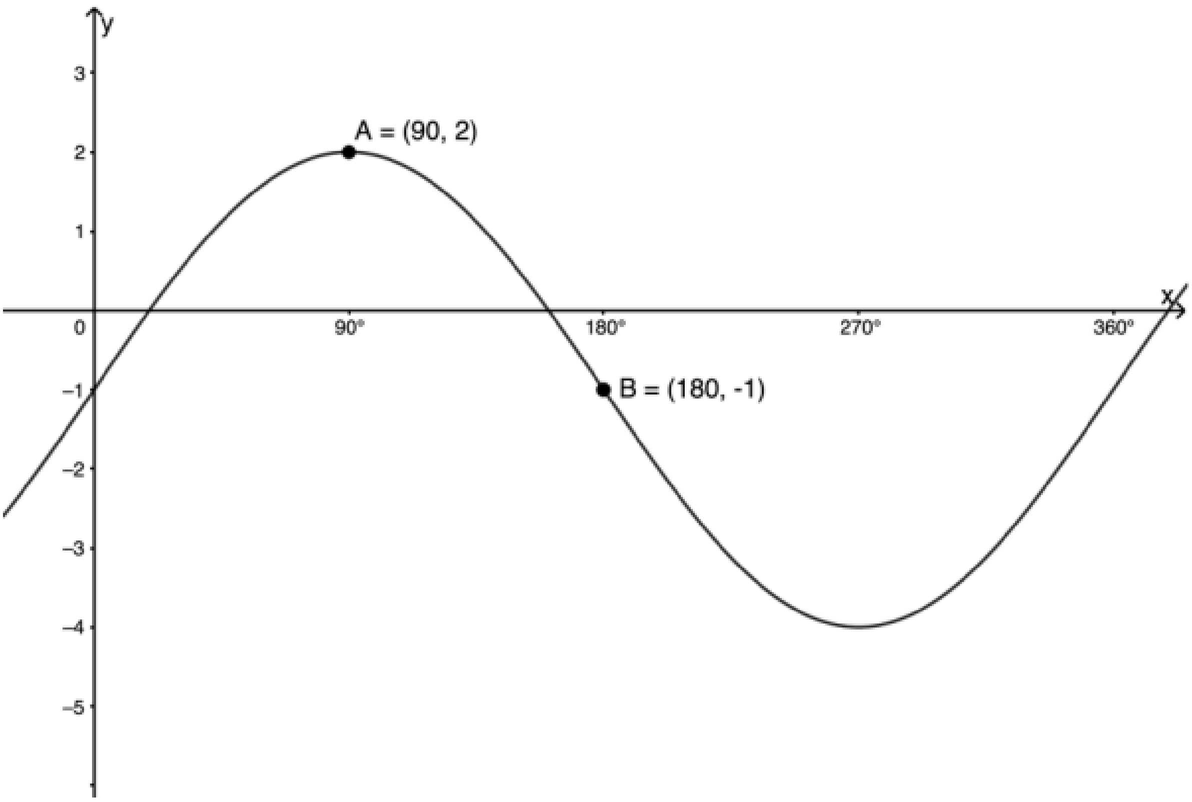


Figure 27: Graph of $f(x) = a \sin x + q$

2. Figure 28 shows the graph of $g(x) = a \cos x + q$ with $\mathrm{C}(0^{\circ}, -2.5)$ and $\mathrm{D}(180^{\circ}, -1.5)$. Find the values of a and q.

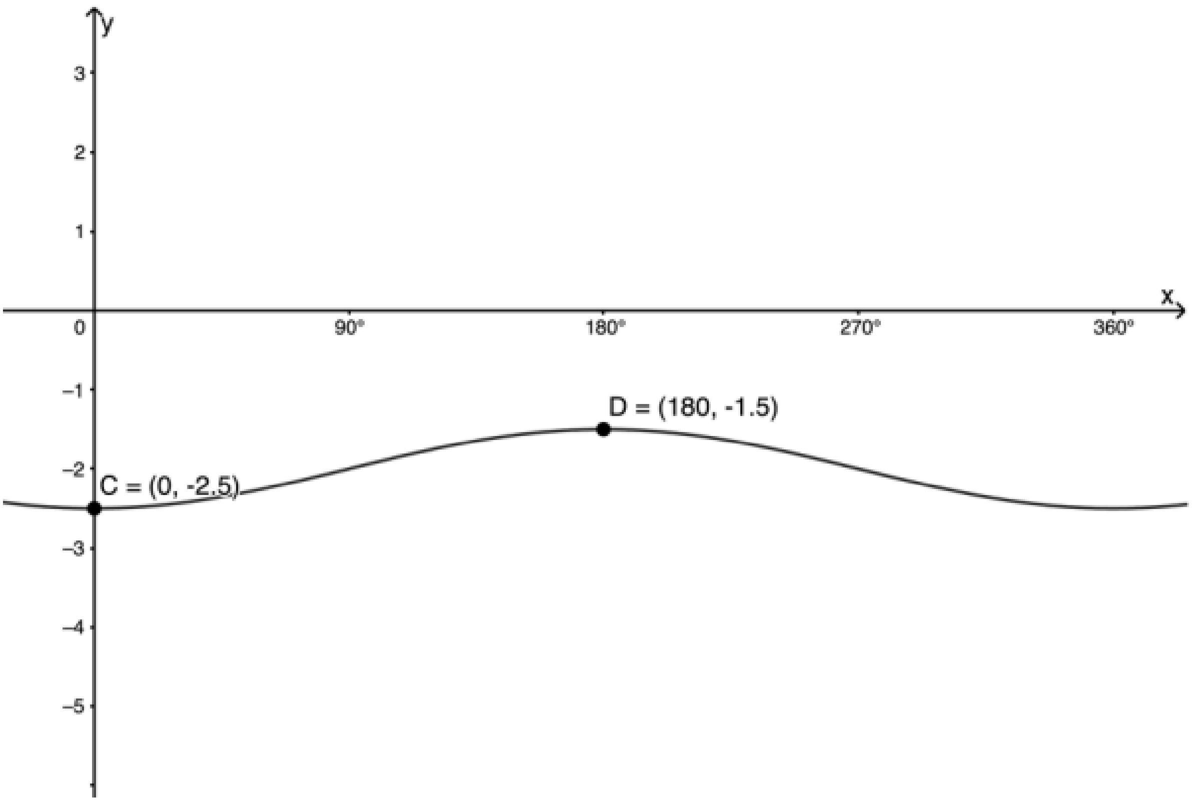


Figure 28: Graph of $g(x) = a\cos x + q$

3. Figure 29 shows the graph of $q(x) = a\tan x + q$ with two points shown with coordinates $(135^{\circ}, 0)$ and $(180^{\circ}, 1.5)$. Find the values of a and q.

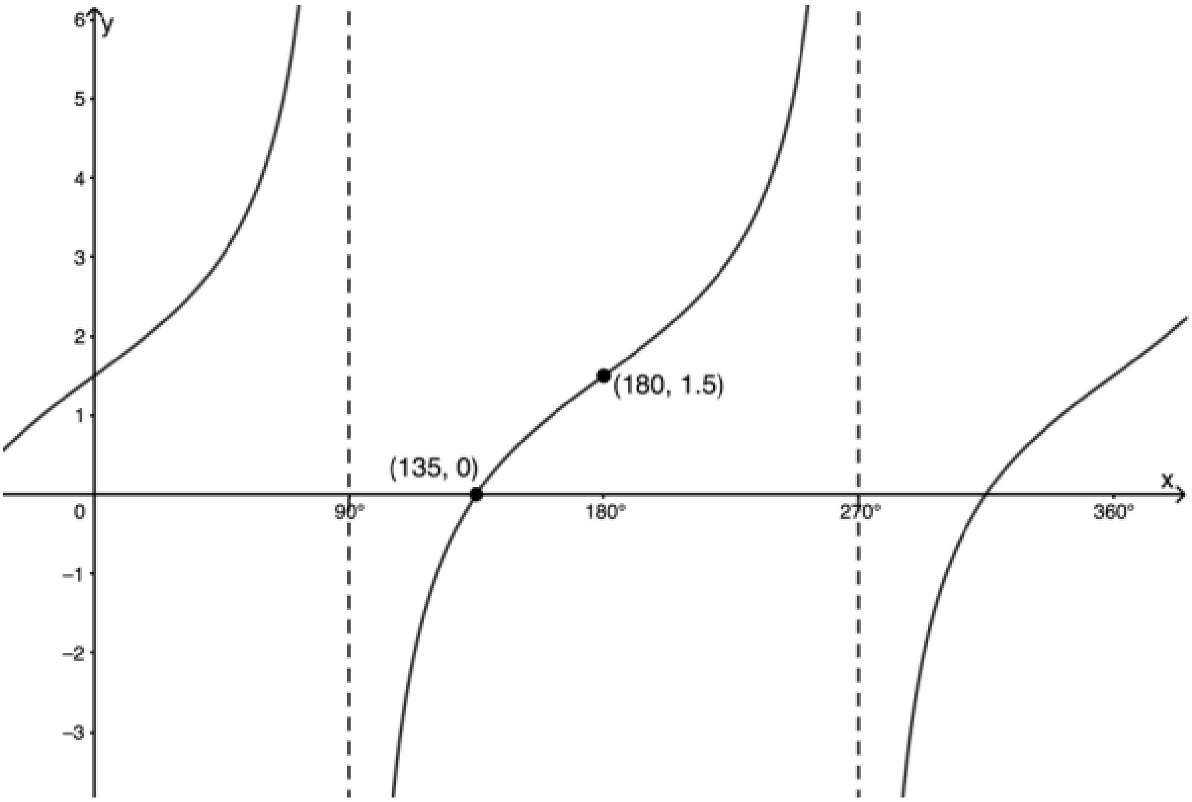


Figure 29: $q(x) = a \tan x + q$

Solutions

1. The graph has a y-intercept of -1. We know that sine normally has a y-intercept of zero. Therefore, $f(x)$ has been shifted down by one unit. Therefore $q = -1$.

   If the line $y = -1$ is the new 'middle' of the graph, we can see that the amplitude of the graph is now 3. Therefore $a = 3$.

   We could also find a algebraically as follows:
   $f(x) = a \sin x - 1$. Substitute in $\mathrm{A}(90^{\circ}, 2)$.
   $$\begin{aligned} 2 &= a \sin 90^{\circ} - 1 \\ \therefore 2 &= a - 1 \qquad \text{Remember that } \sin 90^{\circ} = 1 \\ \therefore a &= 3 \end{aligned}$$

2. The graph has a y-intercept of -2.5. We know that cosine normally has a y-intercept of 1. Therefore, it looks like $g(x)$ has been shifted down by 3.5 units. However, if that were the case, the new 'middle' of the graph would be the line $y = -3.5$ which is not the case. Point C represents a minimum and point D a maximum. We can see that the line half way between these points is $y = -2$. This means that $q = -2$.

   Based on the fact that the new middle of the graph is the line $y = -2$, we can see that the amplitude of the graph is $\frac{1}{2}$. However, cosine normally has a minimum value at 180° but this graph has a maximum value. Therefore $a = -\frac{1}{2}$.

We could also solve for a algebraically.
$g(x) = a\cos x - 2$. Substitute in $\mathrm{A}(0^{\circ}, -2.5)$.
$-2.5 = a\cos 0^{\circ} - 2$
$\therefore -2.5 = a - 2$
$\therefore a = -0.5$

3. The basic tangent function has an x-intercept at $(180^{\circ}, 0)$. This function has this point shifted up by 1.5 units. Therefore, $q = 1.5$.

The shape of the function is the same as the base tangent function. It has not been flipped over the x-axis. Therefore, we know that $a > 0$.

The anchor point at $(135^{\circ}, -1)$ is now at $(135^{\circ}, 0)$. If $a = 1$ this point would be at $(135^{\circ}, 0.5)$. Thus we know that $a \neq 1$. We can solve for a algebraically. $q(x) = a\tan x + 1.5$. Substitute $(135^{\circ}, 0)$.
$0 = a\tan 135^{\circ} + 1.5$
$\therefore 0 = -a + 1.5$
$\therefore a = 1.5$

Exercise 5.4

1. A lies on $f(x) = \cos x - 2$, B lies on $g(x) = -3\sin x + 1$, and$\mathrm{D}$ lies on $j(x) = -2\tan x$.
   a. State the coordinates of A, B, and D.
   b. State the range of $f(x)$, $g(x)$, and $j(x)$.

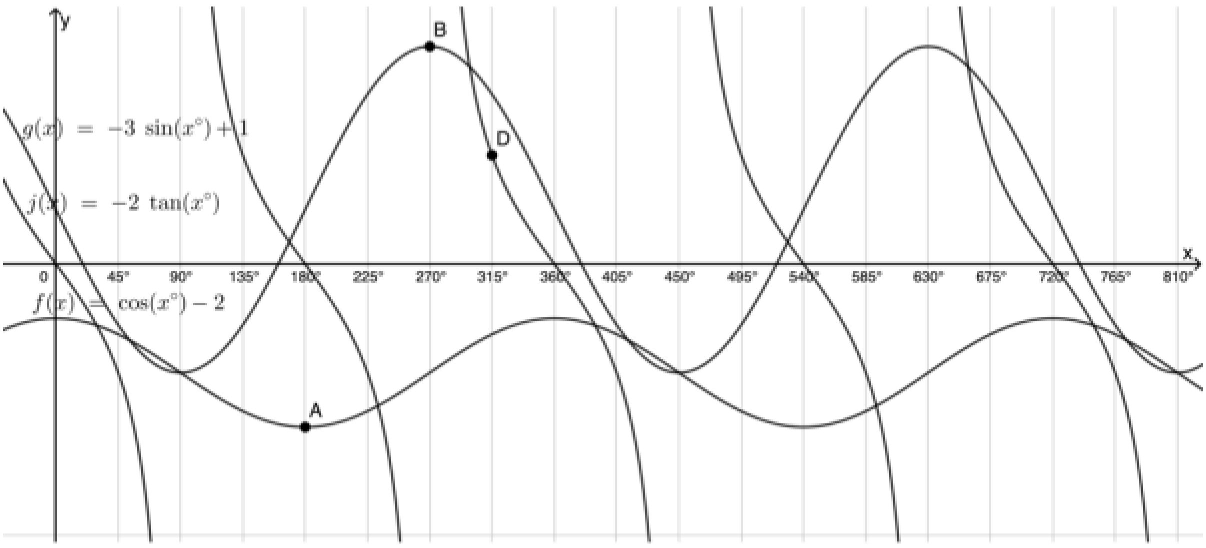


Figure 30: Graph of $y = a\cos x + q$

2. Figure 31 shows the graph of $y = a\cos x + q$. Determine the equation of the function.

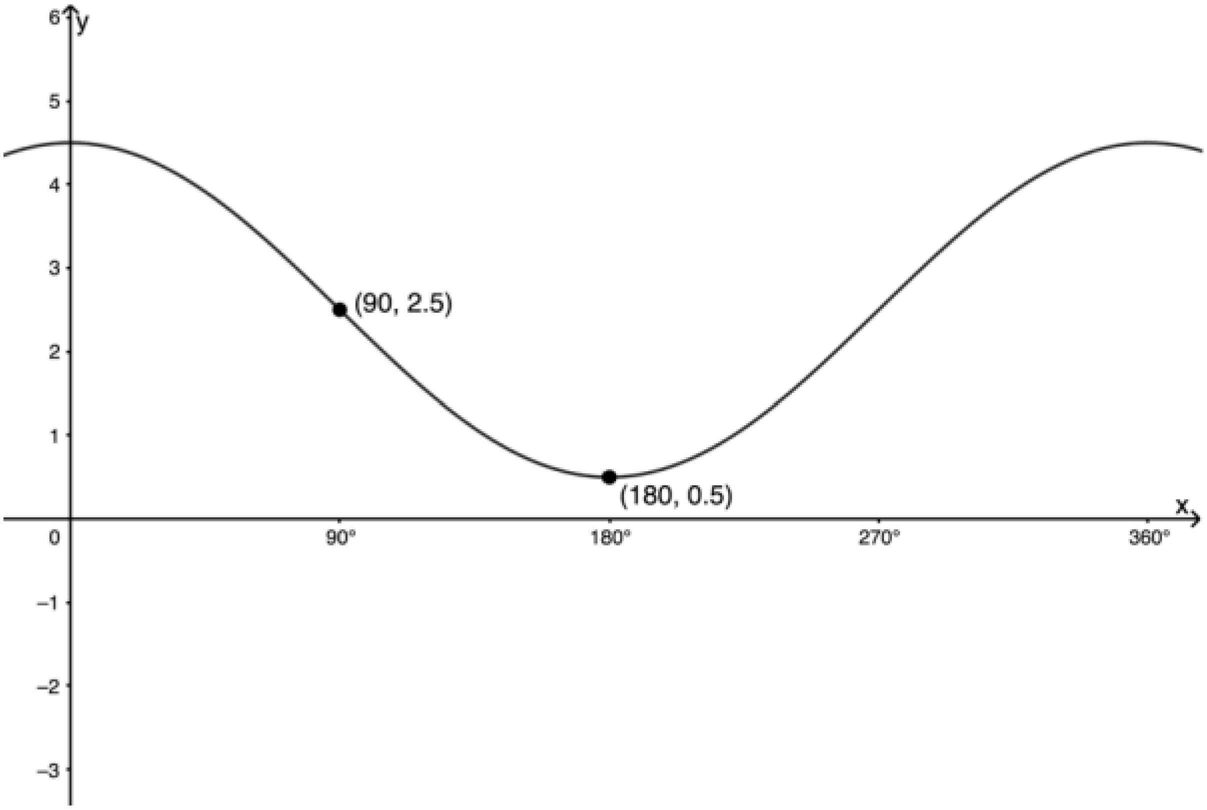


Figure 31: Graph of $y = a \cos x + q$

3. Figure 32 shows a sketch of $r(x) = a \tan x + q$. Determine the values of a and q from the figure.

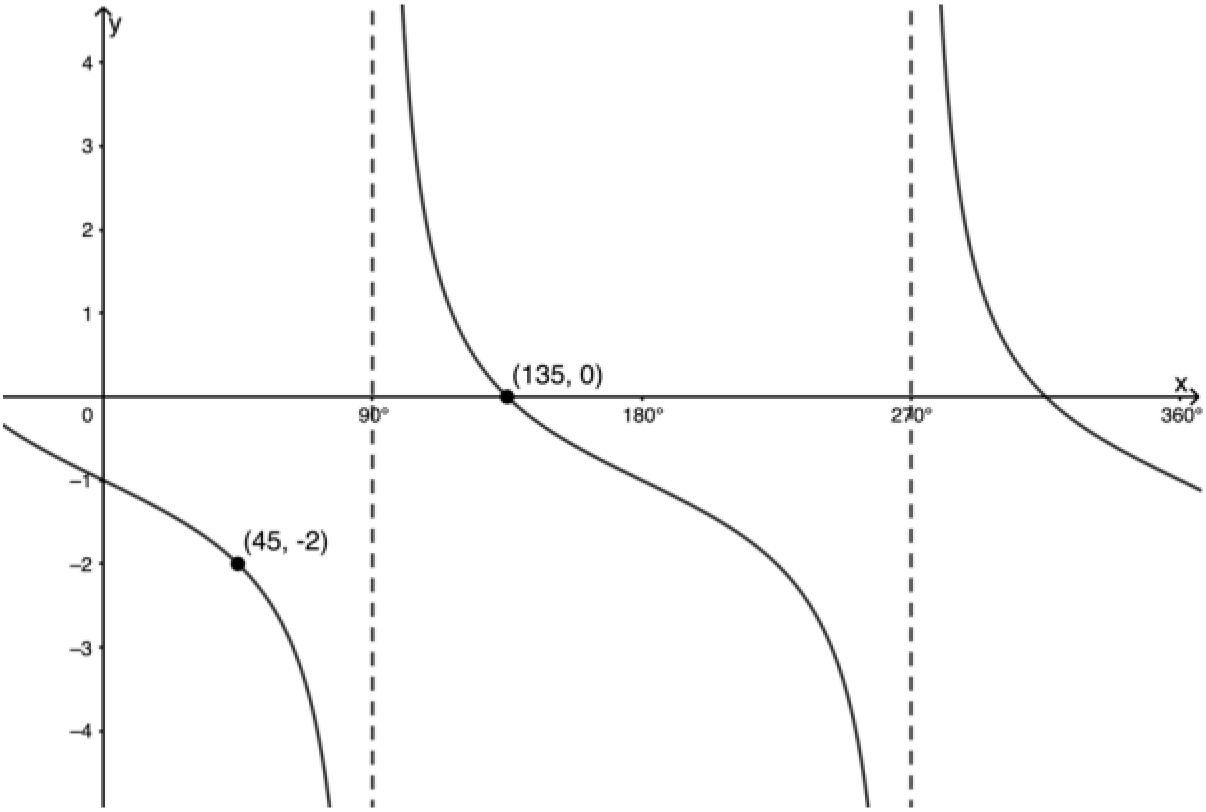


Figure 32: Graph of $r(x) = a\tan x + q$

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to sketch trigonometric functions of the form $y = a\sin x + q$, $y = a\cos x + q$ and $y = a\tan x + q$.
- How to find the equation of graphs of the form $y = a\sin x + q$, $y = a\cos x + q$ and $y = a\tan x + q$.
- The effect of a and q on the shape and position of $y = a\sin x + q$, $y = a\cos x + q$ and $y = a\tan x + q$.
- How to find the asymptotes of functions of the form $y = a\tan x + q$.
- How to determine the domain and range of trigonometric functions of the form $y = a\sin x + q$, $y = a\cos x + q$ and $y = a\tan x + q$.

Unit 5: Assessment

Suggested time to complete: 45 minutes

1. Which of the graphs g, h, p and q in Figure 33 represents the function $y = -\sin x + 1$?

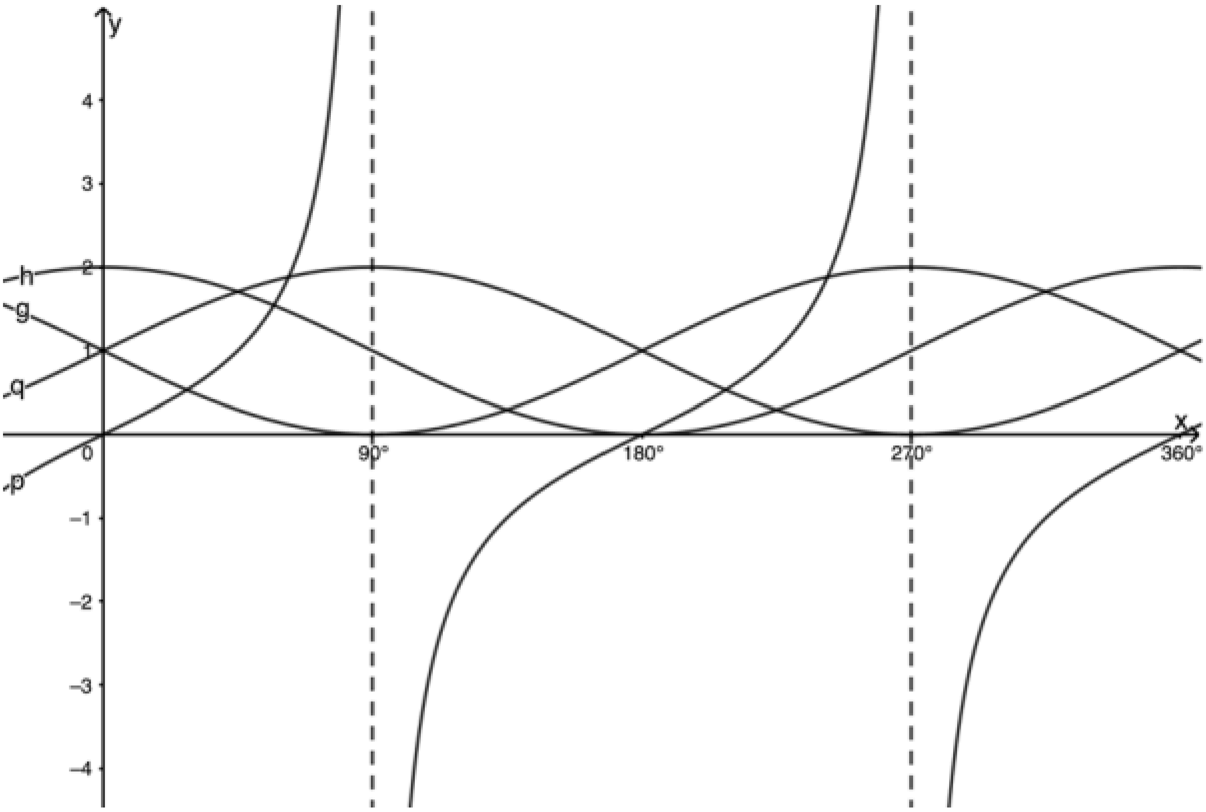


Figure 33

2. Figure 34 shows $f(x) = -3 \sin x$ (dotted line) and $g(x)$. What is the equation of $g(x)$?

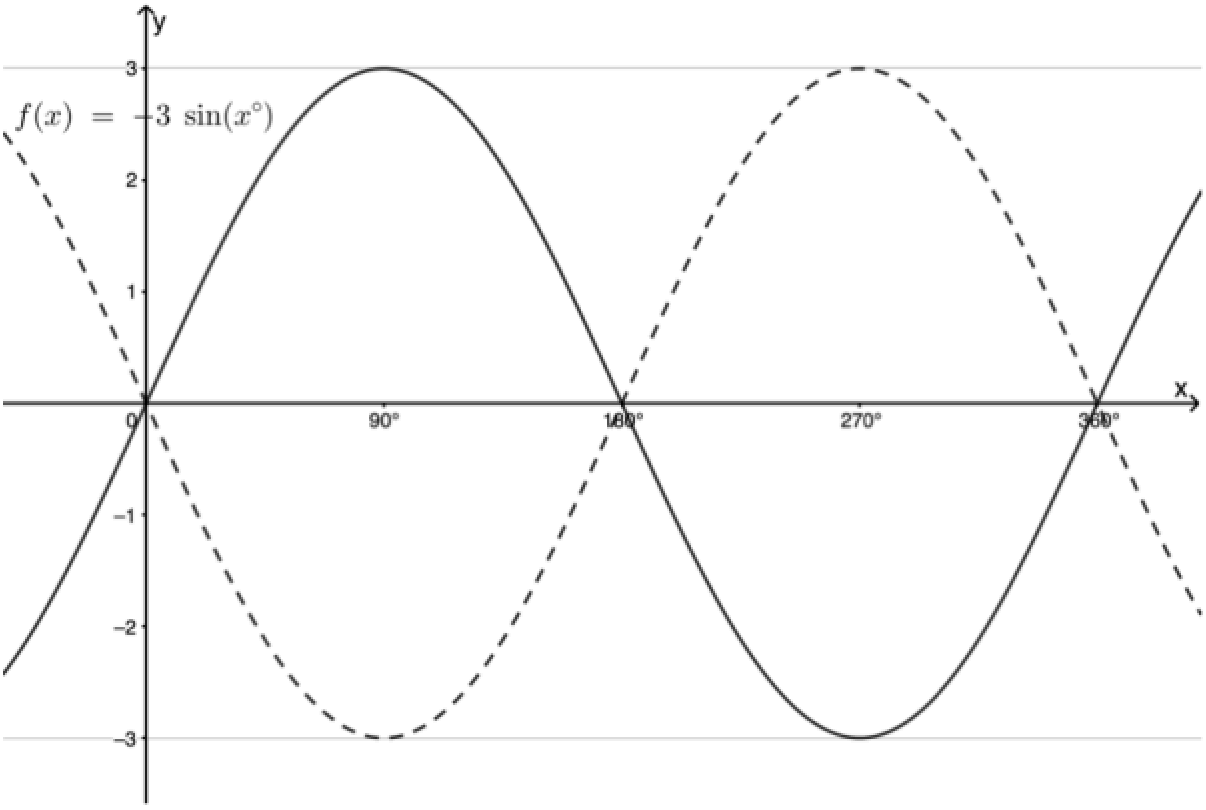


Figure 34: Graphs of $f(x) = -3 \sin x$ (dotted line) and $g(x)$

3. Sketch the following graphs without using a table of values for the interval $0^{\circ} \leq x \leq 360^{\circ}$.
   a. $f(x) = \sin x - 3$
   b. $h(x) = \dfrac{1}{3}\tan x + 1$
4. Given the graphs in Figure 35, answer the following questions.

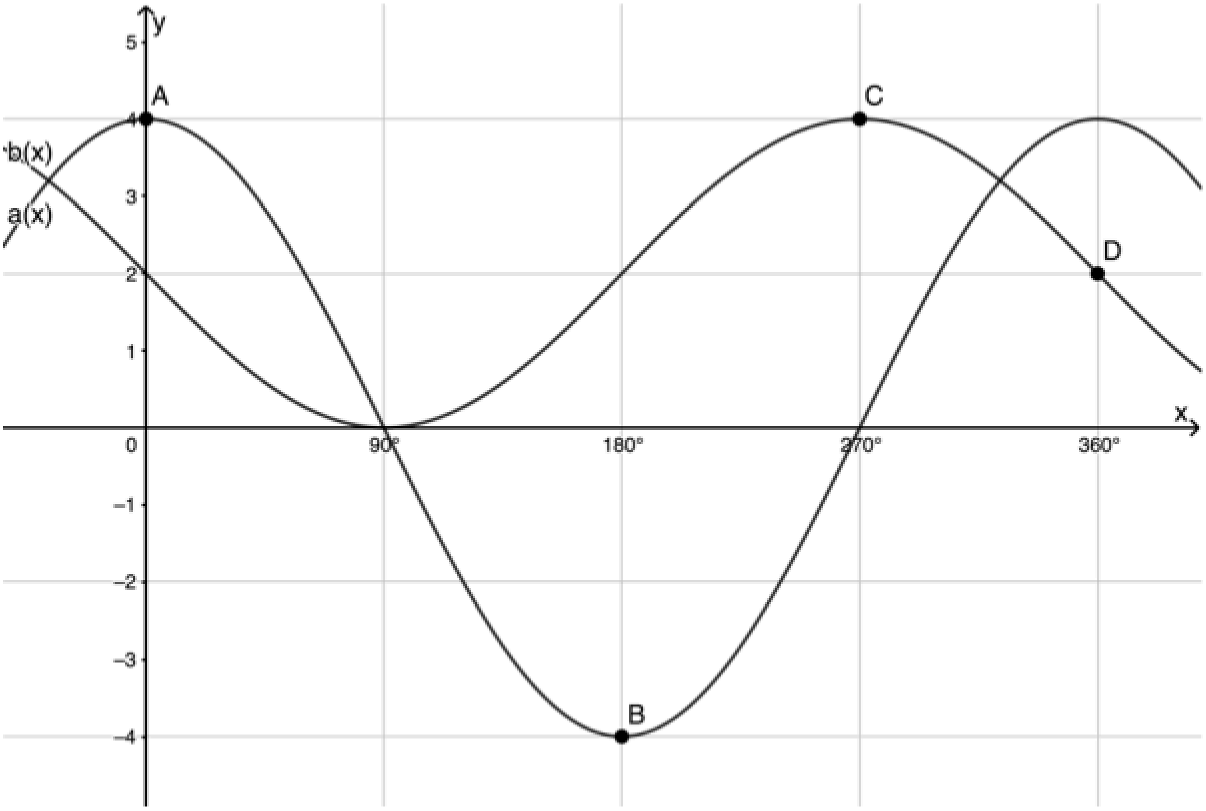


Figure 35: Graphs of $a(x)$ *and* $b(x)$

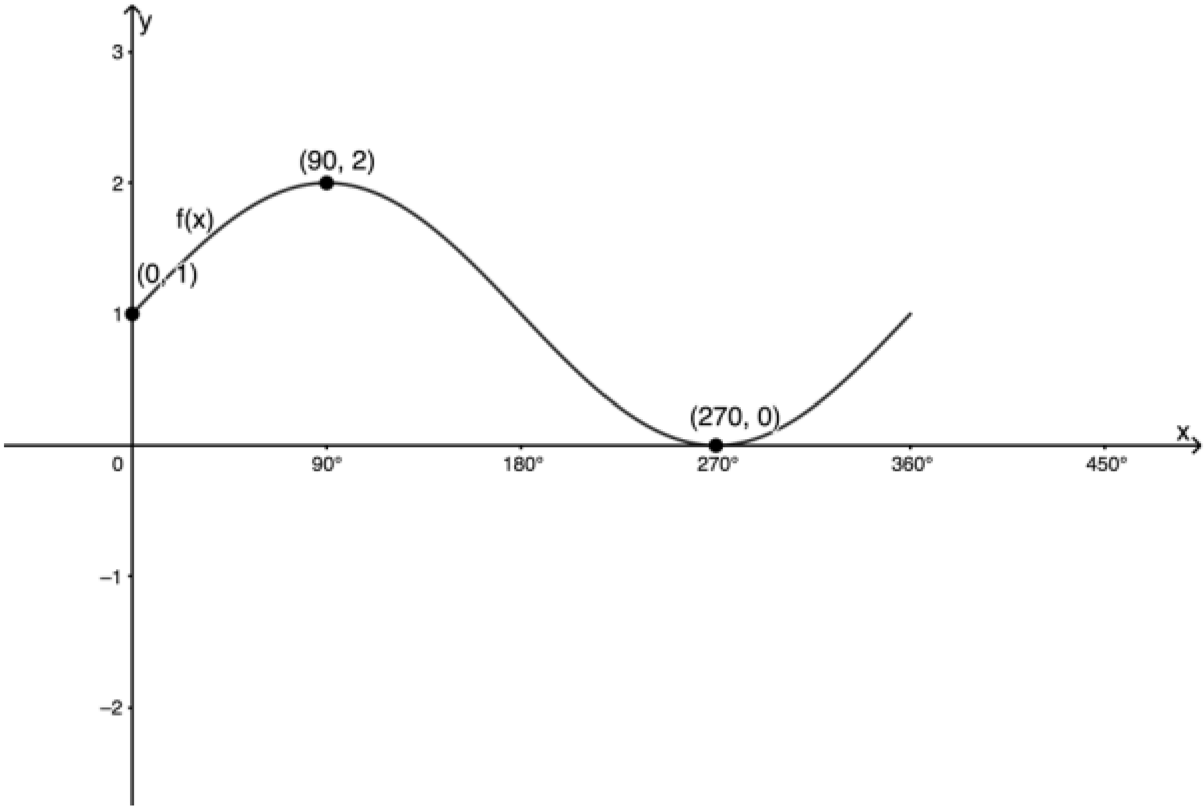
   a. State the coordinates of A, B, C and D.
   b. State the domain and range of each function.
   c. State the amplitude and period of each function.
   d. Evaluate $a(360^{\circ}) - b(360^{\circ})$.
   e. Evaluate $b(180^{\circ}) - a(180^{\circ})$.
   f. In the interval $0^{\circ} \leq x \leq 180^{\circ}$ for what values of x is $a(x) > b(x)$?

The full solutions are at the end of the unit.

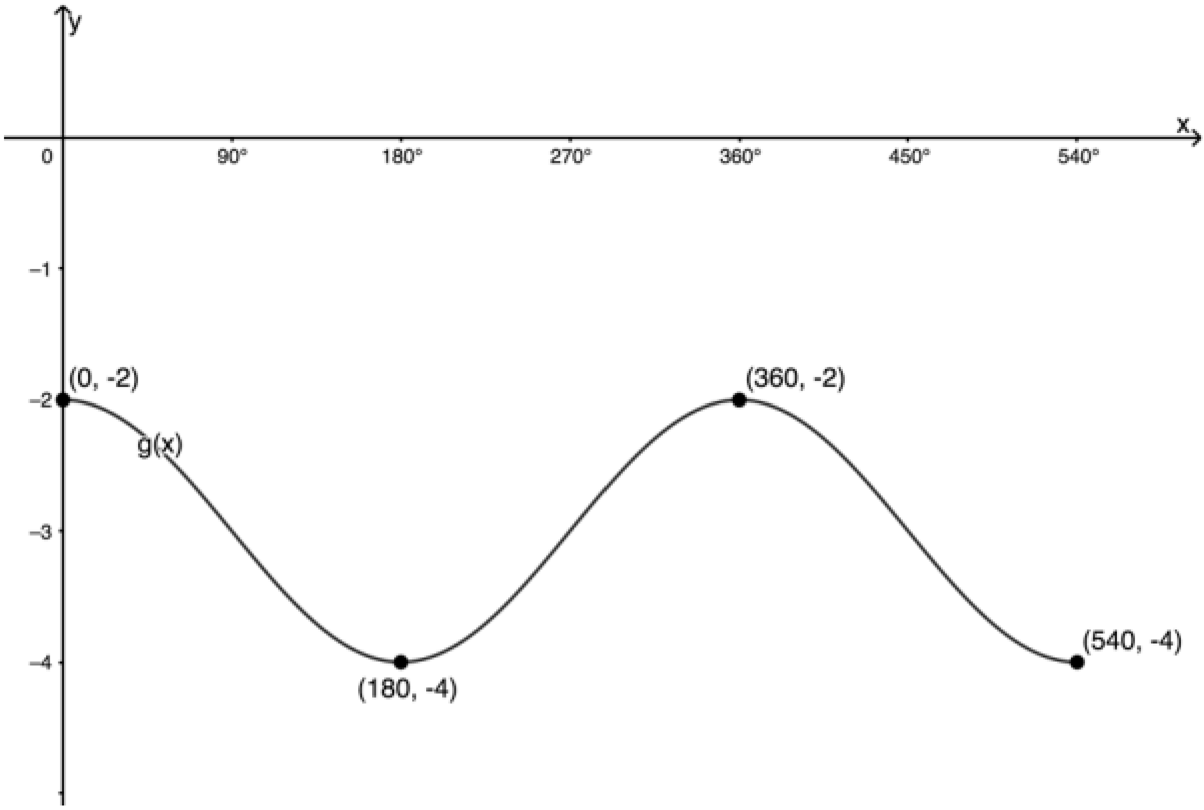
Unit 5: Solutions

Exercise 5.1

1.



2.

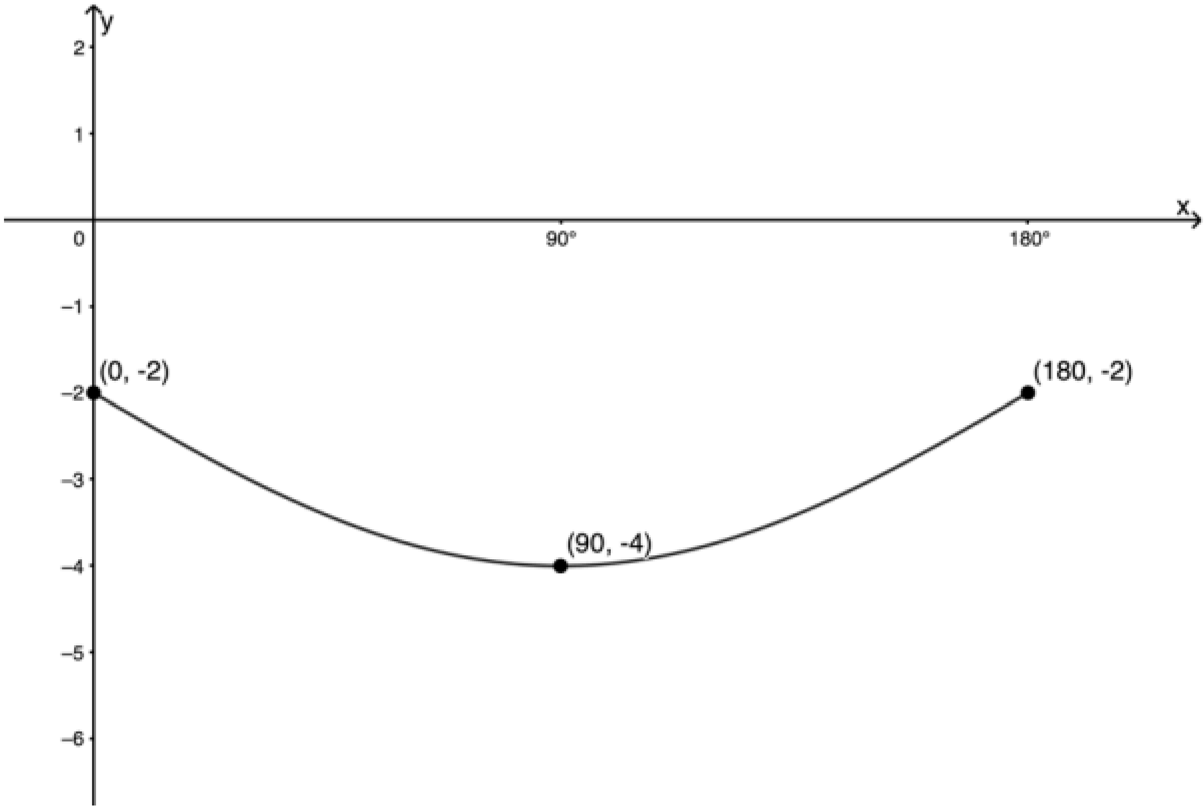


Back to Exercise 5.1

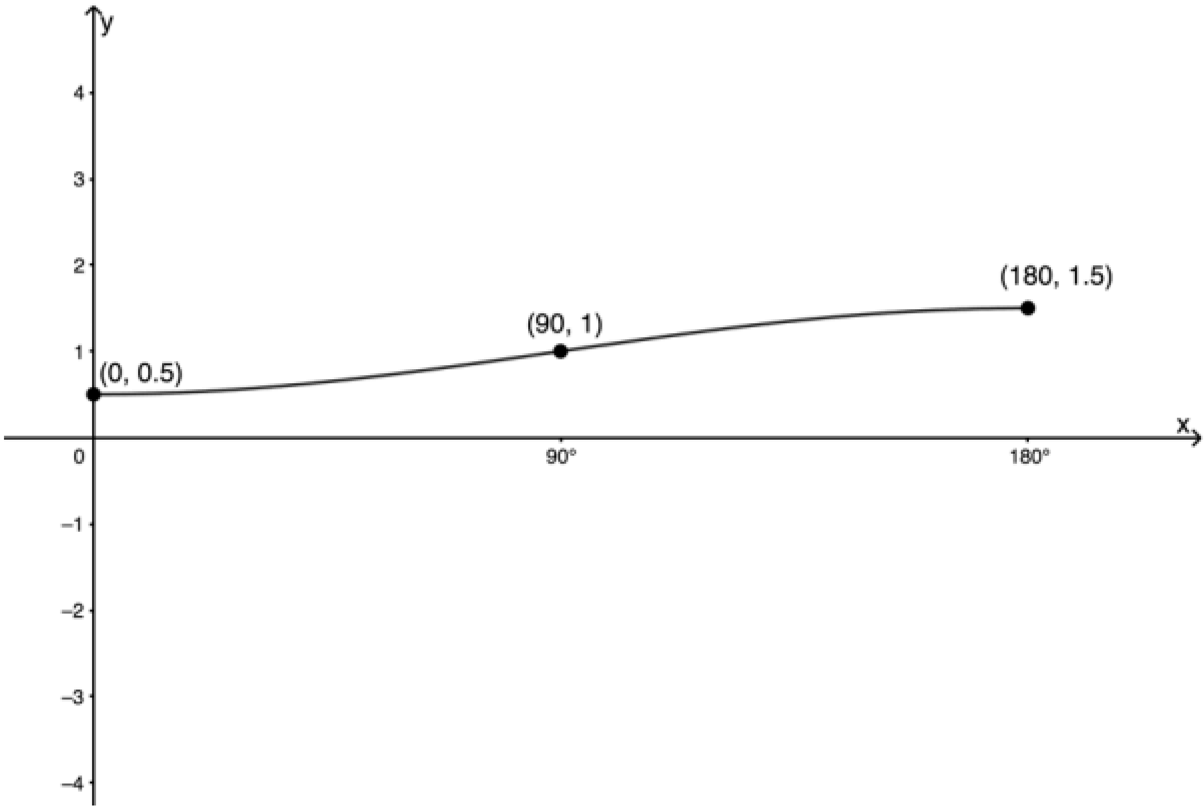
Exercise 5.2

1. $y = -2\sin x - 2$. $a = -2$. The amplitude of the graph will be 2 and it will be flipped over the x-axis. $q = -2$. The whole graph will be moved two units down. Therefore the 'middle' of the graph will now be

the line $y=-2$. Therefore, the point normally at $(0^{\circ},0)$ will move to $(0^{\circ},-2)$. The point normally at $(90^{\circ},1)$ will move to $(90^{\circ},-4)$. The point normally at $(180^{\circ},0)$ will move to $(180^{\circ},-2)$.



2. $y=-\frac{1}{2}\cos x+1$. $a=-\frac{1}{2}$. The amplitude of the graph will be $\frac{1}{2}$ and it will be flipped over the x-axis. $q=1$. The whole graph will be moved one unit up. Therefore the 'middle' of the graph will now be the line $y=1$. Therefore, the point normally at $(0^{\circ},1)$ will move to $(0^{\circ},\frac{1}{2})$. The point normally at $(90^{\circ},0)$ will move to $(90^{\circ},1)$. The point normally at $(180^{\circ},-1)$ will move to $(180^{\circ},\frac{3}{2})$.

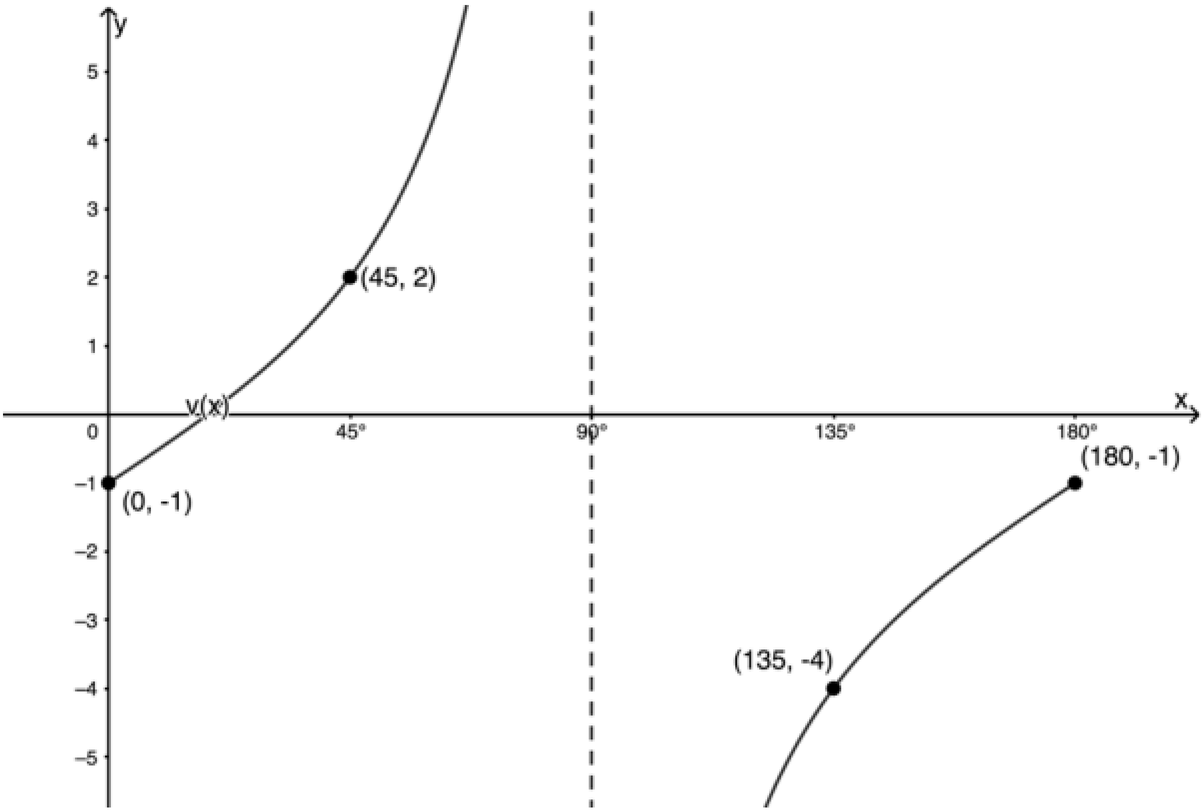


Back to Exercise 5.2

Exercise 5.3

$v(x) = 3\tan x - 1$. $a = 3$ and $q = -1$.

Original point	After a	After q
$(0^{\circ}, 0)$	$(0^{\circ}, 0)$	$(0^{\circ}, -1)$
$(45^{\circ}, 1)$	$(45^{\circ}, 3)$	$(45^{\circ}, 2)$
$(135^{\circ}, -1)$	$(135^{\circ}, -3)$	$(135^{\circ}, -4)$
$(180^{\circ}, 0)$	$(180^{\circ}, 0)$	$(180^{\circ}, -1)$



Back to Exercise 5.3

Exercise 5.4

1.
   a. $\mathrm{A}(180^{\circ}, -3)$, $\mathrm{B}(270^{\circ}, 4)$, $\mathrm{D}(315^{\circ}, 2)$
   b. $f(x) \in [-3, -1]$, $g(x) \in [-2, 4]$, $j(x) \in \mathbb{R}$
2. The graph has a maximum turning point where it intercepts the y-axis. Therefore $a > 0$. The graph of $y = \cos x$ normally cuts the x-axis at 90° which represents the normal 'middle' or point of rest of the function. The minimum turning point is at $(180^{\circ}, 0.5)$. Therefore, the amplitude of the function is 2 and $a = 2$.

   Therefore $y = 2\cos x + q$. We can substitute $(180^{\circ}, 0.5)$ to find q.

$$\frac{1}{2} = 2\cos 180^{\circ} + q$$
$$\therefore \frac{1}{2} = -2 + q$$
$$\therefore q = \frac{5}{2}$$
$$y = 2\cos x + \frac{5}{2}$$

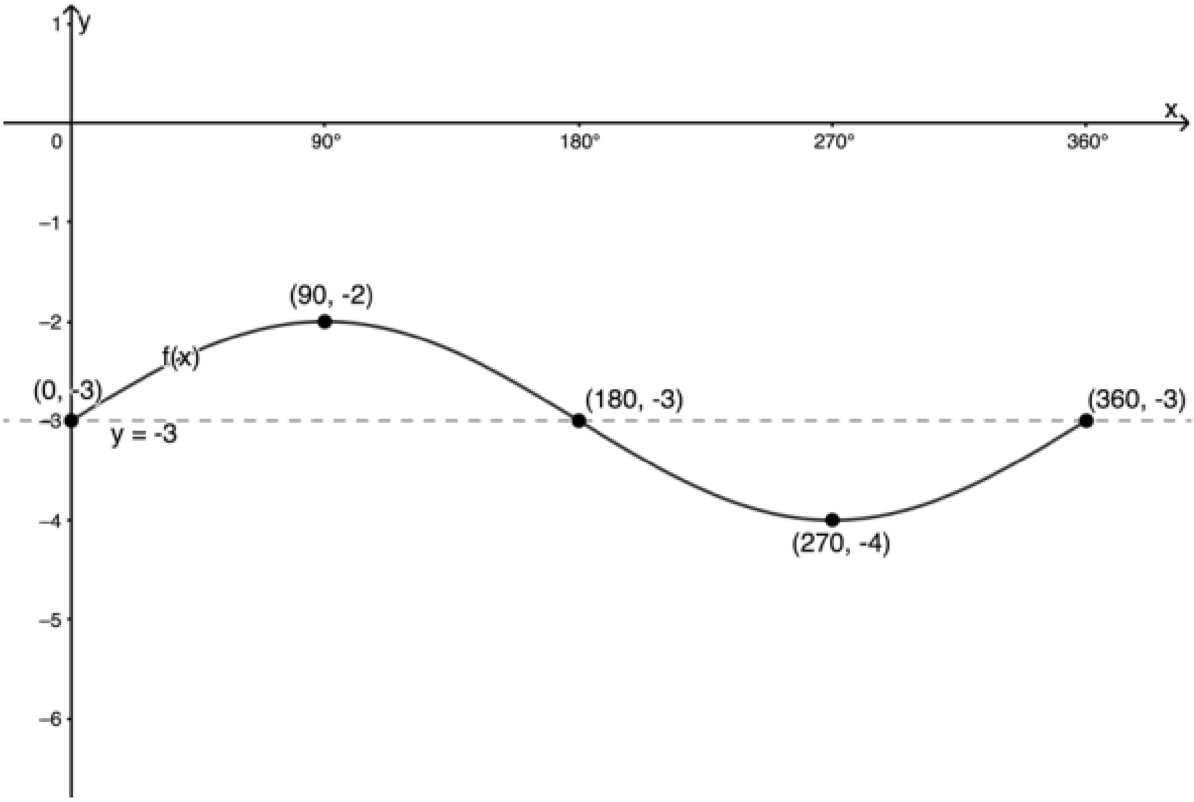
3. The shape of the function is not the same as the basic tangent function. It has been flipped over the x-axis. Therefore, we know that $a < 0$.
   Normally, there are anchor points at $(45^{\circ}, 1)$ and $(135^{\circ}, -1)$. These are 2 units apart. The new anchor points at $(45^{\circ}, -2)$ and $(135^{\circ}, 0)$ are still 2 units apart. Therefore, the graph has not been stretched out. Therefore $a = -1$ and $y = -\tan x + q$.

   We can solve for q algebraically. Substitute $(45^{\circ}, -2)$.
   $$-2 = -\tan 45^{\circ} + q$$
   $$\therefore -2 = -1 + q$$
   $$\therefore q = -1$$

Back to Exercise 5.4

Unit 5: Assessment

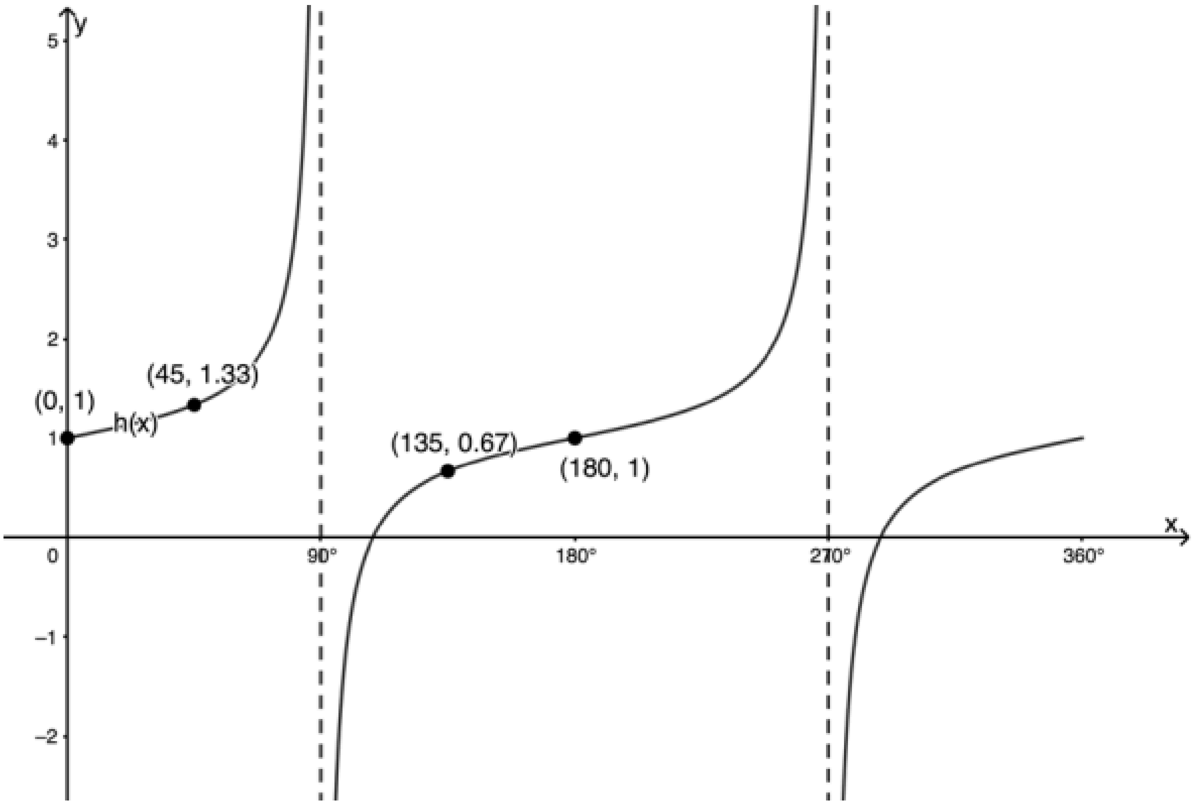
1. It cannot be graph p. This is a graph of the form $y = a\tan x + q$.
   It cannot be graph h. This graph has a maximum at $x = 0^{\circ}$. Therefore it is of the form $y = a\cos x + q$.
   It cannot be graph q. This graph rises to a maximum at $x = 90^{\circ}$. Therefore $a > 0$.
   Graph g represents $y = -\sin x + 1$. The graph falls to a minimum at $x = 90^{\circ}$. The base sine function has a maximum at $x = 90^{\circ}$. Therefore $a < 0$. The graph has an amplitude of 1. Hence $a = -1$ Also, the minimum is not $(90^{\circ}, -1)$ but $(90^{\circ}, 0)$ indicating that the graph has been shifted up by one unit. Hence, $q = 1$.
2. Both graphs have the same amplitude. Therefore the absolute value of a will be the same. $g(x)$ is the same shape as $f(x)$ except that it has been flipped over the x-axis. Therefore it is also of the form $y = a\sin x + q$ but the sign of a will be opposite. $a = -3$ in $f(x)$. Therefore $a = 3$ in $g(x)$ and $g(x) = 3\sin x$.
3.
   a. $f(x) = \sin x - 3$
      The amplitude of the graph is 1 and the graph is shifted three units down. So the new 'middle' of the graph is the line $y = -3$.



b. $h(x) = \frac{1}{3}\tan x + 1$

$a = \frac{1}{3}$ so the graph is going to be squashed vertically. The period has not changed so the asymptotes remain the same as the base tangent function. The whole graph is shifted one unit up. The anchor points will be transformed as follows.

Original point	After a	After q
$(0^{\circ}, 0)$	$(0^{\circ}, 0)$	$(0^{\circ}, 1)$
$(45^{\circ}, 1)$	$(45^{\circ}, \frac{1}{3})$	$(45^{\circ}, \frac{4}{3})$
$(135^{\circ}, -1)$	$(135^{\circ}, -\frac{1}{3})$	$(135^{\circ}, \frac{2}{3})$
$(180^{\circ}, 0)$	$(180^{\circ}, 0)$	$(180^{\circ}, 1)$



4.

a. $\mathrm{A}(0^{\circ}, 4)$
   $\mathrm{B}(180^{\circ}, -4)$
   $\mathrm{C}(270^{\circ}, 4)$
   $\mathrm{D}(360^{\circ}, 2)$

b. $a(x)$: Domain $x \in \mathbb{R}$; Range $a(x) \in [-4, 4]$
   $b(x)$: Domain $x \in \mathbb{R}$; Range $b(x) \in [0, 4]$

c. $a(x)$: Amplitude is 4; period is 360°
   $b(x)$: Amplitude is 2; period is 360°

d. $a(360^{\circ}) - b(360^{\circ}) = 2$

e. $b(180^{\circ}) - a(180^{\circ}) = 6$

f. $x \in (0^{\circ}, 90^{\circ})$

Back to the Unit 5: Assessment

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SUBJECT OUTCOME VI

SPACE, SHAPE AND MEASUREMENT: MEASURE AND CALCULATE PHYSICAL QUANTITIES

Subject outcome 3.1

Measure and calculate physical quantities

Learning outcomes

- Read scales on measuring instruments correctly. Instruments to include are the ruler and protractor.
- Use symbols and Systeme Internationale (SI) units as appropriate to the situation.

Unit outcomes: Unit 1: Scales and measurement

By the end of this unit you will be able to:

- Use a ruler and protractor accurately.
- Understand the SI system of measurement.
- Convert between different units.

Unit 1: Scales and measurement

DYLAN BUSA

Unit outcomes

By the end of this unit you will be able to:

- Use a ruler and protractor accurately.
- Understand the SI system of measurement.
- Convert between different units.

What you should know

Before you start this unit, make sure you can:

- Do basic arithmetic calculations (adding, subtracting, dividing and multiplying) on a calculator.
- Use a ruler to measure lengths.
- Use a protractor to measure angles.

Introduction

Imagine you go to the shops to buy some milk. When you get there all the bottles are different sizes and none of them show you how much milk is in them (figure 1).

Figure 1: A bottle of milk

How would you know how much milk you are getting and whether you are paying a fair price?

Thankfully, this is not how things work. When you go to the shops all the bottles of milk are labelled with how much they contain. We measure the amount of milk (and other liquids) by their volume in litres and millilitres.

We measure solids such as sugar in kilograms and grams (figure 2) and we measure things like fabric and rope in metres and centimetres.

Figure 2: A 1 kg bag of sugar

When we measure something, we give a number to a characteristic of an object or event so that we can more accurately describe it and can compare it with other similar objects or events. Measurement is the cornerstone of trade, science and technology, and research. Think for a moment how much you depend on knowing the measurements of different things in your life.

Systeme Internationale (SI) units

But how do you know that the litre of milk you buy from shop A is exactly the same amount as the litre of milk you buy at Shop B? How do you know that a 1 kg bag of sugar contains 1 kg?

In order to make sure that a metre or a kilogram or a litre is always the same everywhere, we use an international system of measures and units called the Systeme Internationale (SI). This system reduces all physical measurements to a mathematical combination of seven base units. These are the kilogram (mass), metre (length), candela (brightness), second (time), ampere (electricity), kelvin (temperature) and mole (number of particles of a substance). Every other physical measure you can think of (even the old Imperial measures of miles, pounds, feet and gallons) are based on these seven base units.

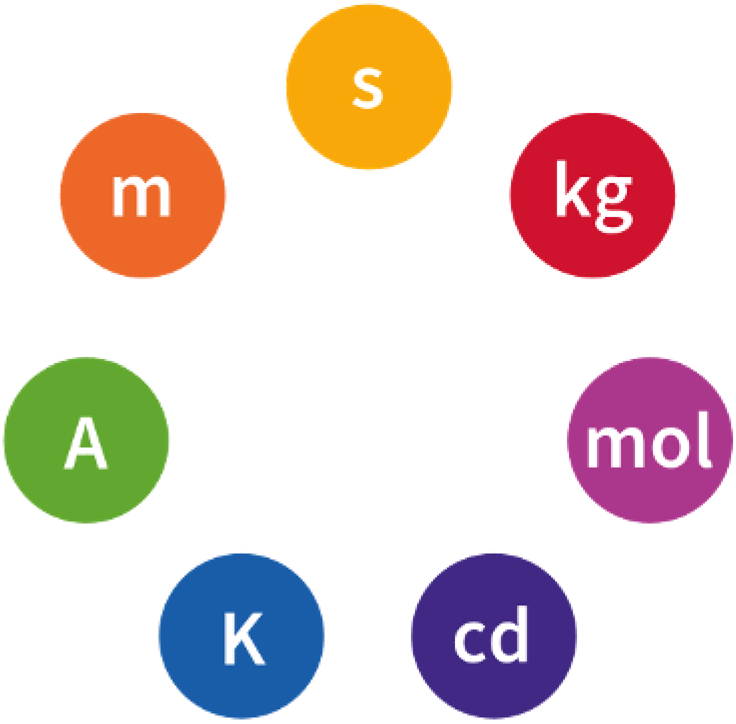


Figure 3: The 7 base units of the Systeme Internationale (SI)

Note

If you want to learn more about each of these units, then visit the SI Units website.

Did you know?

Units like the kilogram and metre used to be based on physical objects kept in Paris. The problem was that the mass and length of these objects changed over time and, if you ever wanted to check your length or mass against these, you had to go all the way to Paris!

First, in 1983, the metre was redefined based on an exact universal constant; the distance travelled by light in a vacuum in $\frac{1}{299\ 792\ 458}$ seconds.

More recently, the kilogram was redefined based on immutable (unchanging) physical constants. If you have an internet connection, watch the video called "How We're Redefining the kg" to see how and why it has been redefined.

How We're Redefining the kg (Duration: 9.48)

Note

Each measure in the SI system has its own specific unit abbreviation.
Kilogram – kg
Metre – m
Second – s
Ampere – A
Kelvin – K
Mole – mol
Candela – cd

Note how some of them are lower case and others are upper case. Also, there should always be a space between the value of the measure and the unit, for example $12\ \mathrm{m}$, $0.45\ \mathrm{kg}$, $1.875\ \mathrm{A}$, and $67.452\ \mathrm{s}$.

Taking measurements

One of the most common measurements in everyday life is the measure of length. We measure the length of many different things. Can you name some of these?

Because we need to measure the lengths of so many different things, it is not always practical to take measurements in metres. Imagine measuring the distance from the earth to the moon or the width of a pencil in metres. It could be done but might not be very accurate.

Fortunately, the SI system includes a set of standard prefixes that allow us to expand the range of all the measurements, for example **kilometres** and **millimetres**. The prefix 'kilo' means $1\ 000$ and the 'milli' means $1\ 000\mathrm{th}$. So $1\ \mathrm{km} = 1\ 000\ \mathrm{m}$ and $1\ \mathrm{mm} = \frac{1}{1\ 000}\ \mathrm{m} = 0.001\ \mathrm{m}$.

There are some other prefixes which we sometimes use like centi ($100\mathrm{th}$) or deci ($10\mathrm{th}$).

Table 1 lists the most common official SI prefixes and how they relate to the base unit. Note that 'centi' is not an official SI prefix but is included because it is commonly used.

Prefix	Symbol	Meaning		Example		
tera	T	1 000 000 000 000	One trillion	TB	terabyte	Storage capacity of a hard drive is about 2 TB
giga	G	1 000 000 000	One billion	GW	gigawatt	Amount of electricity generating capacity in South Africa is about 40 GW
mega	M	1 000 000	One million	MHz	megahertz	The number of cycles a computer executes per second is about 1 000 MHz
kilo	k	1 000	One thousand	kg	kilogram	Mass of an adult male is about 80 kg
centi	c	100	One hundred	cm	centimetre	Width of a doorway is about 60 cm
milli	m	$\frac{1}{1\ 000}$	One thousandth	mm	millimetre	Thickness of a credit card is about 1 mm
micro	μ	$\frac{1}{1\ 000\ 000}$	One millionth	µg	microgram	Mass of a flea 600 µg
nano	n	$\frac{1}{1\ 000\ 000\ 000}$	One billionth	nH	nanometre	Wavelength of red light is about 700 nm
pico	p	$\frac{1}{1\ 000\ 000\ 000}$	One trillionth	pF	picofarad	Width of an atom is about 250 pm

Table 1: Common SI prefixes and centi

Notice how (except for centi) the difference between prefixes is always a factor of one thousand. For example:

- $1\ \text{kg} = 1\ 000\ \text{g}$
- $1\ \text{g} = 1\ 000\ \text{mg}$
- $1\ \text{mg} = 1\ 000\ \mu\text{g}$
- $1\ \mu\text{g} = 1\ 000\ \text{ng}$

Note

If you have an internet connection, you can watch the video called "Metric system: units of distance", which discusses some of the different units we use for measuring length and distance.

Metric system: units of distance (Duration: 6.55)

Did you know?

The human body contains more than $96\ 000\ \text{km}$ of blood vessels and that every year the Sun loses $360\ \text{Tg}$ ($360\ 000\ 000\ 000\ \text{kg}$) of mass.

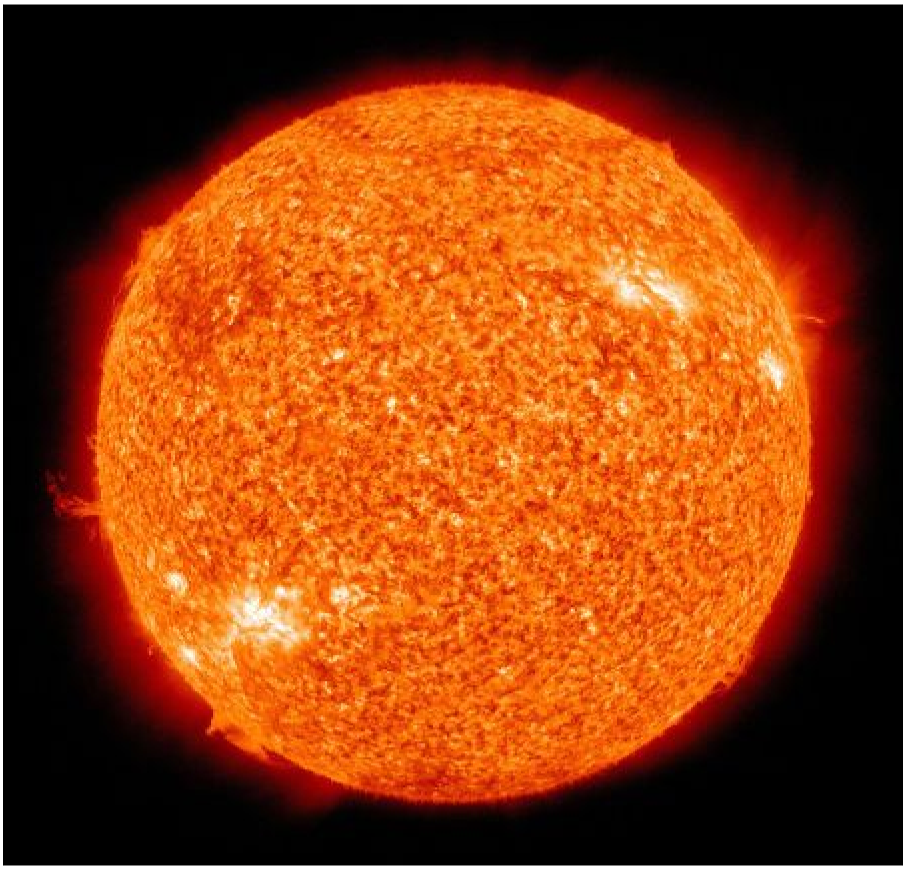


Figure 4: The Sun

Exercise 1.1

What would be the most appropriate SI unit of measure with which to measure the following characteristics including the most appropriate prefix?

1. The height of an adult human
2. The length of a newborn human
3. The height of a building
4. The mass of a truck
5. The distance from Johannesburg to Cape Town
6. The mass of a grain of sand
7. The length of a song
8. The size of a bacterium
9. The size of an atom
10. The thickness of a pane of glass

The full solutions are at the end of the unit.

Converting between units

Sometimes the units in which different measures are given are not convenient to work with or need to be changed for us to perform calculations or comparisons. For example, in Physical Science, speed is measured and calculated in metres per second ($\mathrm{m/s}$) but our everyday experience of speed is kilometres per hour ($\mathrm{km/h}$). In this case, it can be useful to convert between these two units of measurement.

To do most conversions, you need to be familiar with the prefixes in Table 1. You may need to refer to this table while working through the next example. Before you work through the example, watch these two videos on converting between units.

Conversion between metric units (Duration: 5.15)

Imperial to metric conversions (Duration: 3.20)

Example 1.1

1. Convert between the following units:
   a. $4\ 385\ \mathrm{m}$ into km
   b. $7.432\ \mathrm{MV}$ into V
   c. $673\ \mathrm{mm}$ into nm
2. Convert $12\ \mathrm{m/s}$ into $\mathrm{km/h}$.
3. Convert $65.4\ \mathrm{lb}$ (pounds) into g if $1\ \mathrm{lb} = 0.454\ \mathrm{kg}$.

Solutions

1.
   a. We know that $1\ \mathrm{km} = 1\ 000\ \mathrm{m}$. To convert from metres to kilometres, we need to **divide** by one thousand because we are going from a **smaller to a bigger unit**. Therefore $\dfrac{4\ 385\ \mathrm{m}}{1\ 000} = 4.385\ \mathrm{km}$.
   b. We know that $1\ \mathrm{MV} = 1\ 000\ 000\ \mathrm{V}$. To convert from megavolts to volts, we need to **multiply** by

one million because we are going from a **bigger to a smaller unit**. Therefore $7.432 \text{ MV} \times 1\ 000\ 000 \quad = 7\ 432\ 000 \text{ V}$.

c. We know that $1 \text{ mm} = \frac{1}{1\ 000} \text{ m}$ and that $1 \text{ nm} = \frac{1}{1\ 000\ 000} \text{ m}$. This means that $1 \text{ mm} = 1\ 000 \text{ nm}$. To convert from millimetres to nanometres, we need to **multiply** by one thousand because we are going from a **bigger unit to a smaller unit**. Therefore, $673 \text{ mm} \times 1\ 000 \quad = 673\ 000 \text{ nm}$.

2. When we convert 12 m/s into km/h we have to convert the metres into kilometres AND the seconds into hours. Let's start with the metres. We know that $1 \text{ km} = 1\ 000 \text{ m}$. To convert from metres to kilometres, we need to **divide** by one thousand because we are going from a **smaller to a bigger unit**. Therefore, $\frac{12 \text{ m/s}}{1\ 000} = 0.012 \text{ km/s}$.

   Now we can convert from seconds to hours. There are sixty seconds in a minute and sixty minutes in an hour. Therefore, there are $60 \times 60 = 3\ 600$ seconds in an hour. To convert from kilometres per second to kilometres per hour, we need to multiply by $3\ 600$ because we are going from a smaller unit to a bigger unit. Therefore, $0.012 \text{ km/s} \times 3\ 600 = 43.2 \text{ km/h}$.

   Note: the reason that the answer is a bigger number is because you will always travel a shorter distance in one second than you will at the same speed in one hour.

3. We are told that $1 \text{ lb} = 0.454 \text{ kg}$. So, to convert pounds into grams we first need to convert pounds into kilograms and then convert the kilograms into grams. $65.4 \text{ lb} \times 0.454 \text{ kg/lb} = 26.6916 \text{ kg}$. Now we can convert the kilograms into grams by multiplying by one thousand (because $1 \text{ kg} = 1\ 000 \text{ g}$). $26.6916 \text{ kg} \times 1\ 000 = 26691.9 \text{ g}$

Example 1.2

The following conversion questions are all from the Khan Academy video called "Metric system unit conversion examples".

1. Tomas dropped off two packages to be shipped. One package weighed 1.38 kg and the other package weighed 720 g. What was the combined weight of both packages in grams?
2. Julia and her friends are making kites out of paper. For each kite, they need a piece of paper that is 0.65 m long. What length (in centimetres) of paper will they need to make four kites?
3. Omar is pouring five litres of water into two goldfish bowls. He spills 200 ml of water, and then divides the remaining water evenly between the two bowls. How many millilitres of water does Omar pour into each bowl?

Solutions

1. $2\ 100 \text{ g}$
2. 260 cm
3. $2\ 400 \text{ ml}$

Watch the Khan Academy video called "Metric system unit conversion examples" to see the full worked solutions.

Metric system unit conversion examples (Duration: 5.22)

Exercise 1.2

1. Reginald orders two items from an online store which charges $\mathrm{R}15$ for each kilogram or part thereof to ship items. If he buys items weighing $3.67\ \mathrm{kg}$ and $982\ \mathrm{g}$, how much will he have to pay for shipping?
2. A certain type of electric cable weighs $280\ \mathrm{g/m}$. If $3.8\ \mathrm{km}$ of cable is needed, what will be the total weight of the cable, in kilograms?

The full solutions are at the end of the unit.

Measuring lengths with a ruler

You probably already know how to measure with a ruler. Most rulers are marked in centimetres and/or millimetres. The distance between each long hash mark on the ruler in Figure 5 is one centimetre and the distance between each small hash mark is one millimetre. How long, in millimetres, is this ruler?

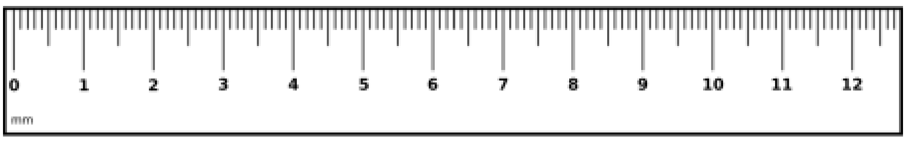


Figure 5: A millimetre ruler

We can see that the length of the blue line in Figure 6 is $9.6\ \mathrm{cm}$ or $96\ \mathrm{mm}$.

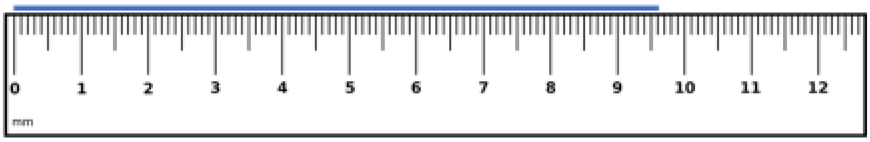


Figure 6: Using a ruler to measure a length of $9.6\ \text{cm}$ *or* $96\ \text{mm}$

We don't always have to measure from zero, so long as we take care to correctly count the distance between the start and end points. Figure 7 shows the same line measured with a different part of the ruler.

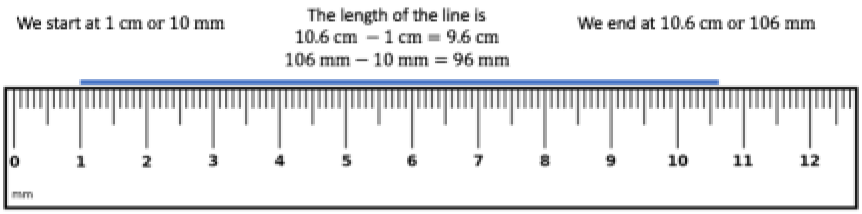


Figure 7: Using a ruler to measure a length of $9.6\ \text{cm}$ *or* $96\ \text{mm}$

Activity 1.1: Measure length with a ruler

Time required: 10 minutes

What you need:

- an internet connection

What to do:

- Open the online interactive activity called Measuring lengths with a ruler.
  Here you will find a ruler in cm with which to measure the lengths of the given boxes.
- Measure the lengths of each of the given lines to the nearest millimetre. You can drag the red cross to move the ruler and the red dot to turn it.
- Now open the online interactive activity called Broken centimetre ruler.
  Here you will find another ruler but this time it is broken.
- Measure the lengths of the given lines and then check your measurements by clicking on the

checkboxes.

What did you find?

The lengths of each of the boxes in the first ‘Measuring lengths with a ruler’ activity are as follows:

A: 120 mm or 12 cm
B: 112 mm or 11.2 cm
C: 39 mm or 3.9 cm
D: 95 mm or 9.5 cm
E: 70 mm or 7 cm

Measuring angles with a protractor

A protractor is a simple instrument for measuring the size of angles (see Figure 8). Most protractors will measure angles in degrees (the symbol for degrees is ∘).

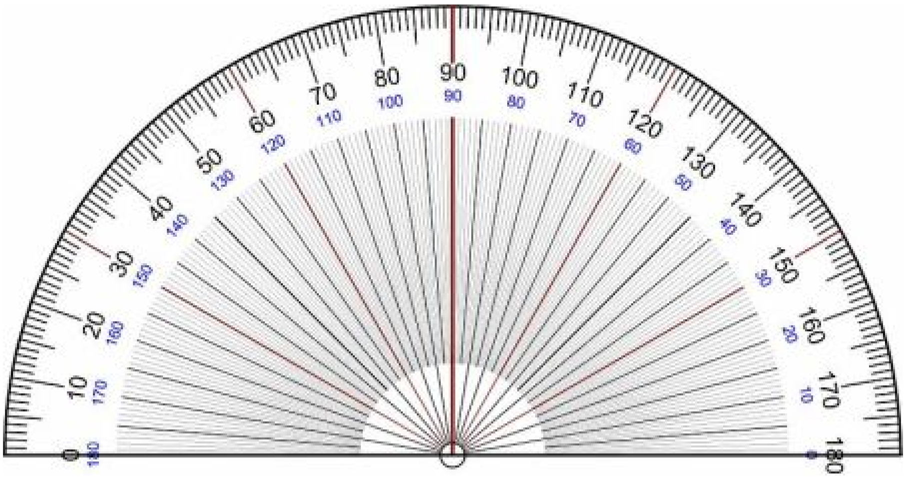


Figure 8: A protractor

As you can see in Figure 8, a protractor can measure angles between 0° and 180°. To use a protractor, place the middle of the protractor on the vertex of the angle (where the two straight lines, or ‘rays’, meet) such that one of the 0° measure lies on one of the lines creating the angle. Then read off the angle created by the other line.

For example, we can see that the size of the angle in Figure 9 is 77°.

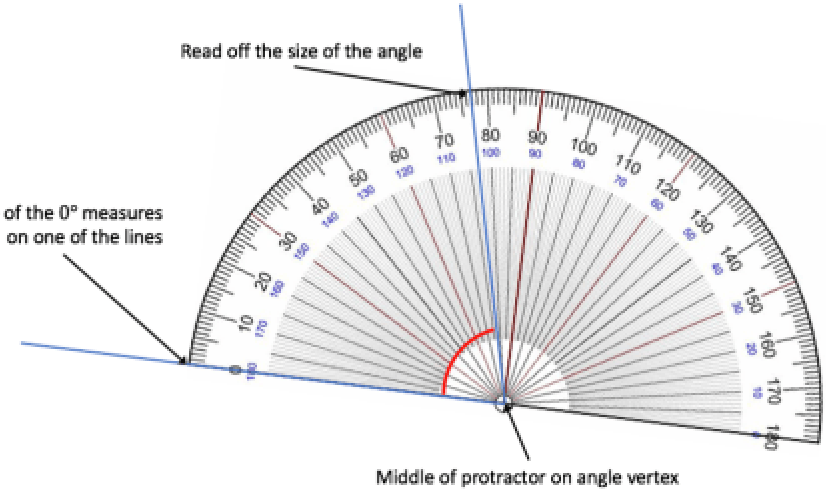


Figure 9: Using a protractor to measure an angle of $77°$

Note

If you have an internet connection, you should watch the video called "Measuring angles using a protractor" that demonstrates how to use a protractor.

Measuring angles using a protractor (Duration: 3.25)

Activity 1.2: Measure angles with a protractor

Time required: 10 minutes

What you need:

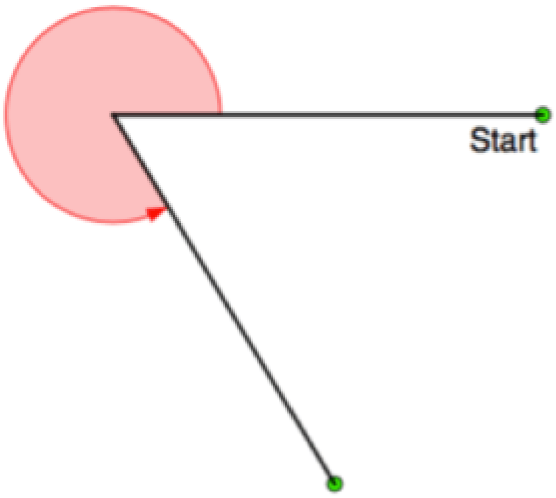
- an internet connection

What to do:

1. Open the online interactive activity called Measuring angles with a protractor.

Here you will find a protractor with which to measure the given angle.

2. Measure the given angle from the line labelled **Start** in the direction of the arrow. You can click and drag the protractor to move it around. Click and drag on the red dot on the protractor to turn it.
3. Check your answer by checking the **Answer** box.
4. Click on the **Random** button to generate a new random angle to measure. Measure as many angles as you need to become confident.
5. Sometimes the angle you need to measure will be greater than 180°.



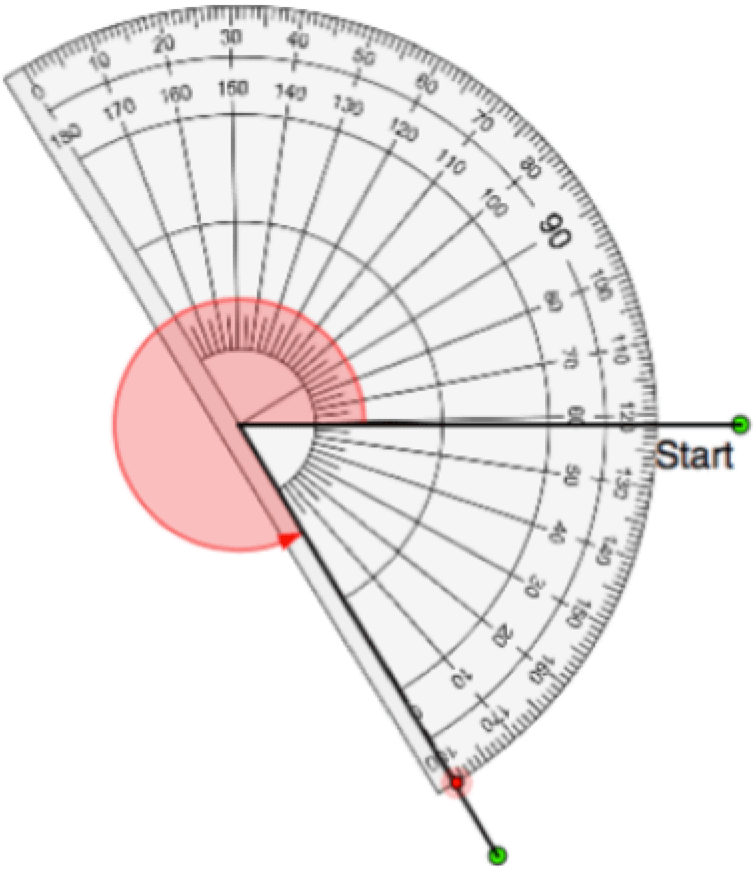
How will you measure the angle in this case shown here? Here's a hint. What if you measured the smaller angle created between the two lines that is less than $180°$? What would you subtract this angle from to find the bigger angle that is greater than $180°$? Open the online interactive activity called Measuring angles between 0 and 360 degrees with a protractor.

Here you will find a protractor with which to measure angles between $0°$ and $360°$.

What did you find?

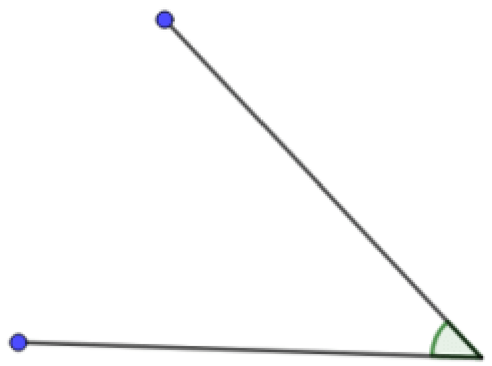
Because the online interactive activity generates random angles to measure, it is not possible to give all the answers. But you can check all your answers with the answer box.

Were you able to work out how to measure angles greater than 180°? To do so, measure the size of the angle smaller than 180° created by the two lines and then subtract this angle from 360° to find the larger angle. We can do this, because the angles around a point always add up to 360°.

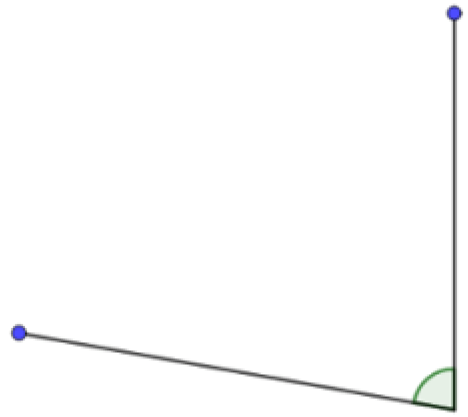


Exercise 1.3

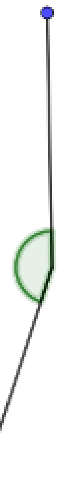
1. Use a ruler to measure the dimensions of the following everyday objects to the nearest mm.
   a. The length and width of a piece of A4 paper
   b. The height of a $2\ \ell$ cool drink bottle
   c. The height, breadth and depth of a $1\ \text{kg}$ packet of sugar
2. Use a protractor to measure the following angles to the nearest degree.
   a.



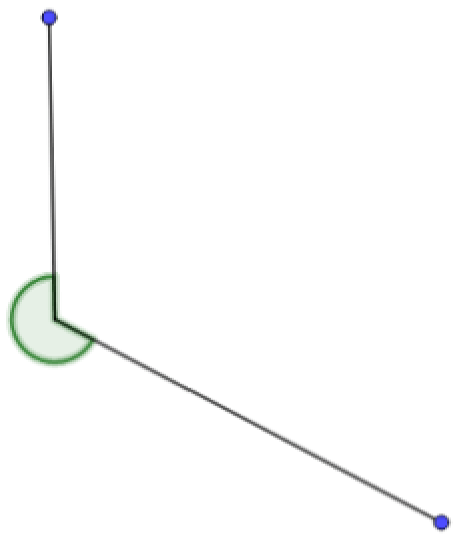
b.



c.



d.



The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- What the SI system of units and measures is.
- How to use a ruler accurately.
- How to use a protractor accurately.

Unit 1: Assessment

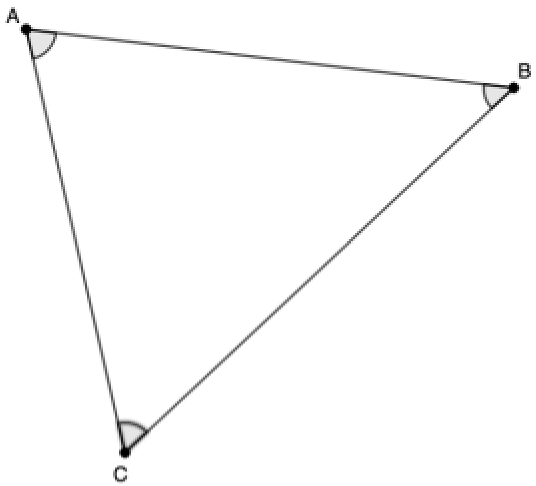
Suggested time to complete: 20 minutes

From what you know about these objects, choose the most appropriate answer for Questions 1 to 5.

1. What is the closest to the width of a grape?
   a. 15 mm
   b. 15 m
   c. 15 km
2. What is the closest to the height of a two-story building?
   a. 30 m
   b. 30 km
   c. 30 cm
3. What is the closest to the mass of an apple?
   a. 55 mg
   b. 55 g

c. 55 kg

4. What would be the best units to use to measure the width of a pen?
   a. metres
   b. centimetres
   c. micrometres
   d. millimetres
5. What would be the best units to measure the distance driven by a car
   a. metres
   b. kilometres
   c. centimetres
   d. megametres
6. Measure the size of all the angles in this triangle to the nearest degree using a protractor.



7. What is the ratio of the lengths of these two lines? In other words, how many times longer is the long line than the short line?

8. Jason accelerates to a speed of $30.56\ \mathrm{km/h}$ in $9.55\ \mathrm{s}$.
   a. To what speed in metres per second does he accelerate?
   b. If $a = \frac{s}{t}$ (acceleration equal to speed divided by time), what is Jason's average rate of acceleration in metres per second per second?

The full solutions are at the end of the unit.

Unit 1: Solutions

Exercise 1.1

1. The height of a human adult would be measured in metres.
2. The length of a newborn human would be measured in centimetres (or maybe millimetres).
3. The height of a building would be measured in metres.
4. The mass of a truck would be measured in kilograms or tons where $1 \text{ ton} = 1\ 000 \text{ kg} = 1 \text{ Mg}$.
5. The distance from Johannesburg to Cape Town would be measured in kilometres.
6. The mass of a grain of sand would be measured in grams or milligrams.
7. The length of a song would be measured in minutes or seconds.
8. The size of a bacterium would be measured in micrometres or nanometres.
9. The size of an atom would be measured in nanometres.
10. The thickness of a pane of glass would be measured in millimetres.

Back to Exercise 1.1

Exercise 1.2

1. $\dfrac{982 \text{ g}}{1\ 000} = 0.982 \text{ kg}$
   Therefore, total weight is $3.67 \text{ kg} + 0.982 \text{ kg} = 4.652 \text{ kg}$
   Shipping is $\text{R}15$ per kilogram or part. Therefore, shipping will be $\text{R}15 \times 5 = \text{R}75$.
2. $3.8 \text{ km} \times 1\ 000 = 3\ 800 \text{ m}$
   Each metre weighs 280 g. Therefore, all the cable will weigh $280 \text{ g/m} \times 3800 \text{ m} = 1\ 064\ 000 \text{ g}$.
   $\dfrac{1\ 064\ 000 \text{ g}}{1\ 000} = 1\ 064 \text{ kg}$.

Back to Exercise 1.2

Exercise 1.3

1.
   a. An A4 paper is 210 mm wide and 297 mm long.
   b. The height of a $2\ \ell$ cool drink bottle is about 327 mm.
   c. A 1 kg packet of sugar: height: 20 cm, breadth: 9 cm , depth: 5.5 cm
2.
   a. 45°
   b. 80°
   c. 161°
   d. 241°

Back to Exercise 1.3

Unit 1: Assessment

1. A)
2. A)
3. B)
4. D)
5. B)
6. Angle A: $70°$, B: $50°$, C: $60°$
7. The longer line is three times as long as the shorter line.
8.
   a. $30.56 \text{ km/h} \times 1\ 000 = 30\ 560 \text{ m/h}$

      There are $3\ 600$ seconds in one hour. Therefore $\dfrac{30\ 560 \text{ m/h}}{3\ 600} = 8.489 \text{ m/s}$

   b. $a = \dfrac{s}{t} = \dfrac{8.489 \text{ m/s}}{9.55 \text{ s}} = 0.889 \text{ m/s/s}$

Back to Unit 1: Assessment

Media Attributions

SUBJECT OUTCOME VII

SPACE, SHAPE AND MEASUREMENT: CALCULATE PERIMETER, SURFACE AREA AND VOLUME IN TWO- AND THREE-DIMENSIONAL GEOMETRICAL SHAPES

Subject outcome 3.2

Calculate perimeter, surface area and volume in two- and three-dimensional geometrical shapes

Learning outcomes

- Calculate the perimeter and surface area of the following laminas:
  - Square
  - Rectangle
  - Circle
  - Triangle
  - Parallelogram
  - Trapezium
  - Hexagon
- Calculate the volume of the following geometric objects:
  - Cubes
  - Rectangular prisms
  - Cylinders
  - Triangular prisms
  - Hexagonal prisms
- Investigate the effect on area of laminas where one or more dimensions are multiplied by a constant factor *k*.
- Investigate the effect on the volume and surface area of right prisms and cylinders, where one or more dimensions are multiplied by a constant factor *k*.

Unit outcomes: Unit 1: Properties of polygons

By the end of this unit you will be able to:

- Name the different types of polygons
- Calculate the perimeter and area of the following:
    - Square
    - Rectangle
    - Circle
    - Triangle
    - Parallelogram
    - Trapezium
    - Hexagon
- Investigate the effect on the perimeter and surface area, where one or more dimensions are multiplied by a constant factor k.

Unit outcomes: Unit 2: Volume of 3-dimensional shapes

By the end of this unit you will be able to:

- Calculate the volume of the following geometric objects:
    - Cubes
    - Rectangular prisms
    - Triangular prisms
    - Hexagonal prisms
    - Cylinders.
- Investigate the effect on the volume of right prisms and cylinders, where one or more dimensions are multiplied by a constant factor k.

Unit outcomes: Unit 3: Total surface area of 3-D shapes

By the end of this unit you will be able to:

- Calculate the surface area of the following geometric objects:
    - Cubes
    - Rectangular prisms
    - Cylinders
    - Triangular prisms
    - Hexagonal prisms
- Investigate the effect on the surface area of right prisms and cylinders, where one or more dimensions are multiplied by a constant factor k.

Unit 1: Properties of polygons

NATASHIA BEARAM-EDMUNDS

Unit outcomes: Unit 1: Properties of polygons

By the end of this unit you will be able to:

- Name the different types of polygons.
- Calculate the perimeter and area of the following:
    - Square
    - Rectangle
    - Circle
    - Triangle
    - Parallelogram
    - Trapezium
    - Hexagon.
- Investigate the effect on the perimeter and surface area, where one or more dimensions are multiplied by a constant factor *k*.

What you should know

Before you start this unit, make sure you can:

Solve algebraic equations. Go over Mathematics Level 2 Subject outcome 2.2 for more on algebraic equations.

Introduction

Some common shapes that you already know are triangles, rectangles, hexagons and squares. The name we give to these types of shapes, formed with straight lines and that can be drawn on a flat (two-dimensional) surface is **polygons**. A circle is another common shape but it is not a polygon. Why do you think that is?

Circles and shapes with curves are not polygons. A polygon, by definition, is made up of **straight lines**. Four-sided polygons are called quadrilaterals. The quadrilaterals that you should be familiar with are parallelograms, rectangles, squares, and trapeziums.

Types of polygons

Polygons are named according to the number of sides they have. They can have three or more sides. The names are based on the prefixes of Greek numbers. In fact, you will find references to Greek numerical prefixes in many everyday objects and this can help you remember the number of sides a polygon has. For example, a decade is 10 years and a decagon has 10 sides; an octopus has eight legs and an octagon has eight sides. Can you think of other examples that use Greek numerical prefixes?

There are two types of polygons, regular polygons and irregular polygons. A regular polygon has sides of equal length with equal angles between each side. In an irregular polygon, the sides have different lengths and the angles between the sides are unequal.

Figure 1 shows the different types of polygons that you may come across and examples of shapes that are not polygons.

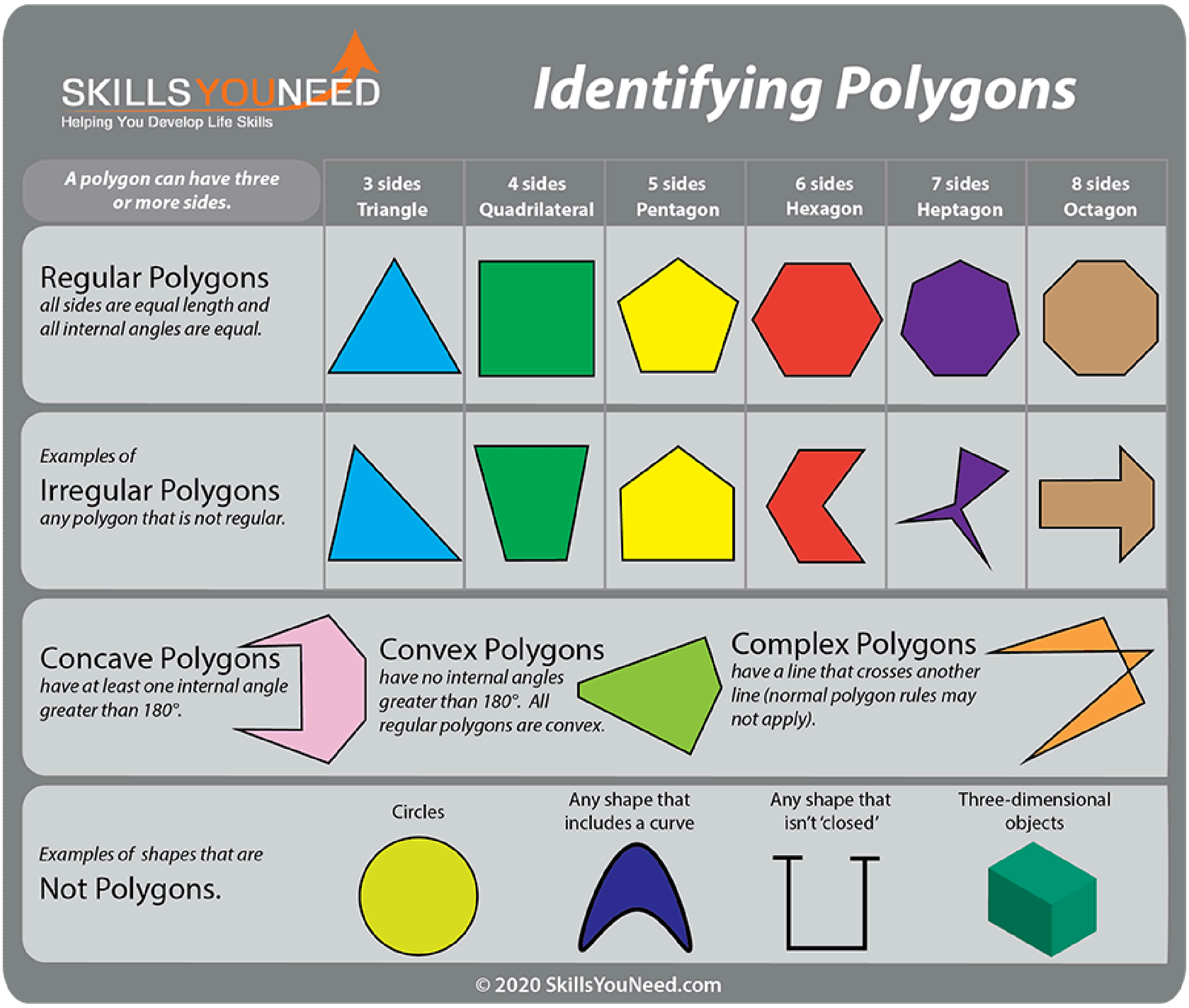


Figure 1: Different types of polygons

Note

When you have an internet connection watch the video called "Introduction to polygons" for a quick review of the different types of shapes.

Introduction to polygons (Duration: 2:24)

Polygons (and other flat shapes) make up the base and sides (or faces) of many three-dimensional (3-D) shapes. In order to work out the volume and surface area of those objects we first need to know how to calculate the area and perimeter of two-dimensional (2-D) shapes.

Calculating the surface area of polygons

You may have already calculated the surface area of objects without realising it. For example, when you cover a workbook or wrap a present you can calculate how much paper is needed to wrap the item so that you do not cut too little or too much paper. You need to understand shapes when doing home improvement, gardening and even when planning a party.

In manufacturing, working out the surface area and volume of objects is especially important to keep costs down and reduce wastage of material. We discuss volume and surface area in Units 2 and 3 of this subject outcome, but for now, we need to focus on the basics of area and perimeter.

Area is the size of the surface of a 2-D shape. It is the number of square units that fit onto a shape.

Perimeter is the total distance around any closed 2-D shape. The word comes from the Greek word peri (around) and meter (measure).

Activity 1.1: Find the area and perimeter of shapes

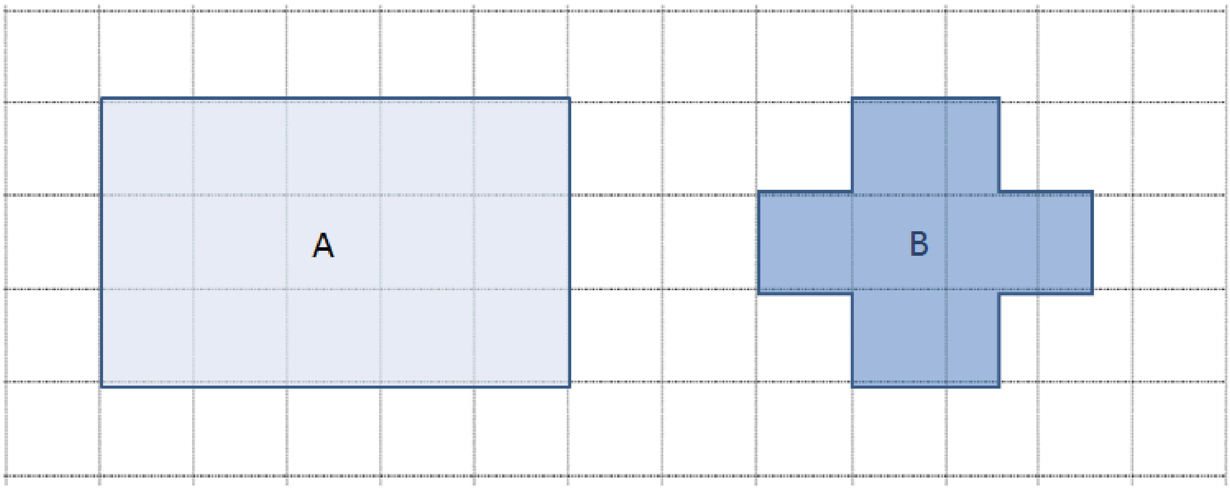
Time required: 15 minutes

What you need:

- a pen and paper

What to do:

Use the diagram below to answer the questions that follow.



1. What is the area of A?
2. Approximately how much surface does B cover? How did you work it out?
3. Is there a difference between finding 'the amount of surface covered' and finding the 'area'?
4. If each square on the graph paper is $1\ \text{cm}$ long and $1\ \text{cm}$ wide, what is the unit of measurement for the area of shapes A and B?
5. What is the perimeter of A?
6. What is the approximate perimeter of B?
7. What is the unit of measurement for the perimeter of shapes A and B?

What did you find?

1. The area of A is 15 square units.
2. B covers about 6.5 square units. We can count 5 full squares and two half squares, which makes a full square and then there is one more half square to add on.
3. Finding 'the surface covered' and finding the 'area' mean the same thing. It is helpful to think of 'the surface covered' when finding areas of faces, or sides, of geometric objects.
4. If each square on the graph paper is $1\ \text{cm}$ long and $1\ \text{cm}$ wide then the area of A and B will be in square centimetres or cm^2.
5. The perimeter of A is 16 units.
6. The approximate perimeter of B is 13 units.
7. The perimeter of A and B will be in centimetres or cm.

Counting squares is not the best or most accurate way to find the area of a shape, especially when edges don't lie exactly on the square unit measures we are using, and this method can take a lot of time. So, mathematicians came up with formulae that we can use to calculate the area of different shapes. See Table 1 for these formulae:

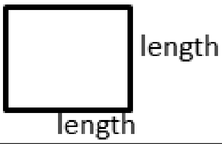
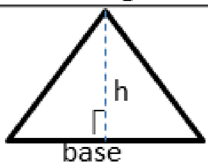
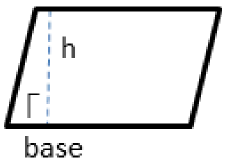
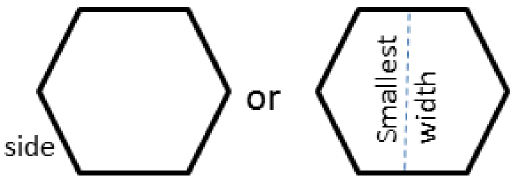
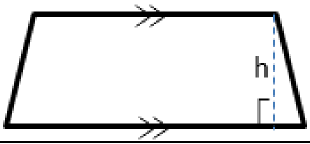
Shape	Formula
	$\text{Area}_{\text{rectangle}} = l \times w$
	$\text{Area}_{\text{square}} = l \times l$
	$\text{Area}_{\text{triangle}} = \frac{1}{2} b \times h$ where h is the perpendicular height
	$\text{Area}_{\text{parallelogram}} = b \times h$ where h is the perpendicular height
	$\text{Area}_{\text{hexagon}} = \frac{3\sqrt{3}}{2} s^2$ Or $\text{Area}_{\text{hexagon}} = \frac{\sqrt{3}}{2} W^2$ where W is the smallest width
	$\text{Area}_{\text{trapezium}} = \frac{1}{2}(\text{sum of parallel sides}) \times h$

Table 1: Formulae to calculate the areas of common shapes

You will notice that the only shape that we have not included in the table of formulae is the circle. This is because the circle deserves special mention. Before we look at the formula for area and perimeter of a circle, let's revise the parts that make up a circle.

Calculating the area of a circle

In a circle, all points on the edge are the same distance from the centre. The **radius**, r, of a circle is the length of a line drawn from the centre of the circle to any point on the circle. The **diameter**, d, is the length of a straight line drawn from one point on the circle through the centre of the circle to a point on the other side of the circle. The diameter is twice as long as the radius of a circle.

The perimeter of a circle is called the **circumference**. The really cool thing about circles is that when you divide the circumference by the diameter you get the same answer all the time! We call this number pi (π). Pi is an irrational number which means that it cannot be written as a fraction and in decimal form, the decimal places go on forever and never repeat.

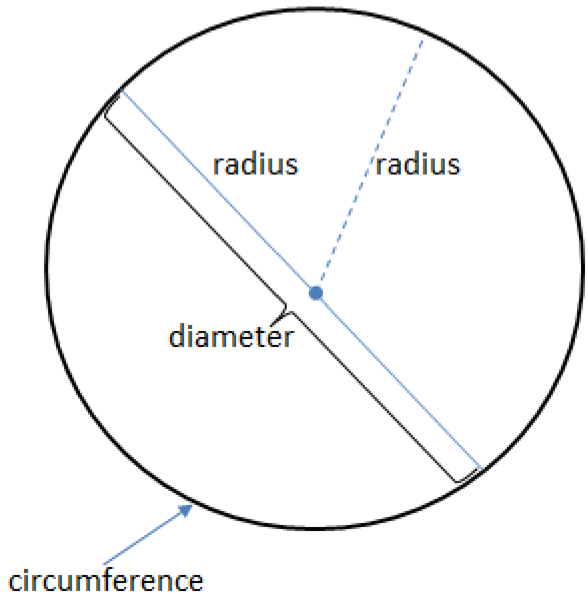


Figure 2: The parts of a circle

$$\frac{\text{circumference}}{\text{diameter}} = \pi \approx 3.14159...$$

So, when the diameter of a circle is one unit the circumference is about 3.14 units (here we have rounded pi to two decimal places).

The circumference of a circle is $\pi \times d$.

We have also seen that $d = 2r$ so it makes sense that:

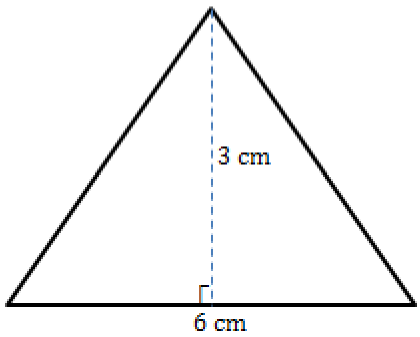
$$\begin{aligned} \text{circumference} &= \pi \times d \\ &= \pi \times 2r \\ &= 2\pi r \end{aligned}$$

The area of a circle is calculated using the formula $\text{Area}_{\text{circle}} = \pi r^2$. For accuracy, always use the π button on your calculator to calculate the area of a circle unless the question specifically states otherwise.

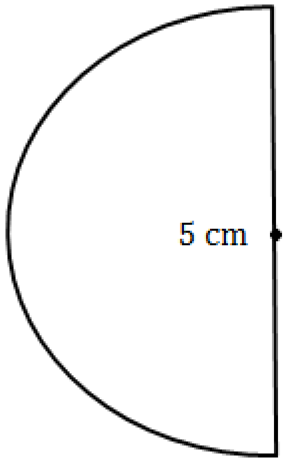
Example 1.1

Find the area of each of the polygons below:

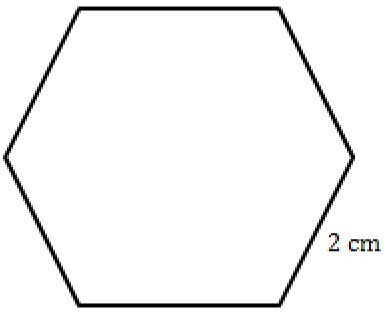
1.



2.



3. (leave answer in surd form)



Solutions

1. We are given the base and the perpendicular height of the triangle so we can substitute straight into the formula for area of a triangle.

$$\begin{aligned}\text{Area} &= \frac{1}{2}bh \\ &= \frac{1}{2}(6)(3) \\ &= 9 \text{ cm}^2\end{aligned}$$

2. Here we have a semicircle, which is half the area of a full circle so we divide the formula by two.

$$
\begin{aligned}
\text{Area} &= \frac{\pi r^2}{2} \text{ remember to use half the length of the diameter for the radius} \\
&= \frac{\pi (2.5)^2}{2} \\
&= 9.82 \text{ cm}^2
\end{aligned}
$$

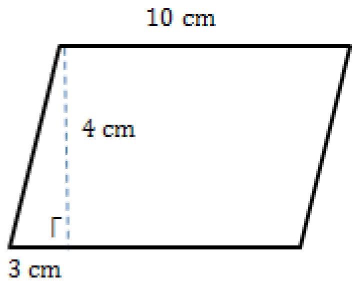
3. We are given the side length of a regular hexagon so we use the formula $\text{Area} = \frac{3\sqrt{3}}{2}s^2$.

$$
\begin{aligned}
\text{Area} &= \frac{3\sqrt{3}}{2}s^2 \\
&= \frac{3\sqrt{3}}{2}(2)^2 \\
&= 6\sqrt{3} \text{ cm}^2
\end{aligned}
$$

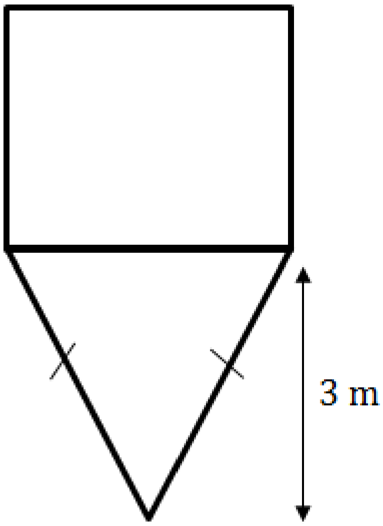
Now, try this exercise on your own.

Exercise 1.1

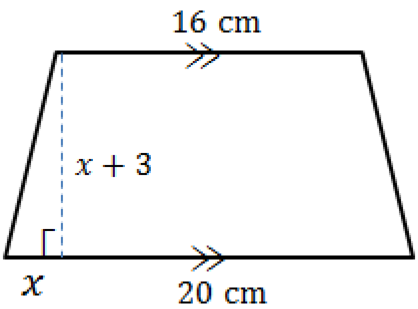
1. Find the area of the following parallelogram.



2. The following diagram shows a square attached to an isosceles triangle. The sides of the square are 2 metres in length, while the perpendicular height of the triangle is 3 metres.
   a. Calculate the area of the shape.
   b. If the equal sides of the triangle measure $\sqrt{10}$ metres calculate the perimeter of the shape (correct to one decimal place).



3. Find the area of the trapezium shown below in terms of x.



The full solutions are at the end of the unit.

Changing dimensions of a 2-D shape

Remember that a 2-D shape has two dimensions. As you have seen when we calculated areas, a rectangle has a length and a width (or breadth), a square has an equal length and a width, triangles and parallelograms have bases and heights, and so on. In simple terms, dimension is the size of the lengths that make up a shape.

Next, we investigate how changing one or both of the dimensions of a shape affects its perimeter and area.

Activity 1.2: Investigate the effect on area and perimeter when multiplying dimensions by a factor of *k*

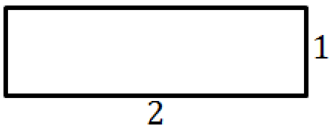
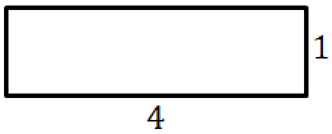
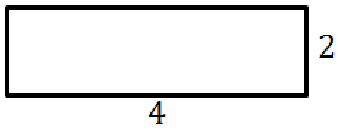
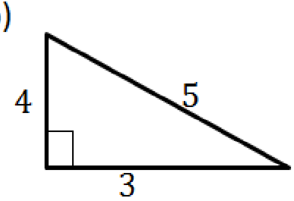
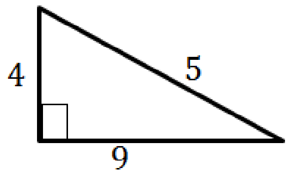
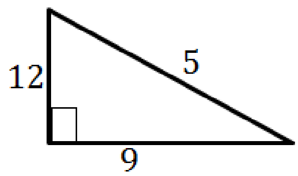
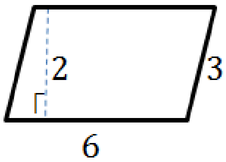
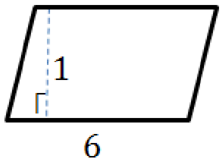
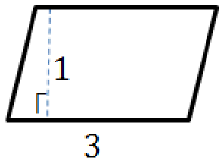
Time required: 20 minutes

What you need:

- a pen and paper

What to do:

Each column below shows an original shape, the shape with one of its dimensions changed and the shape with both dimensions changed by the same value. Use the table to answer the questions that follow.

Original shape	One dimension multiplied by k	Two dimensions multiplied by k
a) 		
b) 		
c) 		

1. State the value of k in each case.
2. Find the area of the shapes in all three columns.
3. Compare the area of the original shape to the area when one dimension is changed. What can you conclude?
4. Compare the area of the original shape to the area when two dimensions are changed. What can you conclude?
5. Find the perimeter of the shapes in all three columns.
6. Compare the perimeter of the original shape to the perimeter when one dimension is changed. What can you conclude?
7. Compare the perimeter of the original shape to the perimeter when the two dimensions used to find the area of a shape are changed. What can you conclude?
8. What would happen to the perimeter if all the sides of a shape are multiplied by a factor of k? Find the perimeter in each case.

What did you find?

1.

a. $k = 2$

b. $k = 3$

c. $k = \frac{1}{2}$

2.

Original shape	One dimension multiplied by k	Two dimensions multiplied by k
a) Area $= 1 \times 2 = 2$	Area $= 1 \times 2(2) = 4$	Area $= 1(2) \times 2(2) = 8$
b) Area $= 3 \times 4 = 12$	Area $= 3(3) \times 4 = 36$	Area $= 3(3) \times 4(3) = 108$
c) Area $= 2 \times 6 = 12$	Area $= 2(\frac{1}{2}) \times 6 = 6$	Area $= 2(\frac{1}{2}) \times 6(\frac{1}{2}) = 3$

3. When you multiply one dimension by k then the new area is $k \times \text{original area}$.
4. When you multiply two dimensions by k then the new area is $k^2 \times \text{original area}$.
5.

Original shape	One dimension multiplied by k	Two dimensions multiplied by k
a) $\text{P} = 2(1 + 2) = 6$	$\text{P} = 2(1 + 4) = 10$	$\text{P} = 2(2 + 4) = 12$
b) $\text{P} = 3 + 4 + 5 = 12$	$\text{P} = 9 + 4 + 5 = 18$	$\text{P} = 9 + 12 + 5 = 26$
c) $\text{P} = 2(3 + 6) = 18$	$\text{P} = 2(3 + 6) = 18$	$\text{P} = 2(3 + 3) = 12$

6. There is no specific pattern that we see with perimeter when only one dimension is multiplied by a factor of k.
7. When the two dimensions used for the area are multiplied by a factor of k there is no pattern observed in the perimeter.
8. If you multiply all the sides of a shape by a factor of k the perimeter is $k \times \text{original perimeter}$.

Original shape	All sides multiplied by k
a) $\text{P} = 2(1 + 2) = 6$	$\text{P} = 2(2 + 4) = 2(6) = 12$
b) $\text{P} = 3 + 4 + 5 = 12$	$\text{P} = 9 + 12 + 15 = 3(12) = 36$
c) $\text{P} = 2(3 + 6) = 18$	$\text{P} = 2(1.5 + 3) = \frac{1}{2}(18) = 9$

We have seen that multiplying one or more dimensions by a factor of k affects the area. If the dimensions that are changed **are part of the area formula** itself then we can conclude that:

1. When you multiply one dimension by k then the new area is $k \times \text{original area}$.
2. When you multiply two dimensions by k then the new area is $k^2 \times \text{original area}$.

We have also found that when we multiply **all the sides** of a shape by a factor of k then the perimeter is $k \times \text{original perimeter}$.

Example 1.2

1. A circular room has a radius of x.
   1. Find the area of the room.
   2. Find the circumference of the room.
   3. If the radius is doubled find the new area and circumference of the room. Compare this to the original values.
2. The area of a rectangular table is $2\ \mathrm{m}^2$. Without doing any further calculations find:
   1. The area of the table if its length is doubled.
   2. The area of the table if the length and width are doubled.

Solutions

1.
   a.
   $$\begin{aligned}\text{Area} &= \pi r^2\\ &= \pi x^2\end{aligned}$$
   b. $Circumference = 2\pi(x)$
   c. The new radius will be $2x$.
   $$\begin{aligned}\text{Area}_{\text{new}} &= \pi(2x)^2\\ &= 4\pi x^2\end{aligned}$$
   When you double the radius, the area increases by four times since the radius is squared in the area formula.

$$\begin{aligned}\text{Perimeter}_{\text{new}} &= 2\pi(2x)\\ &= 4\pi x\end{aligned}$$

When the radius is doubled, the circumference is doubled too.

The area of a circle is directly proportional to the square of the radius of the circle. This means that if the radius is increased by a factor of k, its area will be increased to k^2 times the original area.

The circumference is directly proportional to the size of the radius so if the radius of a circle is increased by a factor of k, its circumference will increase by a factor of k too.

2.
   a. When you multiply one dimension by k then the new area is $k \times \text{original area}$.
   $$\begin{aligned}\text{Area}_{\text{new}} &= 2 \times 2\ \mathrm{m}^2\\ &= 4\ \mathrm{m}^2\end{aligned}$$
   b. When you multiply two dimensions by k then the new area is $k^2 \times \text{original area}$.
   $$\begin{aligned}\text{Area}_{\text{new}} &= (2)^2 \times 2\ \mathrm{m}^2\\ &= 8\ \mathrm{m}^2\end{aligned}$$

Summary

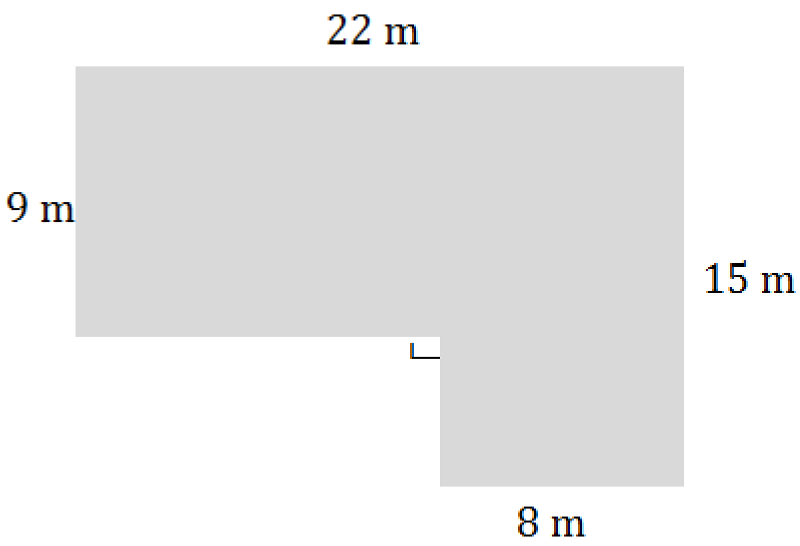
In this unit you have learnt the following:

- How to identify different polygons.
- How to find the area of a square, rectangle, circle, triangle, parallelogram, trapezium and hexagon.
- How multiplying the dimensions by a scale factor of k affects the area.
- How multiplying the dimensions by a scale factor of k affects the perimeter.
- How multiplying all the sides by a scale factor of k affects the perimeter.

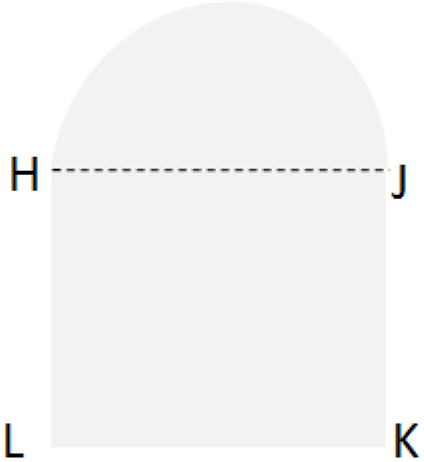
Unit 1: Assessment

Suggested time to complete: 20 minutes

1. The figure below shows the floor plan of a room. Find the area of the room.



2. The shape below is that of a window, consisting of a rectangular section HJKL and a semi-circular top section. $\mathrm{HJ} = 0.5\ \mathrm{m}$ and $\mathrm{JK} = 0.2\ \mathrm{m}$. Calculate the area and perimeter.



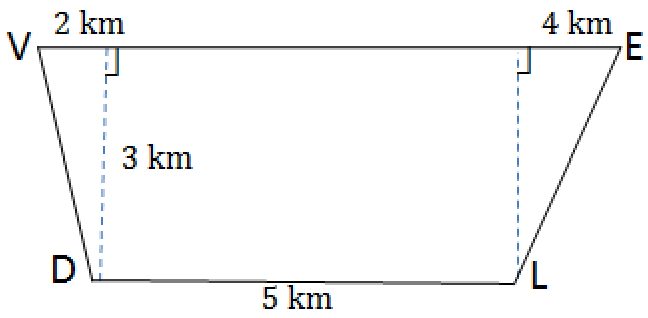
3. A rectangular storeroom has length $3x$ and width $6y$. Write simplified expressions for the:

a. area of the storeroom.

b. perimeter of the storeroom.

c. If the dimensions of the storeroom are tripled, write down the new simplified expression for the area of the storeroom.

4. The following diagram shows a field in the shape of a trapezium. Liza will run in a straight line from V to E to L to D to train for a race.
Use the sketch to calculate the total distance she runs.
(HINT: Use Pythagoras)



The full solutions are at the end of the unit.

Unit 1: Solutions

Exercise 1.1

1. The opposite sides of a parallelogram (parm for short) are equal therefore, the base of the parm is 10 cm.
Use the formula $\text{Area}_{parm} = b \times h$

$$\begin{aligned}\text{Area}_{parm} &= 10\text{ cm} \times 4\text{ cm}\\ &= 40\text{ cm}^2\end{aligned}$$

2. The base of the triangle is a side of the square.

a.

$$\begin{aligned}\text{Total area} &= \text{Area of square} + \text{Area of triangle}\\ &= l^2 + \frac{1}{2}b \cdot h\\ &= 2\text{ m}^2 + \frac{1}{2}(2\text{ m}) \cdot (3\text{ m})\\ &= 2\text{ m}^2 + 3\text{ m}^2\\ &= 5\text{ m}^2\end{aligned}$$

b. Be careful not to count the length that forms part of the boundary of the square and triangle.

$$\begin{aligned}\text{Perimeter} &= 2\text{ m} + 2\text{ m} + 2\text{ m} + \sqrt{10}\text{ m} + \sqrt{10}\text{ m}\\ &= 6 + 2\sqrt{10}\\ &= 12.3\text{ m}\end{aligned}$$

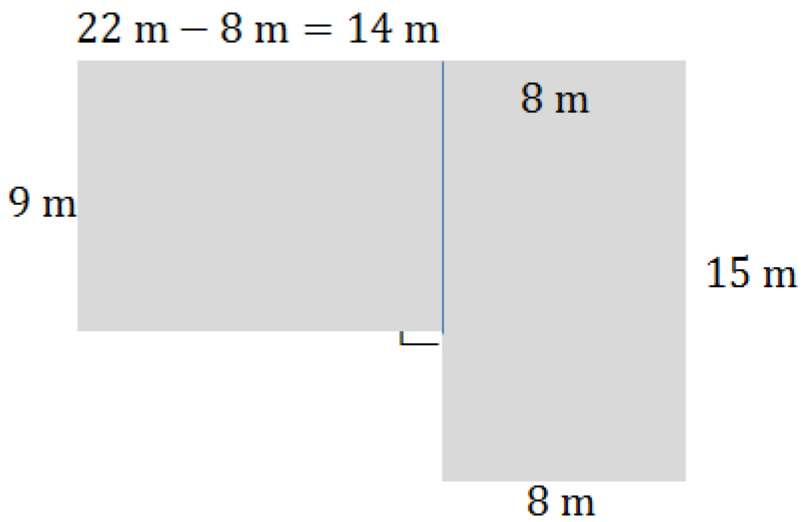
3.

$$
\begin{aligned}
\text{Area}_{\text{trapezium}} &= \frac{1}{2}(\text{sum of parallel sides}) \times h \\
&= \frac{1}{2}[(x+20)+16] \cdot (x+3) \\
&= \frac{1}{2}(x+36)(x+3) \\
&= \frac{1}{2}(x^2+39x+108) \\
&= \frac{1}{2}x^2 + \frac{39}{2}x + 54
\end{aligned}
$$

Back to Exercise 1.1

Assessment

1. We can calculate the area of the room by dividing the floor into two rectangles.



$$
\begin{aligned}
\text{Area} &= (l \times b) + (l \times b) \\
&= (14 \times 9) + (15 \times 8) \\
&= 126 + 120 \\
&= 246\text{ m}^2
\end{aligned}
$$

2.

$$
\begin{aligned}
\text{Area} &= \frac{\pi r^2}{2} + l \times b \\
&= \left(\frac{\pi(0.25)^2}{2}\right) + (0.2 \times 0.5) \\
&= 0.2\text{ m}^2
\end{aligned}
$$

3.

a.

$$
\begin{aligned}
\text{Area} &= 3x \times 6y \\
&= 18xy
\end{aligned}
$$

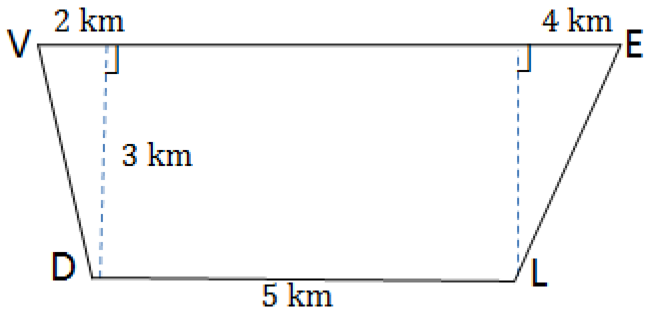
b.

$$
\begin{aligned}
\text{Perimeter} &= 2(3x+6y) \\
&= 6x + 12y
\end{aligned}
$$

c.

$$\begin{aligned}\text{Area}_{\text{new}} &= (3^2)18xy \\ &= 162xy\end{aligned}$$

4. Calculate VD and EL by Pythagoras. The perpendicular distance between parallel lines is always equal, therefore the distance between VE and DL is 3 m.



$$\begin{aligned}\text{VD}^2 &= (2)^2 + (3)^2 \text{ (Pythagoras)} \\ &= 4 + 9 \\ \therefore \text{VD} &= \sqrt{13} \text{ km}\end{aligned}$$

$$\begin{aligned}\text{EL}^2 &= (4)^2 + (3)^2 \text{ (Pythagoras)} \\ &= 16 + 9 \\ \text{VD} &= \sqrt{25} \\ &= 5 \text{ km}\end{aligned}$$

$$\begin{aligned}\text{VD} + \text{VE} + \text{EL} + \text{LD} &= \sqrt{13} \text{ km } + 2 \text{ km} + 5 \text{ km} + 4 \text{ km} + 5 \text{ km} + 5 \text{ km} \\ &= \sqrt{13} \text{ km } + 21 \text{ km} \\ &= 24.6 \text{ km}\end{aligned}$$

Back to Unit 1: Assessment

Media Attributions

Unit 2: Volume of 3-dimensional shapes

NATASHIA BEARAM-EDMUNDS

Unit outcomes: Unit 2: Volume of 3-dimensional shapes

By the end of this unit you will be able to:

- Calculate the volume of the following geometric objects:
    - Cubes
    - Rectangular prisms
    - Triangular prisms
    - Hexagonal prisms
    - Cylinders.
- Investigate the effect on the volume of right prisms and cylinders, where one or more dimensions are multiplied by a constant factor *k*.

What you should know

Before you start this unit, make sure you can:

- Find the area and perimeter of 2-dimensional shapes. You can revise area and perimeter in Unit 1: Properties of polygons of this subject outcome.

Introduction

A 3-dimensional (3-D) shape is a solid or an object with three dimensions; length, width (also called breadth) and height. We saw in Unit 1 that polygons make up the bases and sides, also called faces, of 3-D objects. Every 3-D solid without a curved surface has a **face**, **edge** and **vertex**.

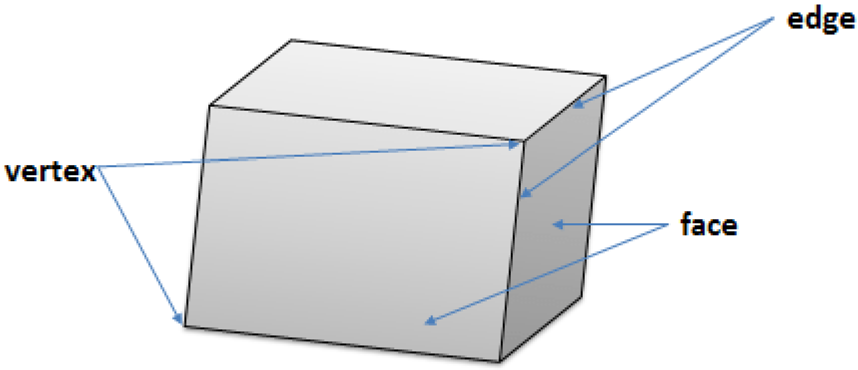


Figure 1: A 3-D object, also called a polyhedron

A face is a flat surface of a 3-D object. An edge is a line segment where two faces meet. A vertex is the point where the edges meet. A shape that has these properties is called a **polyhedron**. Examples of polyhedrons are the square pyramid (see Figure 2) and the dodecahedron (see Figure 3).

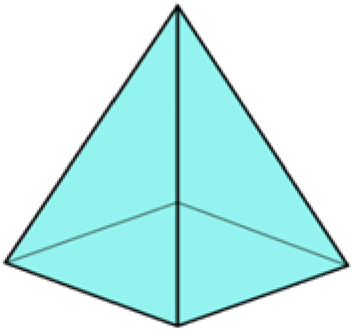


Figure 2: Square pyramid

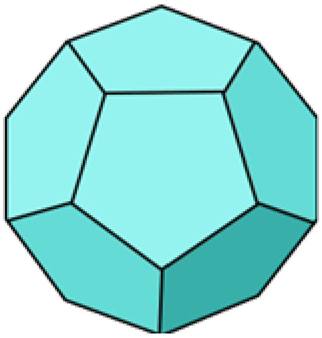


Figure 3 : Dodecahedron

Prisms

There are different types of polyhedrons but in this unit we will only focus on one type called **prisms**. Prisms have the same shape (face) on the top and on the base and these faces are parallel to each other. A parallelepiped (see Figure 4) is an example of a prism. Here the base and top are both rectangles.

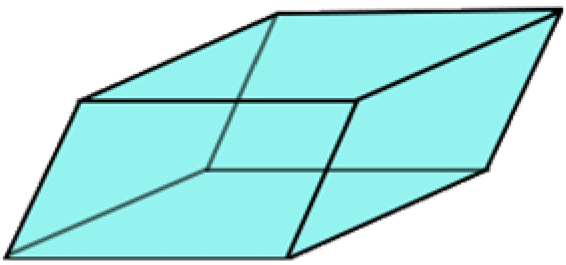


Figure 4: A parallelepiped, which is a type of prism

A cube (see Figure 5) is also an example of a prism. It has square faces on the base and on the top (and on its sides too). But, a cube is a special kind of prism called a **right prism**. In a right prism the side faces are perpendicular or at right-angles to the base. In a right prism, the sides, or faces are always rectangles. Remember that all squares are also rectangles.

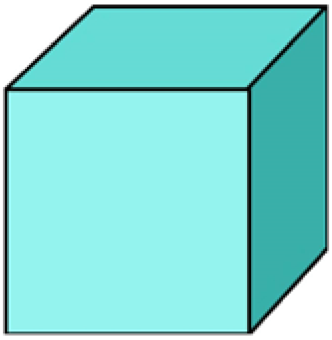


Figure 5: A cube

Solids with curved surfaces, such as cylinders, spheres and cones are **not** polyhedrons.

Figure 6: Non polyhedrons

You come across 3-D objects every day. A toilet paper roll is cylindrical in shape. Dice are cubic. A beach ball is spherical and a party hat is cone shaped. Can you name a few other everyday items that are shaped like cylinders, cubes, spheres or cones?

Volume of prisms

So far we have seen that in prisms the top and the base are the same polygon and are parallel to each other. In right prisms the side faces are perpendicular to the base and top and are rectangles.

There is another type of prism called an **oblique** (slanting) prism. These are like the parallelepiped in Figure 4. In oblique prisms the sides, or faces, are not perpendicular to the base and top. We will only cover finding the volume of right prisms and cylinders in this unit and for exam purposes.

Activity 2.1: Identify prisms

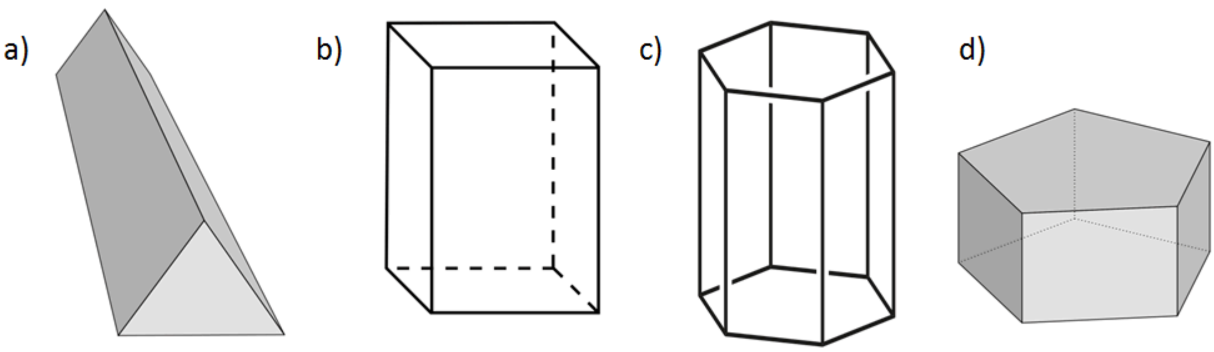
Time required: 10 minutes

What you need:

- a pen and paper

What to do

Study the right prisms below and answer the questions that follow.



1. What are the top and base polygons in each case?
2. How many faces does each object have?
3. How many edges does each object have?
4. How many vertices does each object have?
5. Based on your answer to question 1, and if we name prisms according to the base faces, what do think each prism is called?

What did you find?

The answers are in the table below.

Top and base polygons	Number of faces	Number of edges	Number of vertices	Prism name
a) Triangles	5 faces	9 edges	6 vertices	Triangular base so triangular prism
b) Squares	6 faces	12 edges	8 vertices	Square-based prism, cuboid or rectangular prism
c) Hexagons	8 faces	18 edges	12 vertices	Hexagonal prism
d) Pentagons	7 faces	15 edges	10 vertices	Pentagonal prism

Note

For more on prisms and other 3-D shapes watch the video called “Three dimensional shapes” when you have access to the internet.

Three dimensional shapes (Duration: 08.18)

Table 1 shows a summary of the common prisms and 3-D solids you will come across in this unit and their properties.

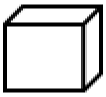
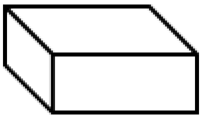
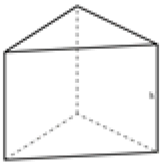
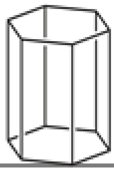
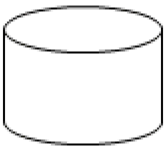
3-D object	Description
	**Cube** A cube has 6 identical square faces. It has 12 edges and 8 vertices.
	**Rectangular prism** A rectangular prism, also called a cuboid, has 6 rectangular faces, 12 edges and 8 vertices.
	**Triangular prism** A right triangular prism has 2 triangular faces and three rectangle faces. It has 5 faces, 9 edges, and 6 vertices.
	**Hexagonal prism** A hexagonal prism has 2 hexagon faces and six rectangular faces. It has 8 faces, 18 edges and 12 vertices.
	**Cylinder** A cylinder is not a prism since it has curved surfaces. It is made up of 2 identical circles and a rectangle with length equal to the circumference of the circles.

Table 1

Recognising different types of 3-D objects so we can find the volume and surface area has important applications in many industries. For example, in manufacturing knowing the volume and surface area is necessary to minimise costs and maximise production.

Volume is the amount of 3-D space taken up by a solid, liquid or gas. Volume is measured in cubes or cubic units.

If you look at the label of any cold drink can, you will see the volume of liquid inside the container. Manufacturers pay close attention to the volume of their products to make sure they deliver an accurate amount every time. You would not be happy if a bottle was marked as containing $400\ \mathrm{ml}$ of a product but only contained $300\ \mathrm{ml}$ when you open it!

Can you tell just by looking at the objects below which one takes up more space?

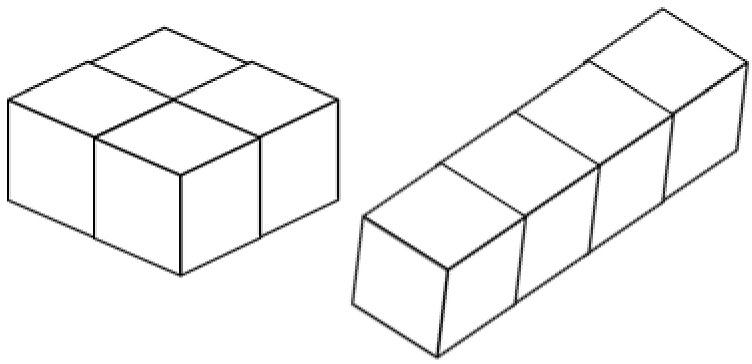


Figure 7: Which object takes up more space?

Count the number of cubes that make up each object. There are 4 cubes that make up the first object and 4 cubes that make up the second object. Both objects occupy the same amount of space because their volumes are both 4 cubic units. So by counting how many cubes it will take to fill up an object we can calculate its volume.

Count the cubes that make up the solids shown in Figure 8. What is the volume of each one? Which object takes up more space?

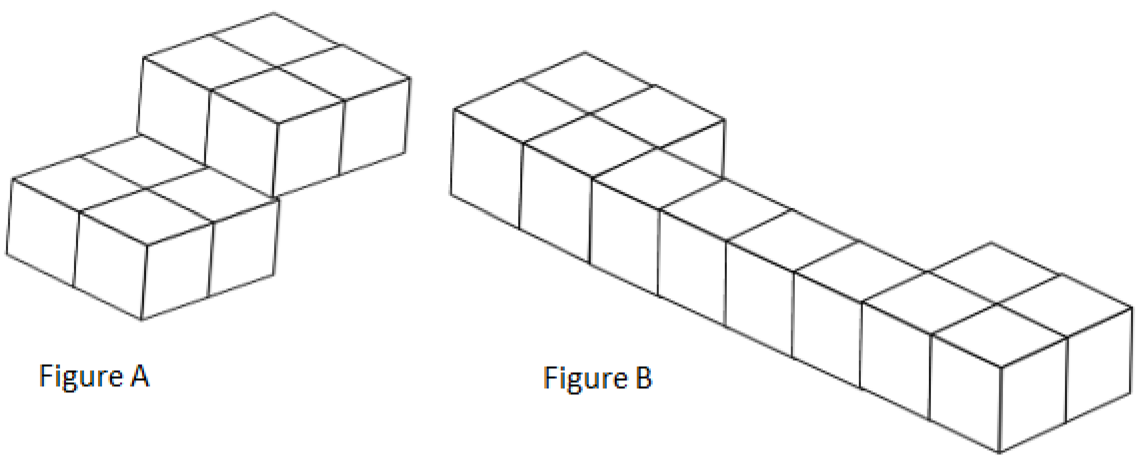


Figure 8 A and B: Calculate the volume of each object

There are 8 cubes that make up Figure 8A so its volume is 8 cubic units. There are 12 cubes that make up Figure 8B so its volume is 12 cubic units. Therefore, Figure 8B takes up more space as it has the bigger volume.

Breaking 3-D objects into cubes and counting them is a long and inefficient way to find the volume of an object. Counting in this way can also lead to careless mistakes. There is a much easier way to calculate the volume using a simple formula, which we discuss in the example that follows.

The volume of a **right prism** is calculated by multiplying the area of the base by the height of the solid.

$\text{V} = \text{Area of base} \times \text{height}$

Example 2.1

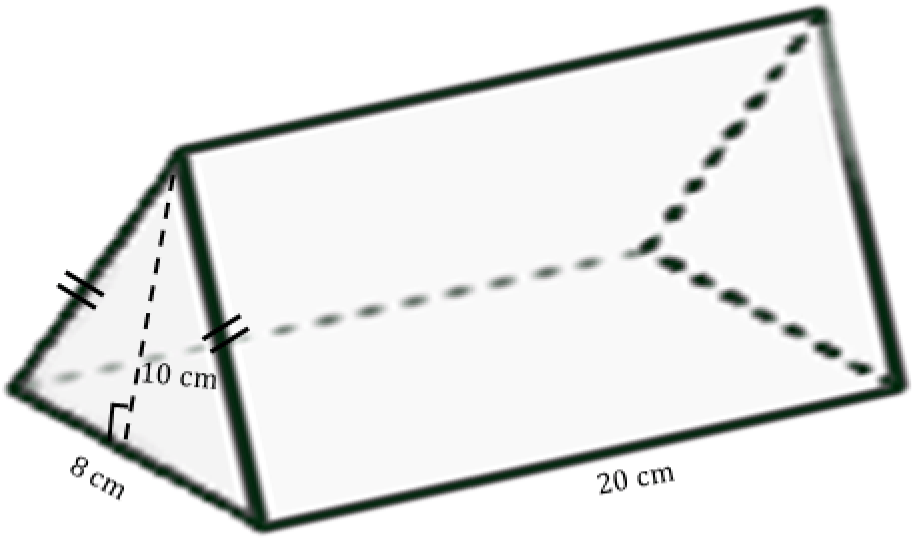
1. An ice cube has a side of 2 cm. What is the volume of the ice cube?
2. A swimming pool is shaped as a rectangular prism. It is 6 m long, 3 m wide and 2 m deep. How much water is needed to completely fill the swimming pool?
3. A cylinder-shaped glass has a radius of 4 cm and height of 6 cm. If the glass is half filled with water what is the volume of water in the glass?

Solutions

1. All the sides and height of a cube have the same measurement.
$$\begin{aligned} \text{V}_{\text{cube}} &= \text{area square base} \times \text{height of cube} \\ &= (2 \text{ cm} \times 2 \text{ cm}) \times 2 \text{ cm} \\ &= 8 \text{ cm}^3 \end{aligned}$$
2. We use the formula $\text{V} = \text{Area of base} \times \text{height}$.
In this case the base is a rectangle.
$$\begin{aligned} \text{V} &= (l \times b) \times h \\ &= (6 \text{ m} \times 3 \text{ m}) \times 2 \text{ m} \\ &= 36 \text{ m}^3 \end{aligned}$$
3. Here we need to remember the formula for the area of a circle because a cylinder has a circular base.
First, calculate the total volume of the glass.
$\text{V} = \text{Area of base} \times \text{height}$
$$\begin{aligned} \text{V}_{\text{glass}} &= \text{Area of circle} \times \text{height} \\ &= \pi r^2 \times \text{h} \\ &= \pi(4 \text{ cm})^2 \times 6 \text{ cm} \\ &= 301.6 \text{ cm}^3 \end{aligned}$$
Next, find a half the volume of the glass to calculate the actual volume of water in the glass.
$$\begin{aligned} \text{V}_{\text{water}} &= 301.6 \text{ cm}^3 \times \frac{1}{2} \\ &= 150.8 \text{ cm}^3 \end{aligned}$$

Exercise 2.1

1. Find the volume of this triangular prism.



2. Find the volume of a cylinder with a radius of $40\ \text{cm}$ and a height of $1.5\ \text{m}$ (correct to 1 decimal place).
3. A cylindrical jug has a diameter of $142\ \text{mm}$ and a height of $28\ \text{cm}$. Will the jug be able to hold $5\ \text{l}$ of water? (HINT $1\ 000\ \text{cm}^3 = 1\ \text{l}$).

The full solutions are at the end of the unit.

Multiplying dimensions by a factor of k

When one or more of the dimensions of a prism or cylinder is multiplied by a constant, the volume will change.

Once we understand the relationship between the change in dimensions and the resulting change in volume it becomes simpler to calculate the new volume of an object when its dimensions are scaled up or down.

Example 2.2

Consider a rectangular prism with dimensions l, b and h. Complete the table to see how multiplying the dimensions by a scale factor affects the volume.

Dimensions	Volume
Original dimensions l, b and h	
Multiply one dimension by 3 l, b and $3\mathrm{h}$	
Multiply two dimensions by 3 $\mathrm{l}, 3\mathrm{b}$ and $3\mathrm{h}$	
Multiply all three dimensions by 3 $3\mathrm{l}, 3\mathrm{b}$ and $3\mathrm{h}$	
Multiply all three dimensions by k $k\mathrm{l}, k\mathrm{b}$ and $k\mathrm{h}$	

Solutions

Dimensions	Volume
Original dimensions l, b and h	$\mathrm{V} = \mathrm{l} \times \mathrm{b} \times \mathrm{h}$ $= \mathrm{lbh}$
Multiply one dimension by 3 l, b and $3\mathrm{h}$	$\mathrm{V}_1 = \mathrm{l} \times \mathrm{b} \times 3\mathrm{h}$ $= 3\mathrm{lbh}$ $= 3\mathrm{V}$
Multiply two dimensions by 3 $\mathrm{l}, 3\mathrm{b}$ and $3\mathrm{h}$	$\mathrm{V}_2 = \mathrm{l} \times 3\mathrm{b} \times 3\mathrm{h}$ $= 3 \times 3\mathrm{lbh}$ $= 3^2\mathrm{V}$
Multiply all three dimensions by 3 $3\mathrm{l}, 3\mathrm{b}$ and $3\mathrm{h}$	$\mathrm{V}_3 = 3\mathrm{l} \times 3\mathrm{b} \times 3\mathrm{h}$ $= 3 \times 3 \times 3\mathrm{lbh}$ $= 3^3\mathrm{V}$
Multiply all three dimensions by k $k\mathrm{l}, k\mathrm{b}$ and $k\mathrm{h}$	$\mathrm{V}_1 = k\mathrm{l} \times k\mathrm{b} \times k\mathrm{h}$ $= k \times k \times k\mathrm{lbh}$ $= k^3\mathrm{V}$

Note

- If one dimension of a prism is multiplied by a factor of k the volume of the prism is changed by a factor of k.
- If two dimensions of a prism are multiplied by a factor of k the volume of the prism is changed by a factor of k^2.
- If all three dimensions of a prism are multiplied by a factor of k the volume of the prism is changed by a factor of k^3.

Using the above information we can easily calculate the new volume of a prism. If, for example, the height is doubled, the new volume will also be twice the original volume.

Exercise 2.2

1. Consider a rectangular prism with a height of 4 cm and base lengths of 3 cm.
    a. Calculate the volume.
    b. Calculate the new volume if the base lengths are multiplied by a constant factor of 4.
    c. Express the new volume as a factor of the original volume.
2. If the volume of a cylinder is $V = 8\pi$, what will the new volume be if the radius is multiplied by 2?
3. Given a prism with a volume of 493 cm^3, find the new volume for a prism if all dimensions are multiplied by a constant factor of 5.

The full solutions are at the end of the unit.

Summary

- In this unit you have learnt the following:
- How to identify different types of prisms based on their properties.
- How to find the volume of right prisms.
- How to find the volume of cylinders.
- How the volume is affected when the dimensions are multiplied by a constant factor.

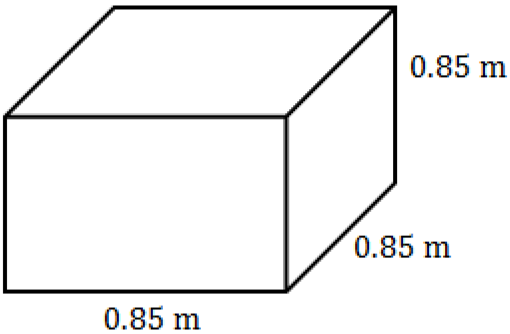
Unit 2: Assessment

Suggested time to complete: 20 minutes

1. Alfred uses a 'piggy bank' that is cylindrical in shape. The 'piggy bank' is constructed from tin and has a slot at the top. The area of the slot is 2 cm^2. The diameter of the base of the 'piggy bank' is 11 cm and the height of the 'piggy bank' is 7 cm.

Determine the volume of the ‘piggy bank’.

2. Thabo has glued together five square pieces of glass to build a fish tank which is open at the top. The length, breadth and height of the tank are all $0.85\ \text{m}$. The completed fish tank is shown below.



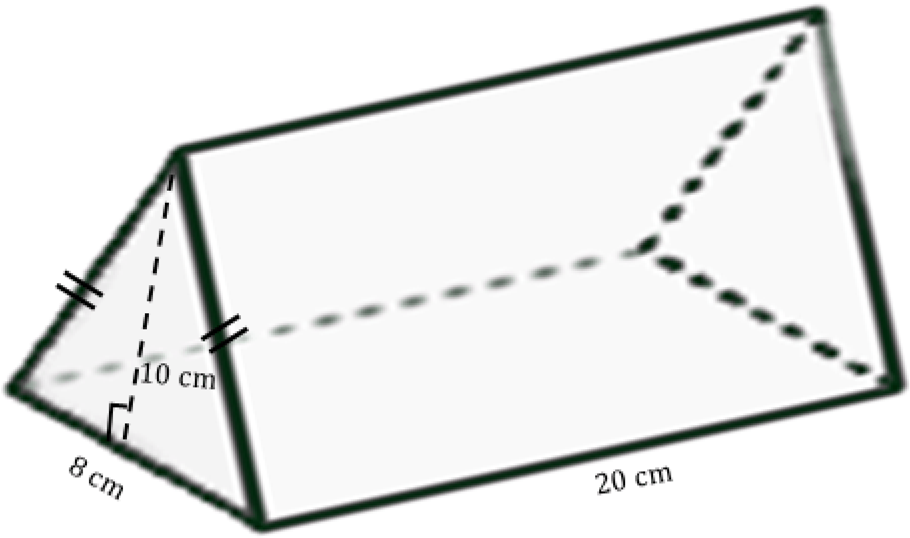
   a. Thabo wants to test his new tank for water leaks. Determine the volume of the water, in m^3, he would need to fill the tank to the top. (Ignore the thickness of the glass in your calculations).
   b. Thabo wants to buy a glass lid for his tank. To prevent the lid from falling into the tank, he decides to buy a lid which is 0.01 times longer than the original length of each side of the tank. Determine the area (in m^2) of the glass Thabo needs to buy for the lid.

The full solutions are at the end of the unit.

Unit 2: Solutions

Exercise 2.1

1.



$$\begin{aligned}
\text{Volume} &= \text{Area of base} \times \text{height} \\
&= \text{Area of triangle} \times \text{height} \\
&= (\frac{1}{2}\text{bh}) \times 20 \\
&= (\frac{1}{2} \cdot 8 \cdot 10) \times 20 \\
&= 800 \text{ cm}^2
\end{aligned}$$

2. Make sure that the radius and height are in the same units before finding the volume. Convert either the radius to metres or the height to centimetres.

$$\begin{aligned}
40 \text{ cm} &= \frac{40}{100} \text{ m } (1 \text{ cm} = 0.01 \text{ m}) \\
&= 0.4 \text{ m} \\
\text{V} &= \text{Area of base} \times \text{height} \\
&= \pi r^2 \times \text{h} \\
&= \pi (0.4)^2 \times 1.5 \\
&= 0.8 \text{ m}^3 \text{ Or } 800\ 000 \text{ cm}^3 \ (1 \text{ m}^3 = 1\ 000\ 000 \text{ cm}^3)
\end{aligned}$$

3. First calculate the volume of the jug, remembering to change the diameter from mm to cm and divide the diameter by two to find the radius. Then convert the volume of water from cm^3 to litres.

$$\begin{aligned}
142 \text{ mm} &= \frac{142}{10} \text{ cm } (1 \text{ mm} = 0.01 \text{ cm}) \\
&= 14.2 \text{ cm} \\
\therefore \text{radius} &= \frac{14.2 \text{ cm}}{2} \\
&= 7.1 \text{ cm} \\
\text{V} &= \pi r^2 \times \text{h} \\
&= \pi (7.1 \text{ cm})^2 \times 28 \text{ cm} \\
&= 4\ 434 \text{ cm}^3
\end{aligned}$$

$$\begin{aligned}
4\ 434 \text{ cm}^3 &= \frac{4\ 434}{1\ 000} \\
&= 4.434 \text{ l}
\end{aligned}$$

The jug will not be able to hold 5 l of water because the most water it can hold is 4.4 l.

Back to Exercise 2.1

Exercise 2.2

1.
   a.
   $$\begin{aligned} V &= l \times b \times h \\ &= 3 \times 3 \times 4 \\ &= 36 \text{ cm}^3 \end{aligned}$$
   b. If two dimensions of a prism are multiplied by a factor of k, the volume of the prism is changed by a factor of k^2.
   $$\begin{aligned} V_{new} &= (4)^2 \times 36 \text{ cm}^3 \\ &= 576 \text{ cm}^3 \end{aligned}$$
   c. $V_{new} = 16 \times V_{original}$
2. $V_{cyclinder} = \pi r^2 h$
   If the radius is multiplied by a factor of k then:
   $$\begin{aligned} V &= \pi (kr)^2 \times h \\ &= k^2 \times \pi r^2 h \end{aligned}$$
   So the new volume is k^2 multiplied by the original volume:
   $$\begin{aligned} V &= 8\pi \\ V_{new} &= (2)^2 \times 8\pi \\ &= 32\pi \end{aligned}$$
3. If all three dimensions of a prism are multiplied by a factor of k the volume of the prism is changed by a factor of k^3.
   $$\begin{aligned} V_{new} &= (5)^3 \times 493 \text{ cm}^3 \\ &= 61\ 625 \text{ cm}^3 \end{aligned}$$

Back to Exercise 2.2

Unit 2: Assessment

1. The diameter of the base of the 'piggy bank' is 11 cm so the radius is 5.5 cm.
   $$\begin{aligned} V &= \pi (5.5 \text{ cm})^2 \times 7 \text{ cm} \\ &= 120.95 \text{ cm}^3 \end{aligned}$$
2.
   a.
   $$\begin{aligned} V &= l \cdot b \cdot h \\ V &= 0.85 \text{ m} \times 0.85 \text{ m} \times 0.85 \text{ m} \\ &= 0.614 \text{ m}^3 \end{aligned}$$
   b. New lid is 0.01 times longer than 0.85 m. So the new lid is $0.85 + 0.01 \times 0.85 = 0.8585 \text{ m}$
   $$\begin{aligned} &\text{Area of lid} = 0.8585 \text{ m} \times 0.8585 \text{ m} \\ &= 0.737 \text{ m}^2 \end{aligned}$$

Back to Unit 2: Assessment

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Unit 3: Total surface area of 3-D shapes

NATASHIA BEARAM-EDMUNDS

Unit outcomes: Unit 3: Total surface area of 3-D shapes

By the end of this unit you will be able to:

- Calculate the surface area of the following geometric objects:
  - Cubes
  - Rectangular prisms
  - Cylinders
  - Triangular prisms
  - Hexagonal prisms.
- Investigate the effect on the surface area of right prisms and cylinders, where one or more dimensions are multiplied by a constant factor *k*.

What you should know

Before you start this unit, make sure you can:

- Find the area and perimeter of 2-D shapes. To revise area and perimeter review Unit 1 of this subject outcome.
- Find the volume of 3-D shapes. For more on volume review Unit 2 of the subject outcome.

Introduction

Have you ever used far too much gift wrapping or not had enough to wrap a gift? If you work out the surface area of the object to be wrapped first, then you will know exactly how much gift wrapping you need. This is a simple example of how surface area can be useful, but there are many more important and practical uses for surface area both in everyday life and manufacturing.

Manufacturers of aluminium cans, for example, need to know how much material is needed to manufacture each can. They also need to calculate the surface area of each can to figure out how big the label for the can should be. If you were to paint the walls in your house, you would need to know what the surface area of the walls is so that you can buy the right amount of paint.

Note

When you have access to an internet connection watch the video called "Surface Area in Real Life" to see another example of the use of surface area in real life.

Surface Area in Real Life (Duration: 15.47)

Surface area of right prisms and cylinders

To find out what surface area is, work through the following activity.

Activity 3.1: Explore surface area

Time required: 20 minutes

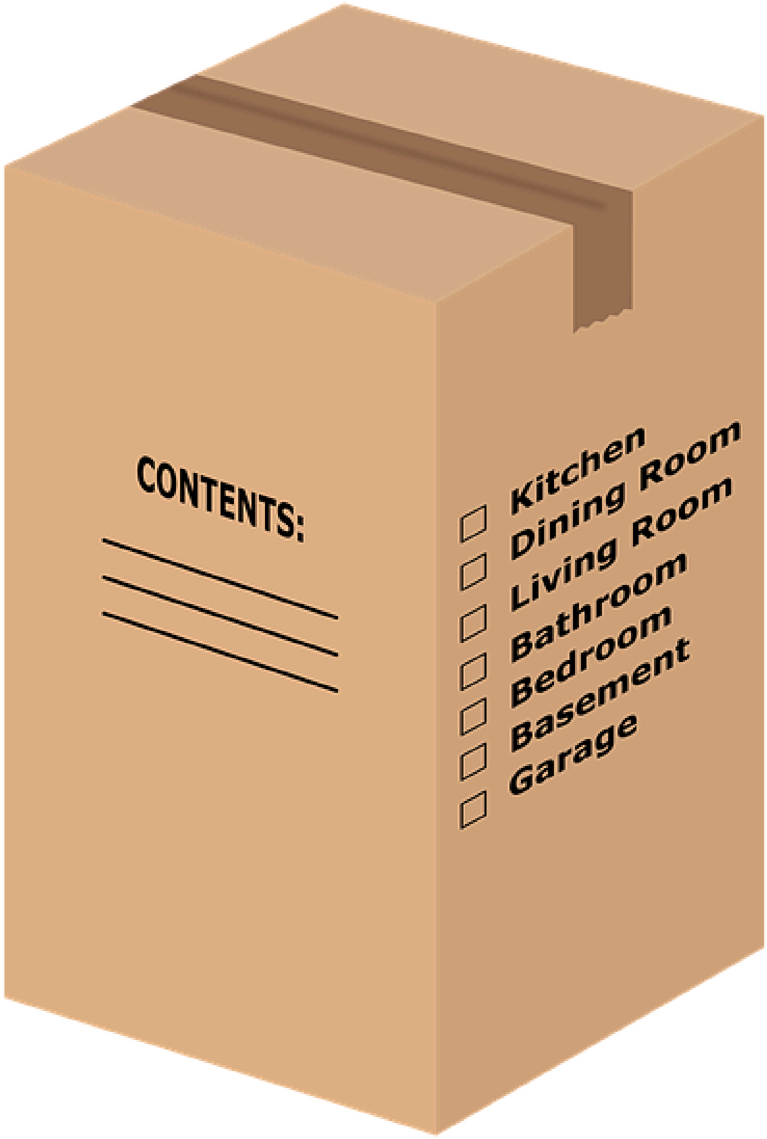
What you need:

- A pen and paper

What to do:

Look at this scenario and answer the questions that follow:

Thandi bought a gift for her mom and decided to put the gift into a box before wrapping it. She found an empty cardboard box, with square bases to use.

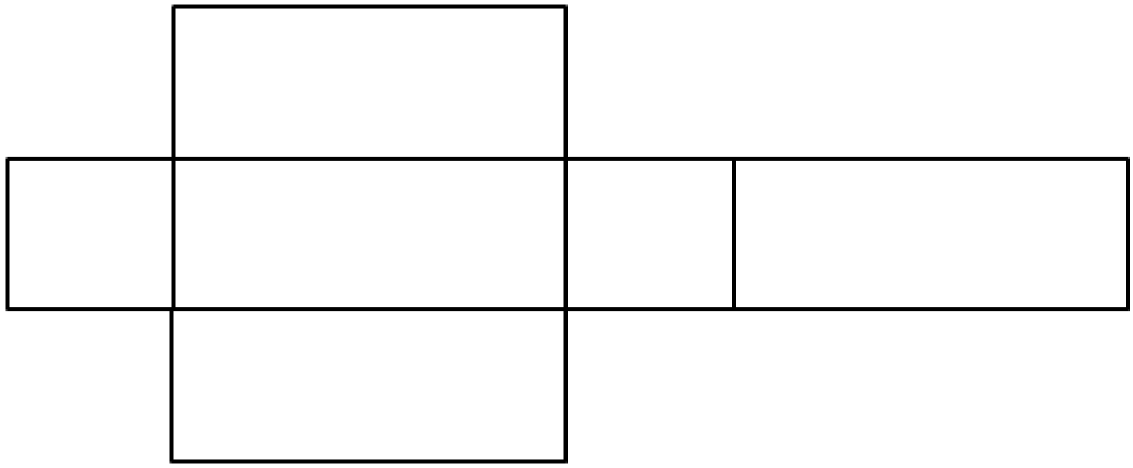


1. How many faces does the box have?
2. Write down the shapes of all the faces and the number of each.
3. What name is given to this type of 3-D shape?
4. Thandi would like to wrap the box. What measurements does she need for each face?
5. Once she has the measurements what calculations will she need to do to cover all the aces?
6. If the box is unfolded, draw each of the faces that make up the box.
7. Thandi measured the box and found that the length and breadth are $0.3\ \text{m}$ and the height is $0.5\ \text{m}$. What is the total surface area of the box?
8. If the gift has to go into a box with a surface area greater than its own, will this box work if the gift has a surface area of $1\ \text{m}^2$?

What did you find?

1. The box has 6 faces.

2. There are 2 square faces and 4 rectangular faces.
3. It is a rectangular prism.
4. To cover the box she would need to measure the length and breadth of each face.
5. To cover all the faces she must calculate the area of each face then add up the areas of each to get the total area.
6. A 3-D shape that is unfolded like this is called a net. When a prism is unfolded into a net, we can clearly see each of its faces. There is more than one way to unfold a 3-D solid into a net.

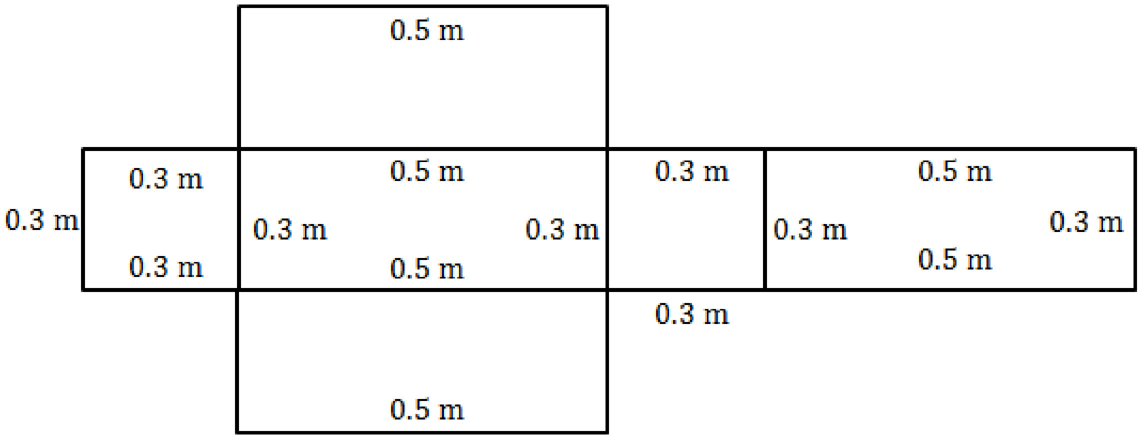


Note

For more on unfolding prisms into nets, you can watch the video called "Nets of polyhedra".

Nets of polyhedra (Duration: 06.47)

7. Let's add in the measurements of each side of the net so we can find the total area.



$$\begin{aligned} \text{Total area} &= 2 \times \text{Area of square} + 4 \times \text{Area of rectangle} \\ &= 2(0.3 \times 0.3) + 4(0.5 \times 0.3) \\ &= 0.78 \text{ m}^2 \end{aligned}$$

8. The box will not work for the gift because its area (1 m^2) is greater than the area of the box and we are told the area of the box must be greater than that of the gift.

The total surface area of a 3-D shape is the sum of the areas of all its faces (or exposed surfaces).

Below are more examples of right prisms and a cylinder that have been unfolded into nets.

A cube unfolded into a net is made up of six identical squares.

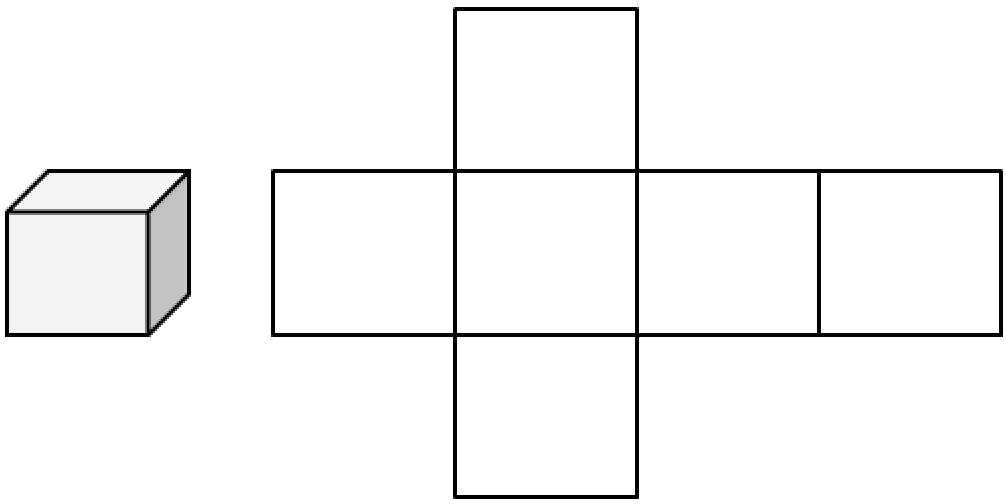


Figure 1: A net of a cube

A triangular prism unfolded into a net is made up of two triangles and three rectangles. The sum of the widths of the rectangles is equal to the perimeter of the triangles.

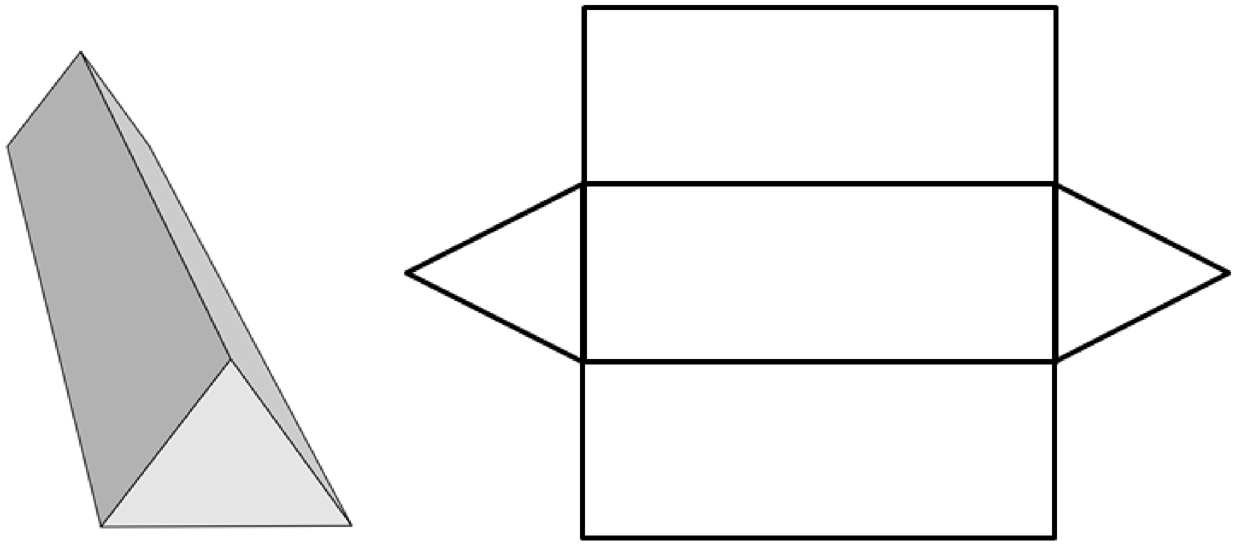


Figure 2: A net of a triangular prism

A hexagonal prism unfolded into a net is made up of two hexagons and six rectangles.

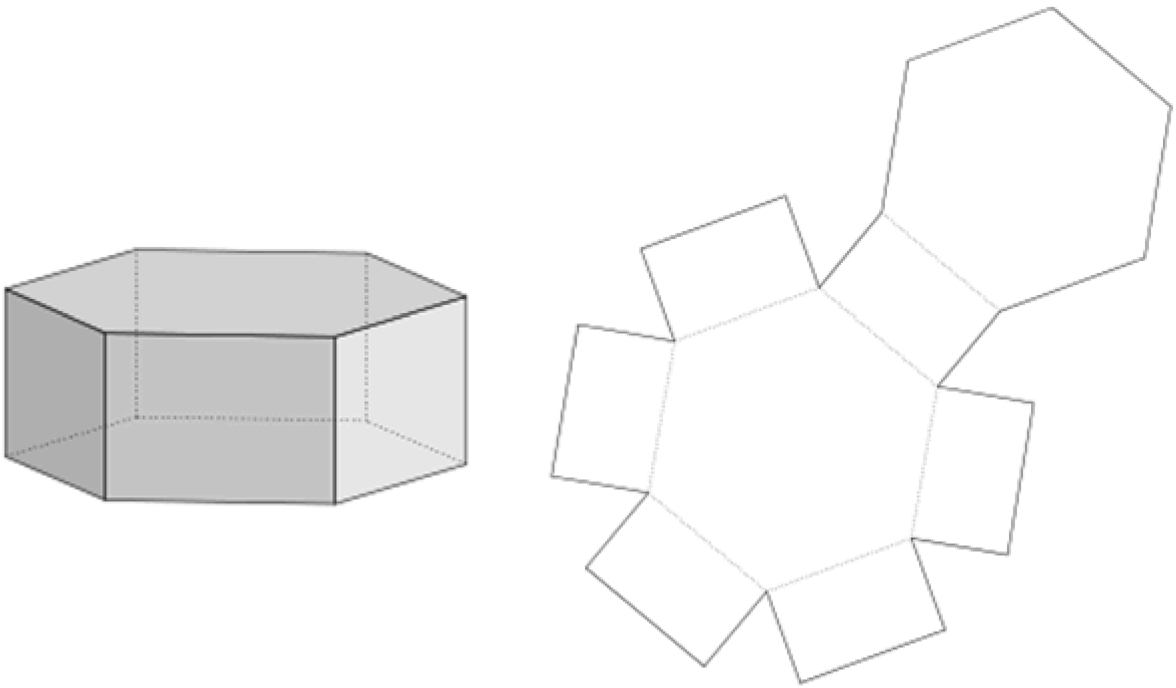


Figure 3: A net of a hexagonal prism

Note

If you have an internet connection, visit this interactive demonstration of the net of the hexagonal prism unfolding and to practise finding the surface area of a hexagonal prism.

A cylinder unfolded into a net is made up of two identical circles and a rectangle with length equal to the circumference of the circles.

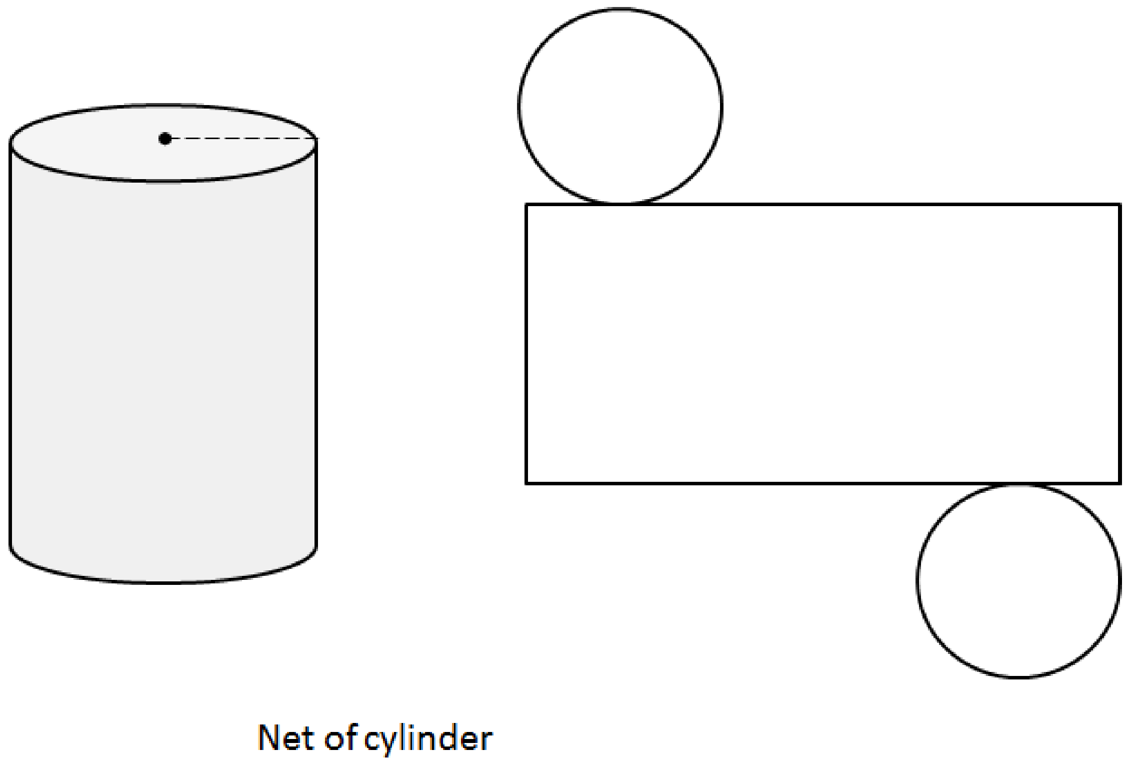
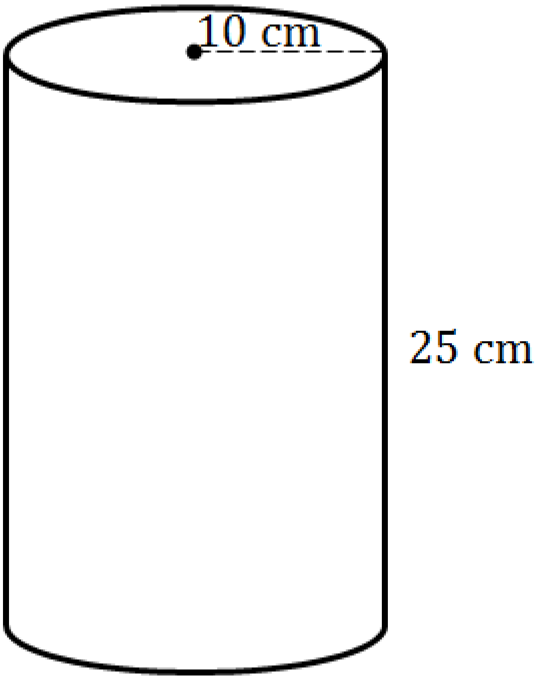


Figure 4: A net of a cylinder

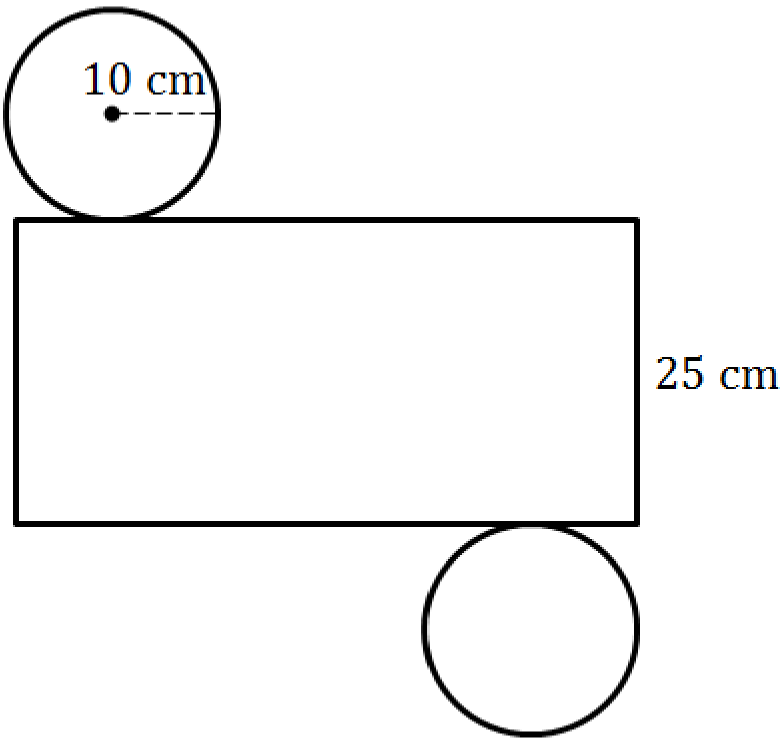
Example 3.1

Find the surface area of the following cylinder (correct to 1 decimal place).



Solution

A cylinder unfolded into a net is made up of two identical circles and a rectangle with a length equal to the circumference of the circles.



Now we find the area of the different shapes of the net and add them together to find the total surface area.

$$\begin{aligned}\text{Area of rectangle} &= \text{circumference of circle} \times \text{length}\\ &= 2\pi r \times l\\ &= 2\pi(10) \times (25)\\ &= 500\pi\ \text{cm}^2\end{aligned}$$

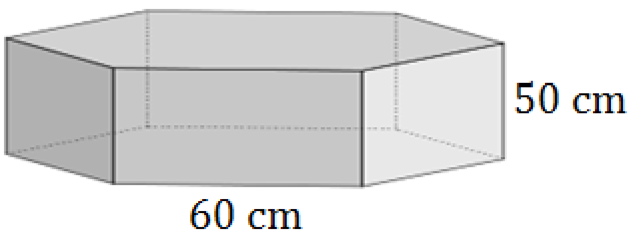
$$\begin{aligned}\text{Area of circle} &= \pi r^2\\ &= \pi(10)^2\\ &= 100\pi\ \text{cm}^2\end{aligned}$$

$$\begin{aligned}\text{Total surface area} &= \text{Area of rectangle } + 2(\text{area of circle})\\ &= 500\pi\ \text{cm}^2 + 2(100\pi\ \text{cm}^2)\\ &= 500\pi\ \text{cm}^2 + 200\pi\ \text{cm}^2\\ &= 700\pi\ \text{cm}^2\\ &= 2199.1\ \text{cm}^2\end{aligned}$$

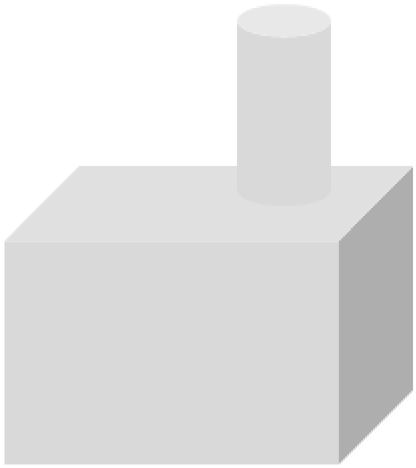
The surface area of the cylinder is $2199.1\ \text{cm}^2$. Round off in the last step only.

Exercise 3.1

1. If a litre of paint covers an area of $2\ \text{m}^2$, how much paint (in whole litres) does a painter need for one coat of paint on the following objects:
   a. A hexagonal table with base sides that are $60\ \text{cm}$ long and a perpendicular height of $50\ \text{cm}$. Round off to one decimal place.



   b. A closed cylindrical water tank with a diameter of $140\ \text{cm}$ and a height of $2.2\ \text{m}$. Round off to the nearest whole number.
   (Hint: $1\ \text{m}^2 = 10\ 000\ \text{cm}^2$)
2. Find the surface area of the following shape, which is made up of a rectangular prism with a cylinder on top. The rectangular prism is $9\ \text{m}$ long, $3\ \text{m}$ high and $6\ \text{m}$ wide. The cylinder has a diameter of $2\ \text{m}$ and its height is the same as the rectangular prism's height. Round off to one decimal place.



The full solutions are at the end of the unit.

Note

There is a formula we can use to easily find the total surface area of a closed right prism or cylinder.

$\text{TSA} = 2 \times \text{Area of the base} + \text{Perimeter of the base} \times \text{height}$

This formula works well for any closed 3-D prism or cylinder. But, for combined shapes and shapes that are open on the top or base you will need to modify the formula based on the question.

Example 3.2

Find the surface area of a fish tank that is the shape of a rectangular prism and is open on the top. The dimensions of the tank are $5\ \text{m}$ by $3\ \text{m}$ by $2.5\ \text{m}$.

Solution

For this solution we will use the formula for total surface area.

$\text{TSA} = 2 \times \text{Area of the base} + \text{Perimeter of the base} \times \text{height}$

We are told that this tank is open on top so there is no need to multiply the area of the base by two as there is only one face with that measurement.

$$\begin{aligned}\text{TSA} &= \text{Area of the base} + \text{Perimeter of the base} \times \text{height}\\ &= l \times b + 2(l+b)h\\ &= 5 \times 3 + 2(5+3) \times 2.5\\ &= 55\ \text{m}^2\end{aligned}$$

To practise using the formula to find total surface area do Activity 3.1 again.

Multiplying dimensions by a constant factor

We saw in Unit 2 of this subject outcome that multiplying one or more of the dimensions of a prism or cylinder by a constant changed the volume. A change in dimensions will also affect the surface area of a 3-D shape. By understanding the relationship between the change in dimensions and its effect on surface area we can find simpler ways to calculate the new surface area of an object when its dimensions are scaled up or down.

Activity 3.2: Investigate the effect of scaling dimensions on surface area

Time required: 10 minutes

What you need:

- a pen and paper

What to do:

Consider a rectangular prism with dimensions l, b, h. Complete the table to see the effect on surface area when we multiply one, two and three of its dimensions by a constant factor of 3.

1.

Dimensions	Surface area
a) l, b, h	$\text{A} = 2[(l \times h) + (l \times b) + (b \times h)]$
b) $l, b, 3h$	$\text{A}_1 =$
c) $l, 3b, 3h$	$\text{A}_2 =$
d) $3l, 3b, 3h$	$\text{A}_3 =$
e) kl, kb, kh	$\text{A}_k =$

2. What effect does multiplying one dimension by a constant factor have on surface area?
3. What effect does multiplying two dimensions by a constant factor have on surface area?
4. If all three dimensions are multiplied by a constant factor of k what is the effect on surface area?

What did you find?

1.

Dimensions	Surface area
a) l, b, h	$\mathrm{A} = 2[(l \times h) + (l \times b) + (b \times h)]$ $= 2(lh + lb + bh)$
b) $l, b, 3h$	$\mathrm{A}_1 = 2[(l \times 3h) + (l \times b) + (b \times 3h)]$ $= 2(3lh + lb + 3bh)$
c) $l, 3b, 3h$	$\mathrm{A}_2 = 2[(l \times 3h) + (l \times 3b) + (3b \times 3h)]$ $= 2(3lh + 3lb + 3^2bh)$ $= 2 \times 3(lh + lb + 3h)$
d) $3l, 3b, 3h$	$\mathrm{A}_3 = 2[(3l \times 3h) + (3l \times 3b) + (3b \times 3h)]$ $= 2(3^2lh + 3^2lb + 3^2bh)$ $= 3^2 \times 2(lh + lb + h)$ $= 3^2\mathrm{A}$
e) kl, kb, kh	$\mathrm{A}_k = 2[(kl \times kh) + (kl \times kb) + (kb \times kh)]$ $= 2(k^2lh + k^2lb + k^2bh)$ $= k^2 \times 2(lh + lb + h)$ $= k^2\mathrm{A}$

2. When one dimension is multiplied by a constant factor we cannot see a relationship between the original surface area and the new surface area.
3. When two dimensions are multiplied by a constant factor we cannot see a relationship between the original surface area and the new surface area.
4. When three dimensions are multiplied by a constant factor the new surface area is $k^2 \times$ the original surface area.

We can use the relationship found between multiplying dimensions by a constant factor and the effect of surface area to easily calculate new surface areas without too much additional working out as shown in the next example.

Example 3.3

Consider a rectangular prism with a height of 4 cm and base lengths of 3 cm.

1. Calculate the surface area and volume.
2. Calculate the new surface area and volume if all dimensions are multiplied by a constant factor of 5.
3. Express the new surface area and volume as a factor of the original surface area and volume.

Solutions

1. Original surface area and volume:

$$\begin{aligned} A &= 2\text{Area of base} + \text{Perimeter of base} \times h \\ &= 2(3 \times 3) + 2(3+3) \times 4 \\ &= 66 \text{ cm}^2 \end{aligned}$$

$$\begin{aligned} V &= \text{Area of base} \times \text{h} \\ &= (3 \times 3) \times 4 \\ &= 36 \text{ cm}^3 \end{aligned}$$

2.

$$\begin{aligned} A_{\text{new}} &= 5^2 A \\ &= 25 \times 66 \\ &= 1\ 650 \text{ cm}^2 \end{aligned}$$

$$\begin{aligned} V_{\text{new}} &= 5^3 V \\ &= 125 \times 1\ 650 \\ &= 206\ 250 \text{ cm}^3 \end{aligned}$$

3.

$$\begin{aligned} \frac{A_{\text{new}}}{A} &= \frac{1\ 650}{66} \\ &= 25 \\ \therefore A_{\text{new}} &= 5^2 A \end{aligned}$$

$$\begin{aligned} \frac{V_{\text{new}}}{V} &= \frac{206\ 250}{1\ 650} \\ &= 125 \\ \therefore V_{\text{new}} &= 5^3 V \end{aligned}$$

Exercise 3.2

1. Given a prism with surface area of $2\ 520 \text{ m}^2$, calculate the new surface area if all dimensions are tripled.
2. Consider a cylinder with a radius of r and a height of h. Calculate the surface area (in terms of r and h) if the radius is multiplied by a constant factor of k.

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to find the surface area of right prisms and cylinders.
- How to draw the nets of right prisms and cylinders to calculate the surface area.
- How to use the surface area formula.
- How to find the new surface area when dimensions are multiplied by a constant factor.

Unit 3: Assessment

Suggested time to complete: 20 minutes

1. Here we look at Alfred's 'piggy bank' again. Alfred's 'piggy bank' is cylindrical in shape, is constructed from tin and has a slot at the top. Remember that the area of the slot is $2\ \text{cm}^2$, the diameter of the base of the 'piggy bank' is $11\ \text{cm}$ and the height of the 'piggy bank' is $7\ \text{cm}$.

Calculate the surface area of material that was used to manufacture this 'piggy bank'.
2. If the length of each side of a hexagonal prism is quadrupled, what will the surface area of the new prism be?

The full solutions are at the end of the unit.

Unit 3: Solutions

Exercise 3.1

1.
   a. We need to find the surface area of the hexagonal table to find out how much paint is needed to paint it. A hexagonal prism is made up of two hexagons and six rectangles.

$$\begin{aligned}\text{Surface area} &= 2(\text{Area of hexagon}) + 6(\text{Area of rectangle}) \\ &= 2(\frac{3\sqrt{3}}{2}s^2) + 6(l \times b) \text{ The breadth of the rectangles is the same as the height of the prism} \\ &= 3\sqrt{3}(60)^2 + 6(60 \times 50) \\ &= 36\ 706.1\ \text{cm}^2\end{aligned}$$

Convert from cm^2 to m^2.

$$\begin{aligned}1\ \text{m}^2 &= 10\ 000\ \text{cm}^2 \\ 36\ 706.1\ \text{cm}^2 &= \frac{36\ 706.1}{10\ 000}\ \text{m}^2 \\ &= 3.67061\ \text{m}^2\end{aligned}$$

Therefore, the painter would need 2 litres of paint for one coat of the table since 2 litres of paint would cover an area of $4\ \text{m}^2$, which is more than the area of the table.

b. The diameter of the tank is $140\ \text{cm}$ so the radius is $70\ \text{cm}$. Remember to use the same units throughout. Therefore, convert $70\ \text{cm}$ to $0.7\ \text{m}$. The height $2.2\ \text{m}$.

$$\begin{aligned}\text{Area of rectangle} &= \text{circumference of circle} \times \text{length} \\ &= 2\pi r \times l \\ &= 2\pi(0.7) \times (2.2) \\ &= \frac{77}{25}\pi\ \text{m}^2\end{aligned}$$

$$\begin{aligned}\text{Area of circle} &= \pi r^2 \\ &= \pi(2.2)^2 \\ &= \frac{121}{25}\pi\ \text{m}^2\end{aligned}$$

$$\begin{aligned}\text{Total surface area} &= \text{Area of rectangle } + 2(\text{area of circle}) \\ &= \frac{77}{25}\pi\ \text{m}^2 + 2(\frac{121}{25}\pi\ \text{m}^2) \\ &= \frac{319}{25}\pi\ \text{m}^2 \\ &= 40.09\ \text{m}^2\end{aligned}$$

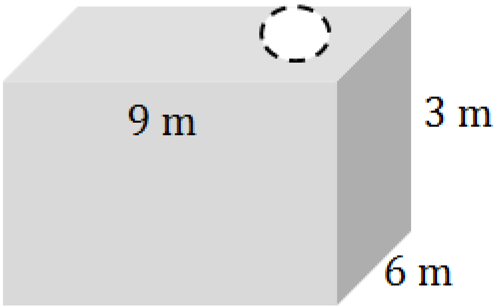
The painter would need to cover $\frac{40}{2} = 20\ \text{m}^2$.

1 litre of paint would cover an area of $2\ \text{m}^2$, so he therefore needs 10 litres of paint.

2. The surface area by definition only takes into account exposed surfaces. So, in this question we must not include two circles when we calculate the surface area of the cylinder as only the top circle is exposed.

$$\begin{aligned}\text{Total surface area}_{\text{cylinder}} &= \text{Area of rectangle } + \text{area of circle} \\ &= 2\pi rl + \pi r^2 \\ &= 2\pi(1)(3) + \pi(1)^2 \\ &= 7\pi\ \text{m}^2\end{aligned}$$

We must also exclude the base circle of the cylinder from surface area of the rectangular prism. To do that, we must remember that the section where the cylinder meets the top face of the prism will not be exposed. So, we must subtract the area of the circle from the area of the top rectangular face to find the rectangular prism's surface area.



$$\begin{aligned}\text{Total surface area}_{\text{rectangular prism}} &= (9 \times 6 - \pi r^2) + (9 \times 6) + 2(9 \times 3) + 2(6 \times 3) \\ &= 54 - \pi(1)^2 + 54 + 54 + 36 \\ &= (198 - \pi)\ \text{m}^2\end{aligned}$$

For total surface area of the combined shape add the surface area of the cylinder to the surface area of the rectangular prism, and exclude the area of the circular base twice.

$$\begin{aligned}\text{Total surface area} &= 2\pi rl + \pi r^2 + (198 - \pi) \\ &= 2\pi(1)(3) + \pi(1)^2 + (198 - \pi) \\ &= 8\pi + \pi + 198 - \pi \\ &= 223\ \text{m}^2\end{aligned}$$

Back to Exercise 3.1

Exercise 3.2

1.

$$\begin{aligned}\text{A}_{\text{new}} &= 3^2(2\ 520) \\ &= 22\ 680\ \text{m}^2\end{aligned}$$

2.

$$\begin{aligned}\text{Surface Area} &= 2(\pi r^2) + 2\pi rh \\ \text{Surface Area new} &= 2\pi(kr)^2 + 2\pi krh \\ &= 2(\pi r^2) \cdot k^2 + 2\pi rh \cdot k \\ &= k(2\pi r^2 k + 2\pi rh)\end{aligned}$$

Back to Exercise 3.2

Unit 3: Assessment

1. The area of the slot of $2\ \text{cm}^2$ must be subtracted from the area of the top circle since that is not covered in tin. The diameter of the base of the 'piggy bank' is $11\ \text{cm}$ so the radius is $5.5\ \text{cm}$ and the height of the 'piggy bank' is $7\ \text{cm}$.

$$\begin{aligned}\text{Area} &= \pi r^2 + (\pi r^2 - 2) + 2\pi rh \\ &= 2\pi r^2 - 2 + 2\pi rh \\ &= 2\pi(5.5)^2 - 2 + 2\pi(5.5)(7) \\ &= 429.97\ \text{cm}^2\end{aligned}$$

2.

$$\begin{aligned}\text{Surface area} &= 2(\text{Area of hexagon}) + 6(\text{Area of rectangle})\\ &= 2(\frac{3\sqrt{3}}{2}s^2) + 6(l \times b)\\ &= 3\sqrt{3}s^2 + 6lb\end{aligned}$$

$$\begin{aligned}\text{Surface area}_{\text{new}} &= 3\sqrt{3}(4s)^2 + 6(4l)(4b)\\ &= 3\sqrt{3}(16s^2) + 6(16lb)\\ &= 16(3\sqrt{3}s^2 + 6lb)\\ &= 4^2 \text{ original surface area}\end{aligned}$$

Back to Unit 3: Assessment

Media Attributions

SUBJECT OUTCOME VIII

SPACE, SHAPE AND MEASUREMENT: USE THE CARTESIAN CO-ORDINATE SYSTEM TO DERIVE AND APPLY EQUATIONS

Subject outcome 3.3

Use the Cartesian co-ordinate system to derive and apply equations

Learning outcomes

- Use the Cartesian co-ordinate system to plot points, lines and polygons.
- Use the Cartesian co-ordinate system to calculate the distance between two points.
- Use the Cartesian co-ordinate system to find the gradient of the line joining two points.
- Use the Cartesian co-ordinate system to find the co-ordinates of the midpoint of a line segment joining two points.

Unit outcomes: Unit 1: Plotting points on the Cartesian plane

By the end of this unit you will be able to:

- Use the Cartesian co-ordinate system to plot points, lines and polygons.
- Find co-ordinates from labelled points.

Unit outcomes: Unit 2: Distance, gradient and midpoints

By the end of this unit you will be able to:

- Calculate the distance between two points on the Cartesian plane.
- Calculate the gradient between two points on a straight line.
- Work with the gradients of parallel and perpendicular lines.
- Calculate the midpoint between two points on the Cartesian plane.

Unit 1: Plotting points on the Cartesian plane

DYLAN BUSA

Unit outcomes: Unit 1: Plotting points on the Cartesian plane

By the end of this unit you will be able to:

- Use the Cartesian co-ordinate system to plot points, lines and polygons.
- Find co-ordinates from labelled points.

What you should know

Before you start this unit, make sure you can:

- Read a numberline.
- State the key features of the square, rectangle, hexagon, trapezium, and triangle. Refer to level 2 subject outcome 3.2 unit 1 if you need help with this.

Introduction

Where in the world are you? How would you explain to someone else on the other side of the world where you are right now? You might say you are in your house or at college. But where is that? You might give the address of your house or college for the person to look up on the internet. But how do satellite navigation systems or Google Maps know where your address actually is? It all comes down to co-ordinates.

Co-ordinates

Co-ordinates are a set of two numbers that describe your position exactly and consistently. These numbers are your **longitude** (how far East or West you are) and your **latitude** (how far North or South you are).

The Union Buildings in Pretoria, for example, have the following co-ordinates:
Latitude: -25.740610122680664
Longitude: 28.21186637878418

This means that they are 28.21186637878418 degrees East of (or to the right of) the Prime Meridian and -25.740610122680664 degrees South of (or below) the Equator (see Figure 1). The Prime Meridian is an imaginary line, like the Equator, but that runs from the North Pole to the South Pole dividing the Earth into East and West.

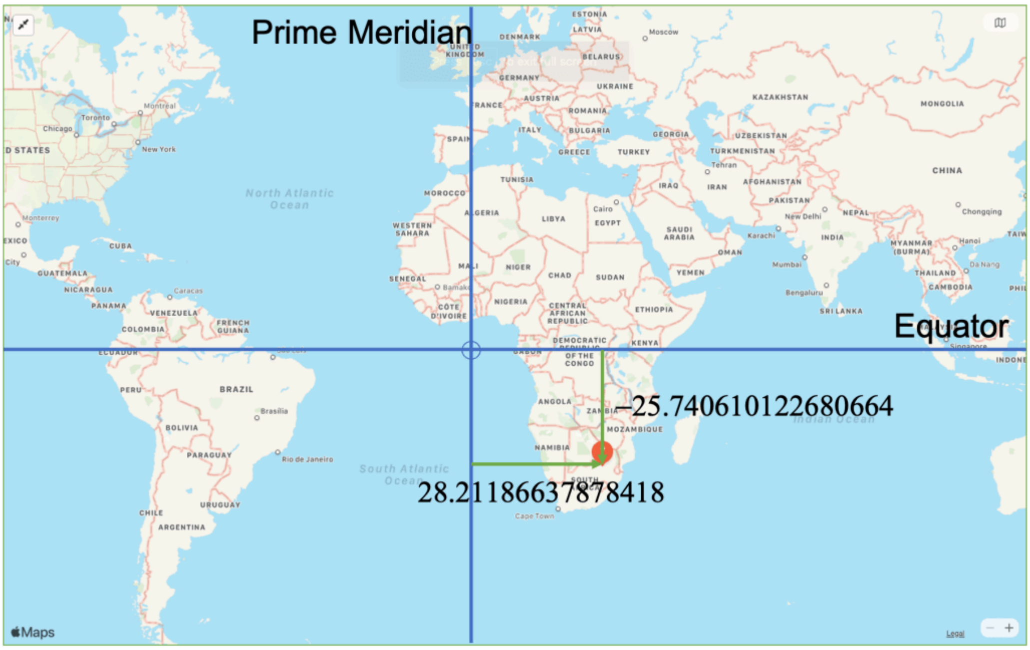


Figure 1: Location of the Union Buildings in Pretoria, South Africa

We can write these co-ordinates as an **ordered pair**.

$$(\overbrace{-25.740610122680664}^{\text{Latitude}}, \overbrace{28.21186637878418}^{\text{Longitude}})$$

We call this an ordered pair because there are two numbers (a pair) and they are always given in the same order (latitude then longitude).

Every time you have used a map or navigation system you have used this global positioning system. Such a system, which is defined by two perpendicular number lines, is called a Cartesian system in honour of the mathematician, Rene Descartes, who formalised its use in mathematics.

Did you know?

The place on Earth with co-ordinates $(0, 0)$ is called Null Island . There is a little weather buoy (see Figure 2) anchored in this location. You can learn more about Null Island on Wikipedia.

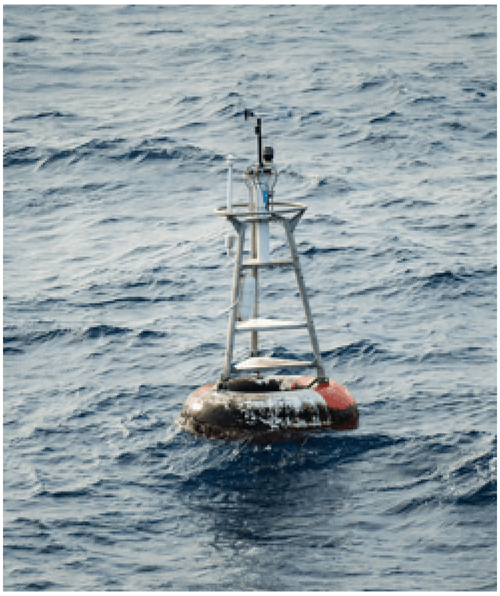


Figure 2: Null Island

The Cartesian plane

The Cartesian plane is a two-dimensional surface defined by two perpendicular number lines which we call **axes**. The horizontal axis (running from left to right) is the **x-axis** and the vertical axis (running from top to bottom) is the **y-axis**. Each axis extends forever in both directions this is why we draw them with arrows on the ends.

We call the point where the axes meet the **origin**.

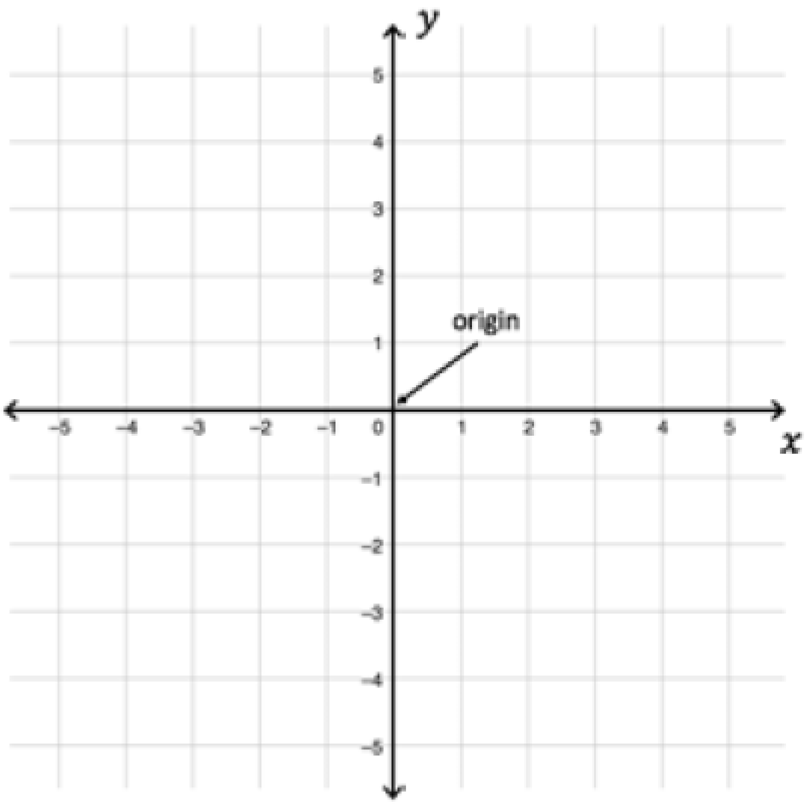


Figure 3: The Cartesian plane

We can describe the position of any point on the Cartesian plane with an ordered pair. The first number is the **x-coordinate** (how far left or right of the origin the point is) and the second number is the **y-coordinate** (how far up or down from the origin the point is).

Figure 4 shows the position of the points $\mathrm{A}(2, 4)$, $\mathrm{B}(-3, 1)$ and $\mathrm{C}(4, -3)$ on the Cartesian plane. We can see that point A is 2 units to the right of the origin and 4 units up from the origin. Point B is 3 units to the left of the origin (it is a negative number) and 1 unit up from the origin. Point C is 4 units to the right of the origin and 3 units below the origin (it is a negative number).

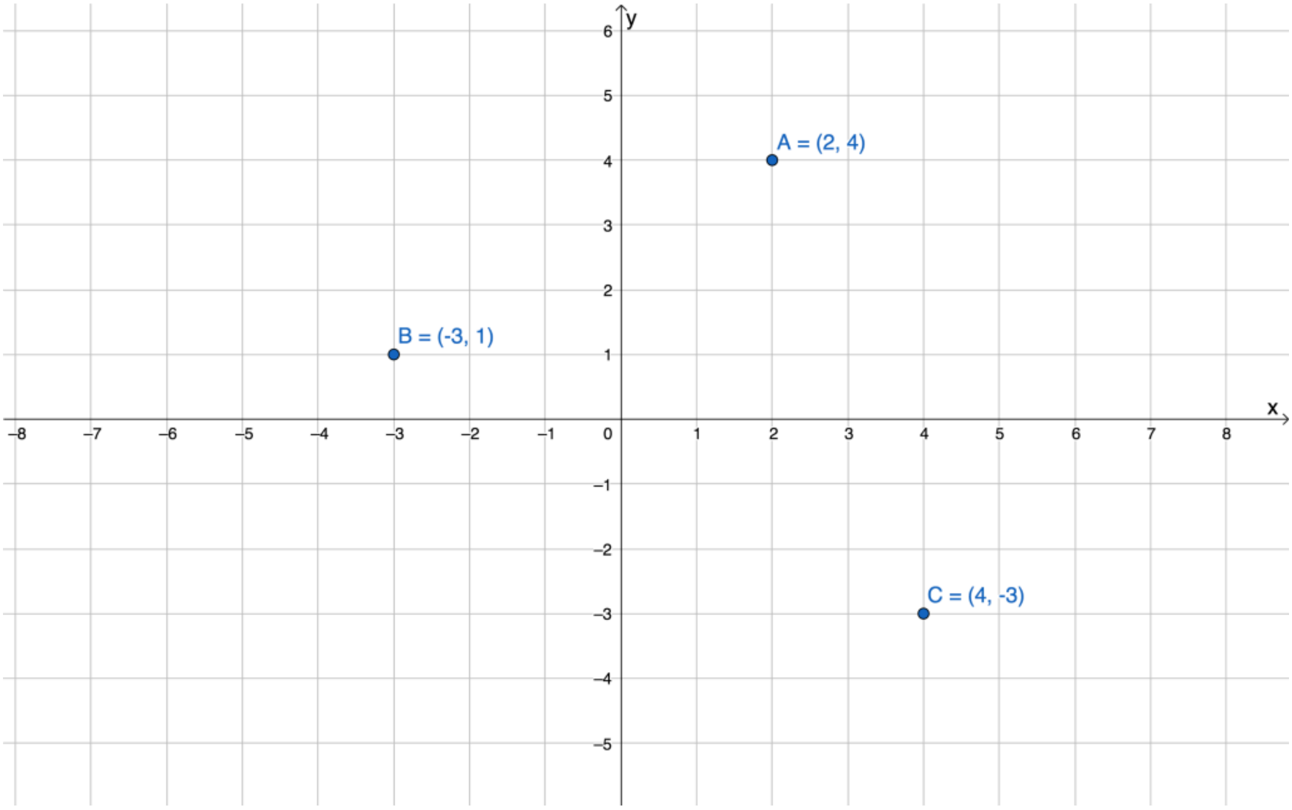
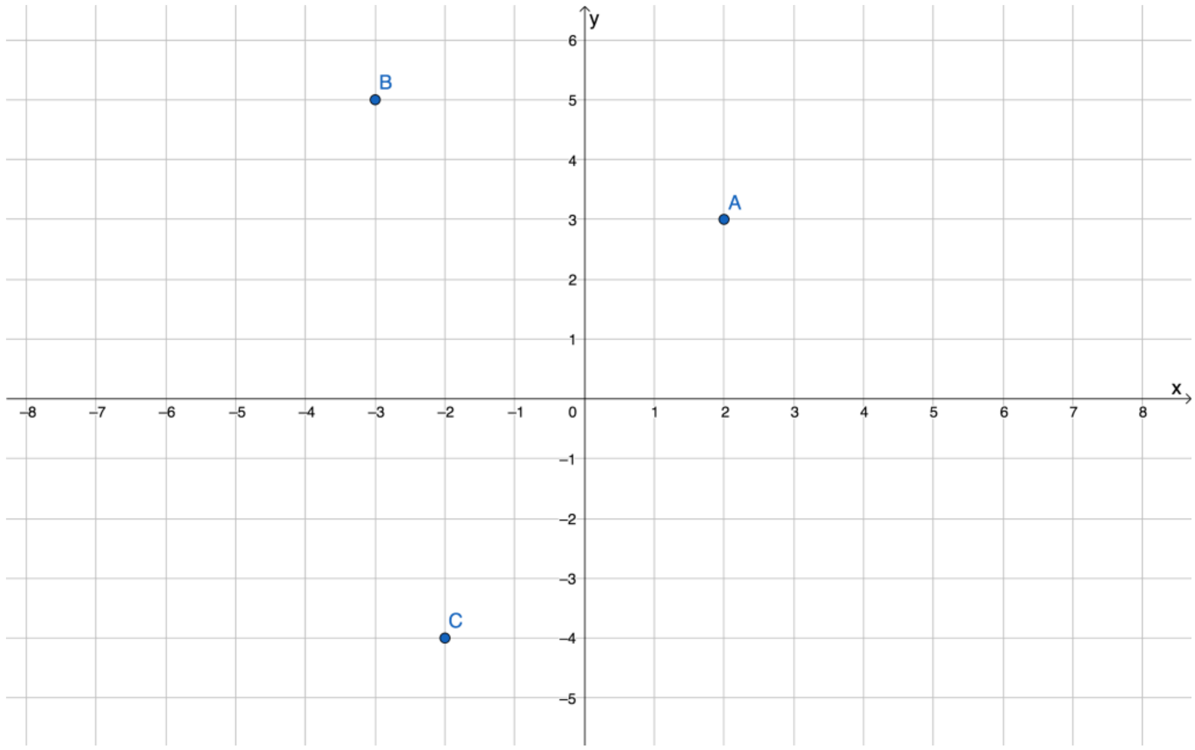


Figure 4: The locations of points A, B *and* C *on the Cartesian plane*

Example 1.1

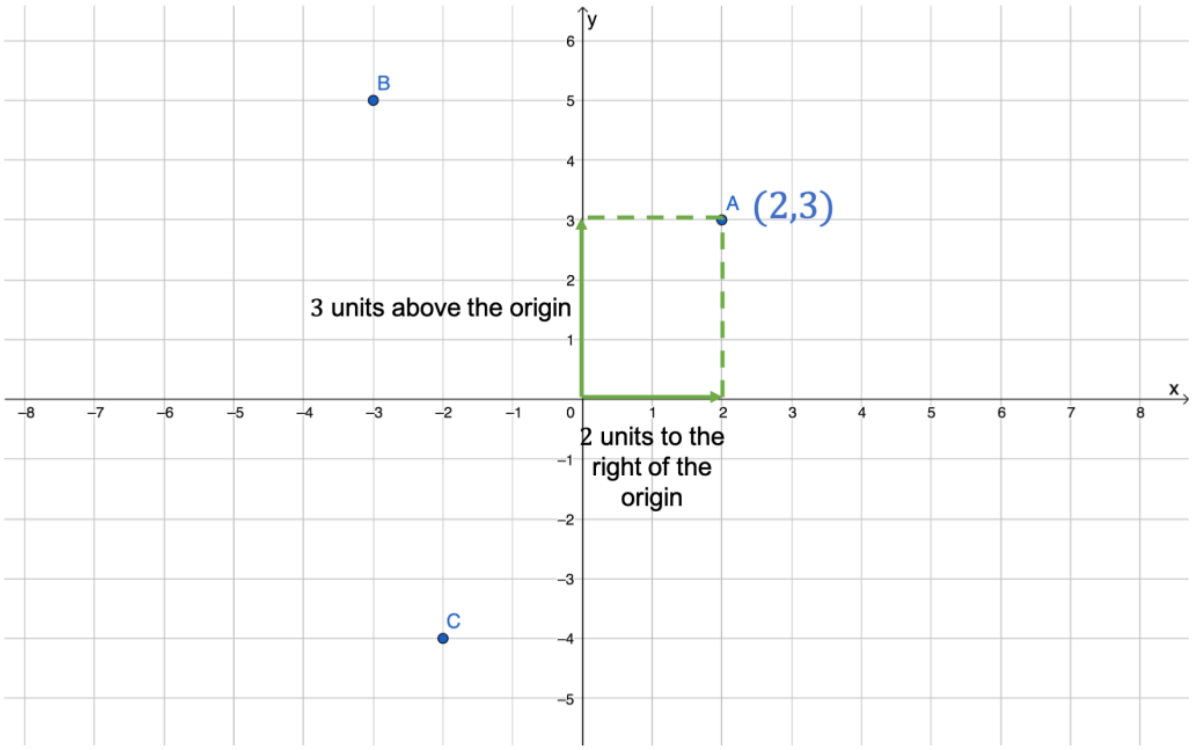
1. Determine the coordinates of points A, B and C on the Cartesian plane.



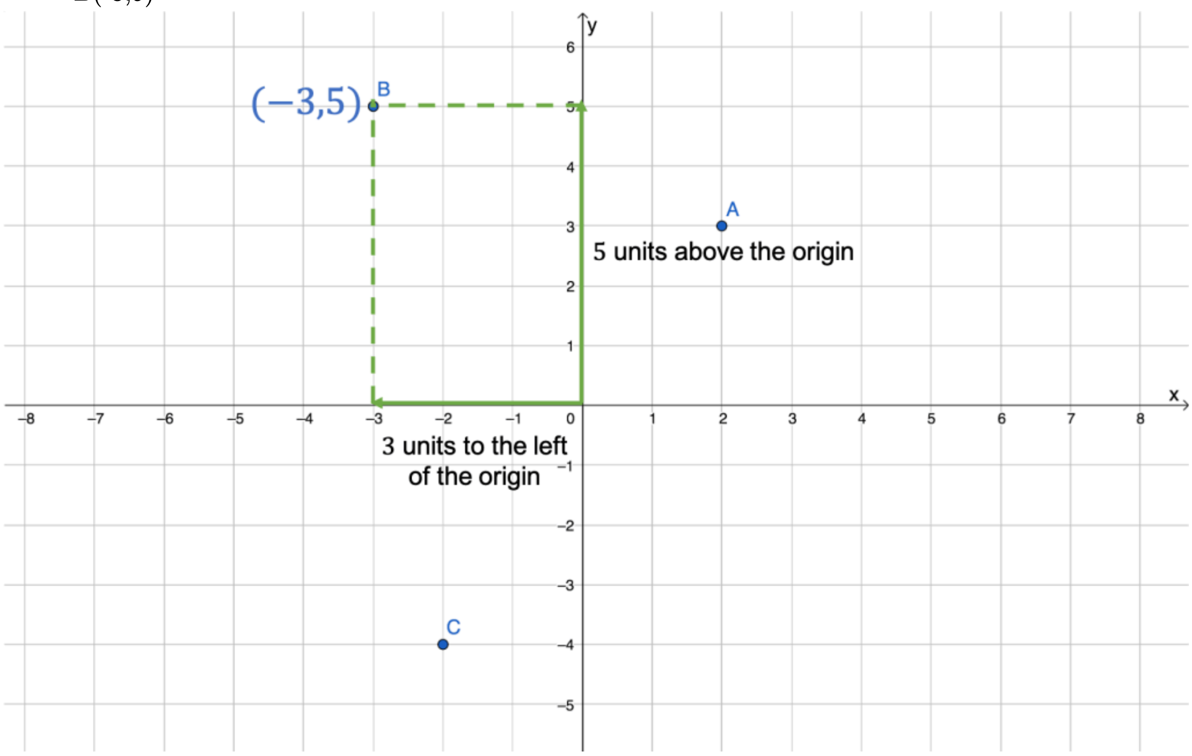
2. Plot the following points on the same Cartesian plane.
   a. $\mathrm{P}(4,7)$
   b. $\mathrm{Q}(1,-7)$
   c. $\mathrm{R}(-3,-4.5)$
   d. $\mathrm{S}\,(-3,0)$

Solutions

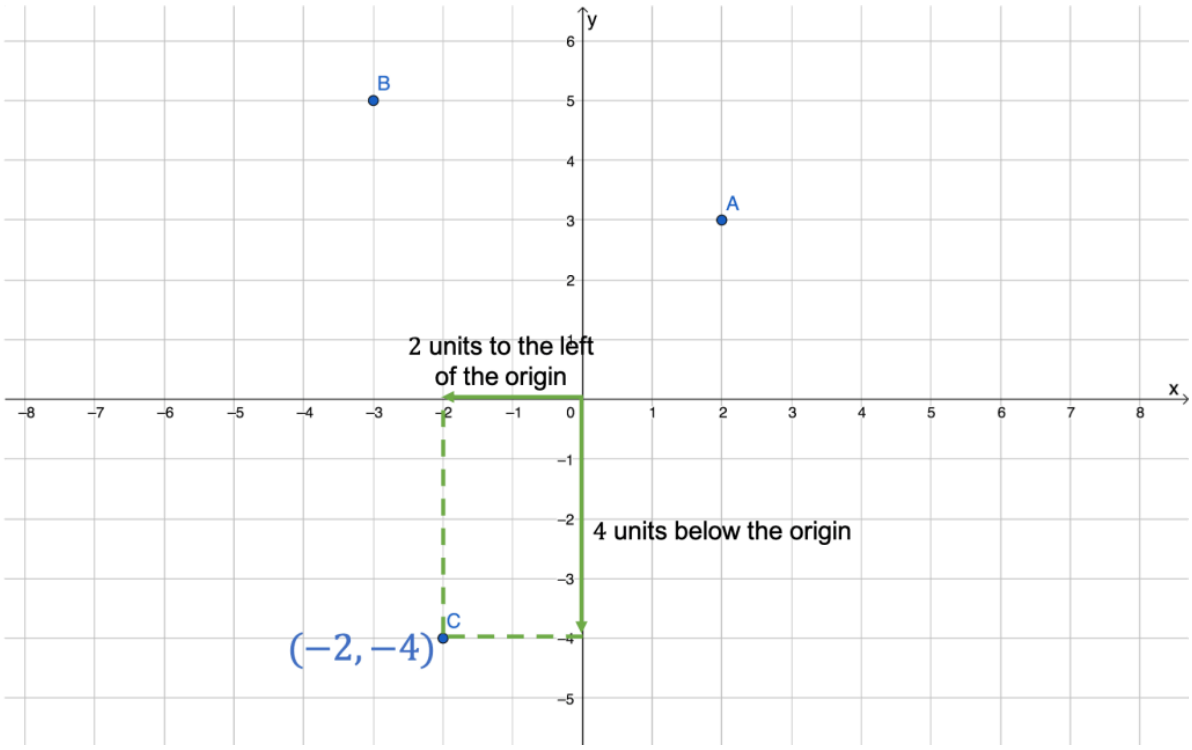
1. Point $\mathrm{A}(2{,}3)$:



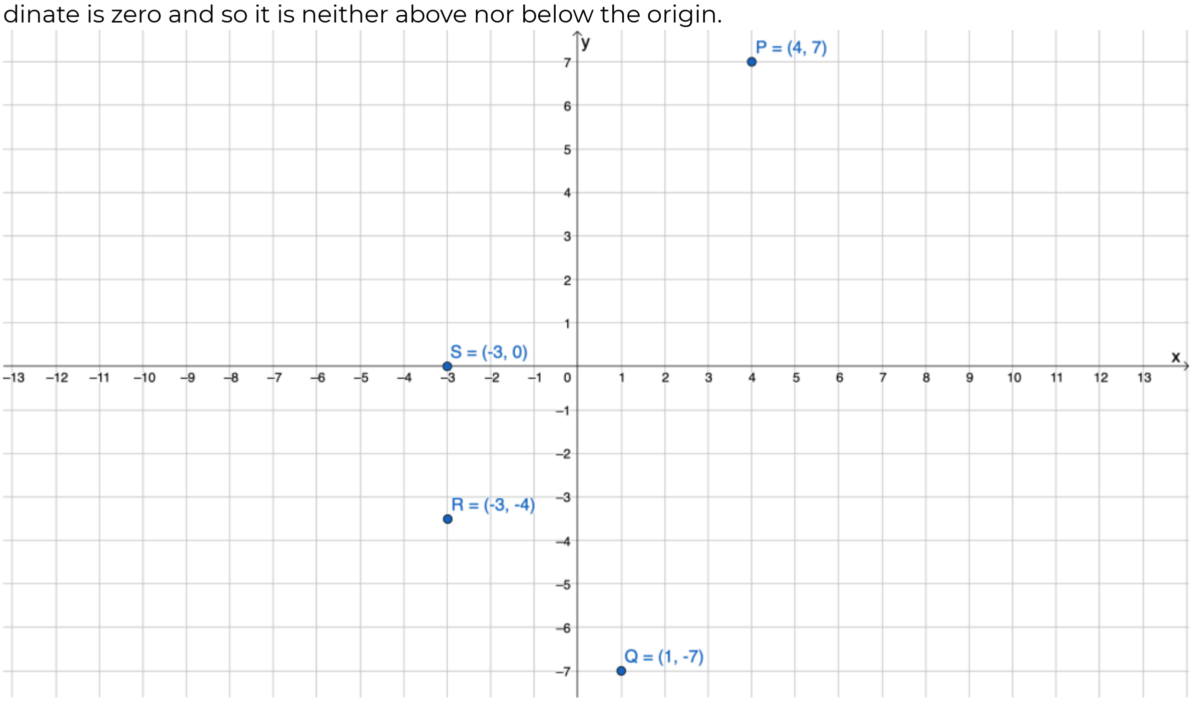
Point B(-3,5):



Point C(-2,-4):



2. Here are the points plotted on the same Cartesian plane. Notice how $R(-3, -4.5)$ is halfway between -4 and -5 on the y-axis. Notice how $S\,(-3, 0)$ lies on the x-axis. This is because its y-coordinate is zero and so it is neither above nor below the origin.



Note

In Question 2 in Example 1.1, it was necessary for us to draw our own Cartesian plane. When you draw your own Cartesian plane, take note of the following properties.

- The axes must be perpendicular (at right angles) to each other.
- There must be an arrowhead at both ends of each axis.
- You must label the axes x and y.
- You must label the origin 0.
- You must add a scale to each axis and label them with whole numbers.

You may have noticed that the two axes on the Cartesian plane divide the plane into four pieces. We call these **quadrants** and label them **quadrant 1** (or quadrant I), **quadrant 2** (or quadrant II), **quadrant 3** (or quadrant III), and **quadrant 4** (or quadrant IV) as shown in Figure 5.

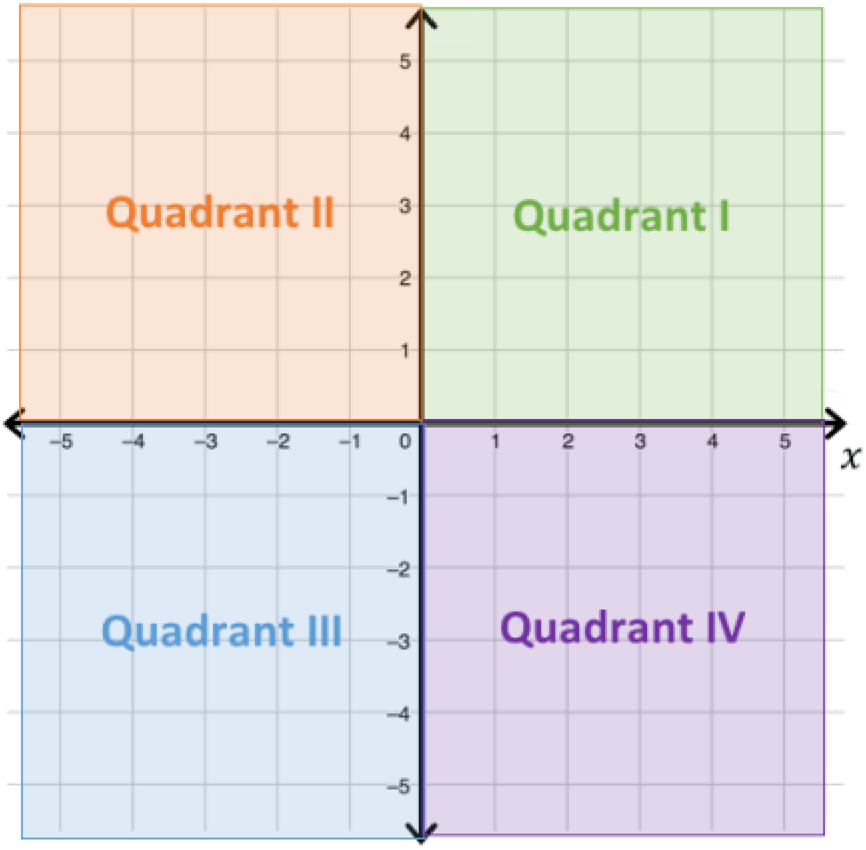


Figure 5: The four quadrants of the Cartesian plane

- Coordinates in quadrant 1 have positive x-value and y-values.
- Coordinates in quadrant 2 have negative x-values and positive y-values.
- Coordinates in quadrant 3 have negative x-values and y-values.
- Coordinates in quadrant 4 have positive x-values and negative y-values.

Note

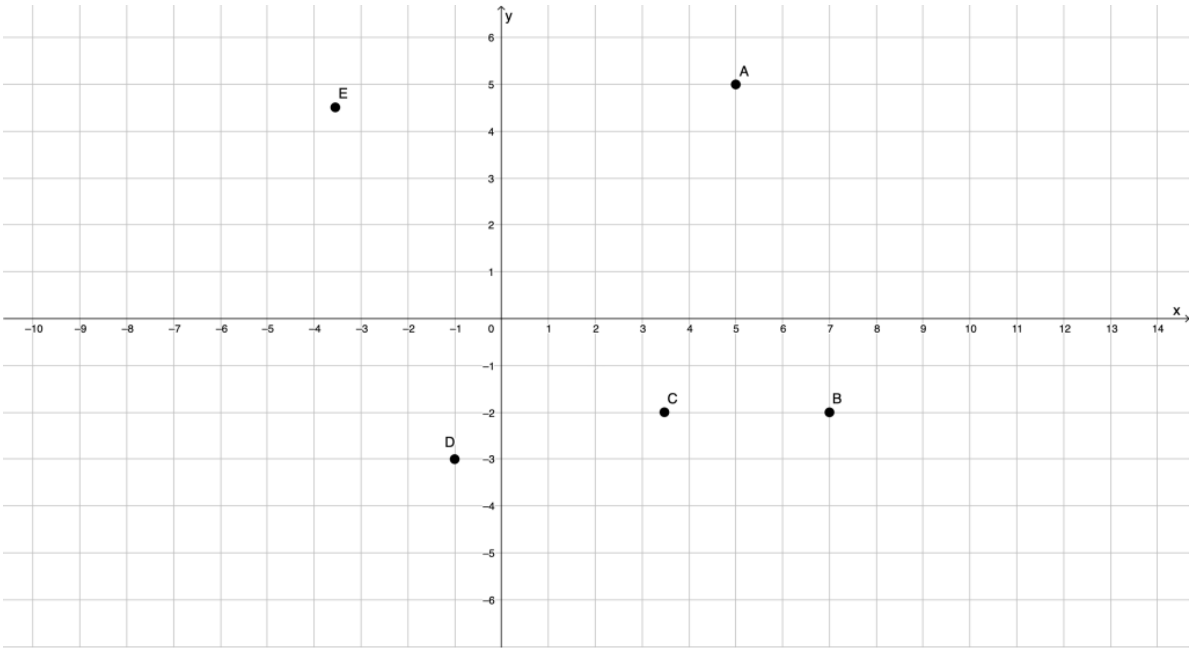
If you would like to practise plotting points on the Cartesian plane try the interactive activities called Plot points in quadrant 1

and Plot points in all quadrants.

If you would like more practise finding the coordinates of points on the Cartesian plane try the interactive activity called Plotting and finding coordinates.

Exercise 1.1

1. Plot the following points on the same system of axes (this means on the same Cartesian plane).
   a. $\mathrm{H}(-2, 1.5)$
   b. $\mathrm{I}(0, -4)$
   c. $\mathrm{J}(6, 0)$
   d. $\mathrm{K}(-4, -5)$
   e. $\mathrm{L}(0, 0)$
2. What are the coordinates of each of the points A to F.



3. In which quadrant will the following points be located?
   a. $(6, 7)$
   b. $(-6, -7)$
   c. $(-1, 3)$
   d. $(\pi, -4)$

The full solutions are at the end of the unit.

Note

If you have an Internet connection, watch these two videos for a summary of what we have learned so far.

Axis and Coordinates (Duration: 2.46)

The Cartesian Plane (Duration: 7.44)

Lines and polygons on the Cartesian plane

In Subject outcome 3.2, Unit 1 you learnt about polygons. Do you recall that a polygon is any closed shape with straight sides? Figure 6 shows some examples of polygons. To revise more about polygons, refer back to that unit.

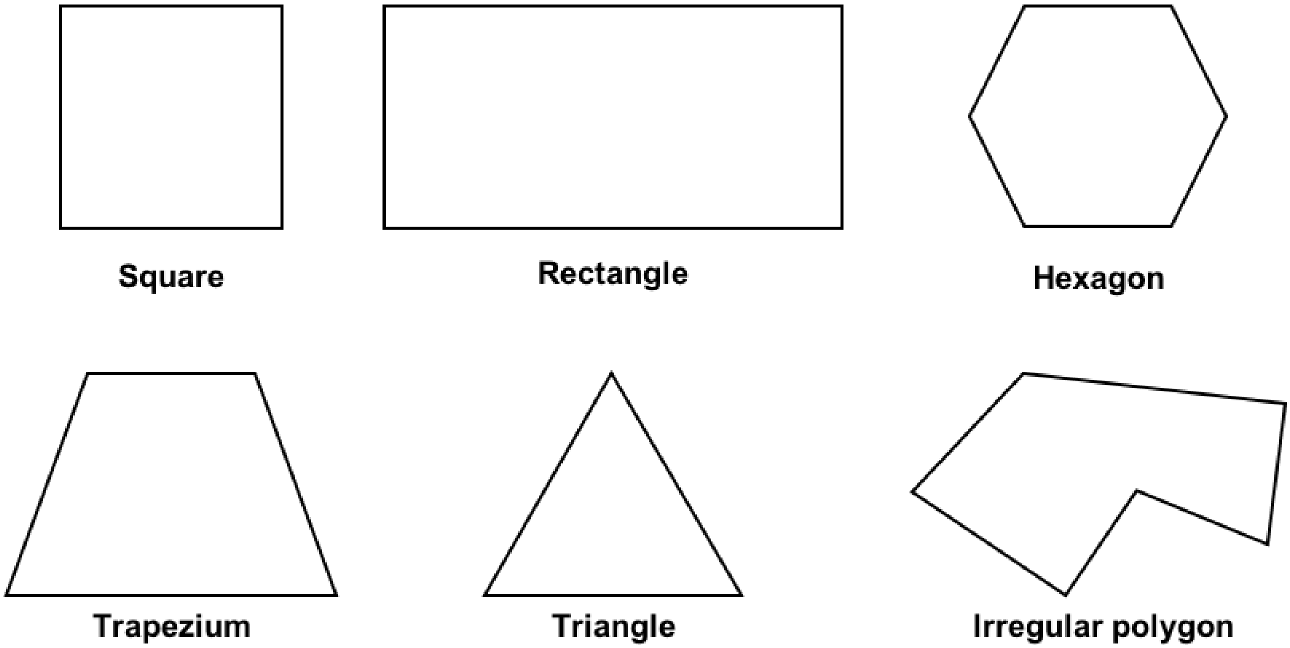


Figure 6: Examples of polygons

In the next activity we will create polygons on a Cartesian plane by plotting points and joining them with straight lines to form shapes.

Activity 1.1: Join the dots

Time required: 20 minutes

What you need:

- a piece of paper
- coloured pens or pencils

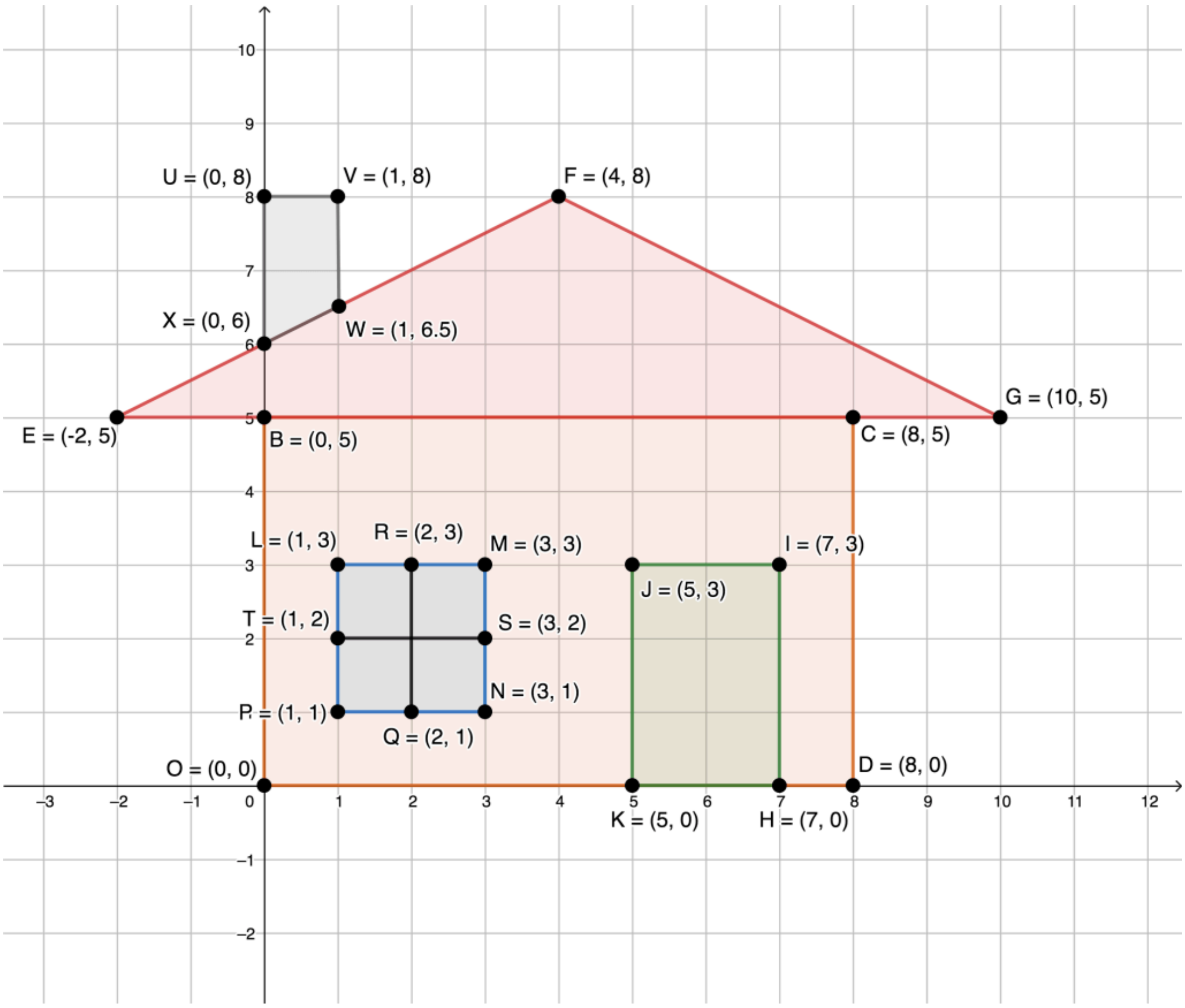
What to do:

Draw your own Cartesian plane on a piece of paper. You need to make sure that the x-axis extends from -3 to 13 and that the y-axis extends from -1 to 9.

1. Plot the following points: $O(0, 0)$, $B(0, 5)$, $C(8, 5)$, $D(8, 0)$. Join these points in the following order: O,B,C,D. What shape does this make?
2. Plot the following points: $E(-2, 5)$, $F(4, 8)$, $G(10, 5)$. Join these points in the following order: E,F,G. What shape does this make?
3. Plot the following points: $H(7, 0)$, $I(7, 3)$, $J(5, 3)$, $K(5, 0)$. Join these points in the following order: H,I,J,K. What shape does this make?
4. Plot the following points: $L(1, 3)$, $M(3, 3)$, $N(3, 1)$, $P(5, 0)$. Join these points in the following order: L,M,N,P. What shape does this make?
5. Plot the following points: $Q(2, 1)$, $R(2, 3)$. Join these points in the following order: Q,R. What shape does this make?
6. Plot the following points: $S(3, 2)$, $T(1, 2)$. Join these points in the following order: S,T. What shape does this make?
7. Plot the following points: $U(0, 8)$, $V(1, 8)$, $W(1, 6.5)$, $X(0, 6)$. Join these points in the following order: U,V,W,X. What shape does this make?
8. Colour $UVWX$ in brown. Colour $LMNP$ in blue. Make QR and ST black. Colour $HIJK$ green. Colour EFG red. Colour the rest of $OBCD$ blue. What have you drawn?

What you found:

1. $OBCD$ is a rectangle.
2. EFG is a triangle.
3. $HIJK$ is a rectangle.
4. $LMNP$ is a square.
5. QR is a straight line segment.
6. ST is a straight line segment.
7. $UVWX$ is a quadrilateral.
8. Here is the picture you should have drawn.



From Activity 1.1, we can see that we can use sets of ordered pairs on the Cartesian plane to draw polygons and lines. In fact, we can draw any polygon we like. Remember a polygon is a closed shape with straight sides. The more sides there are, the more ordered pairs we need.

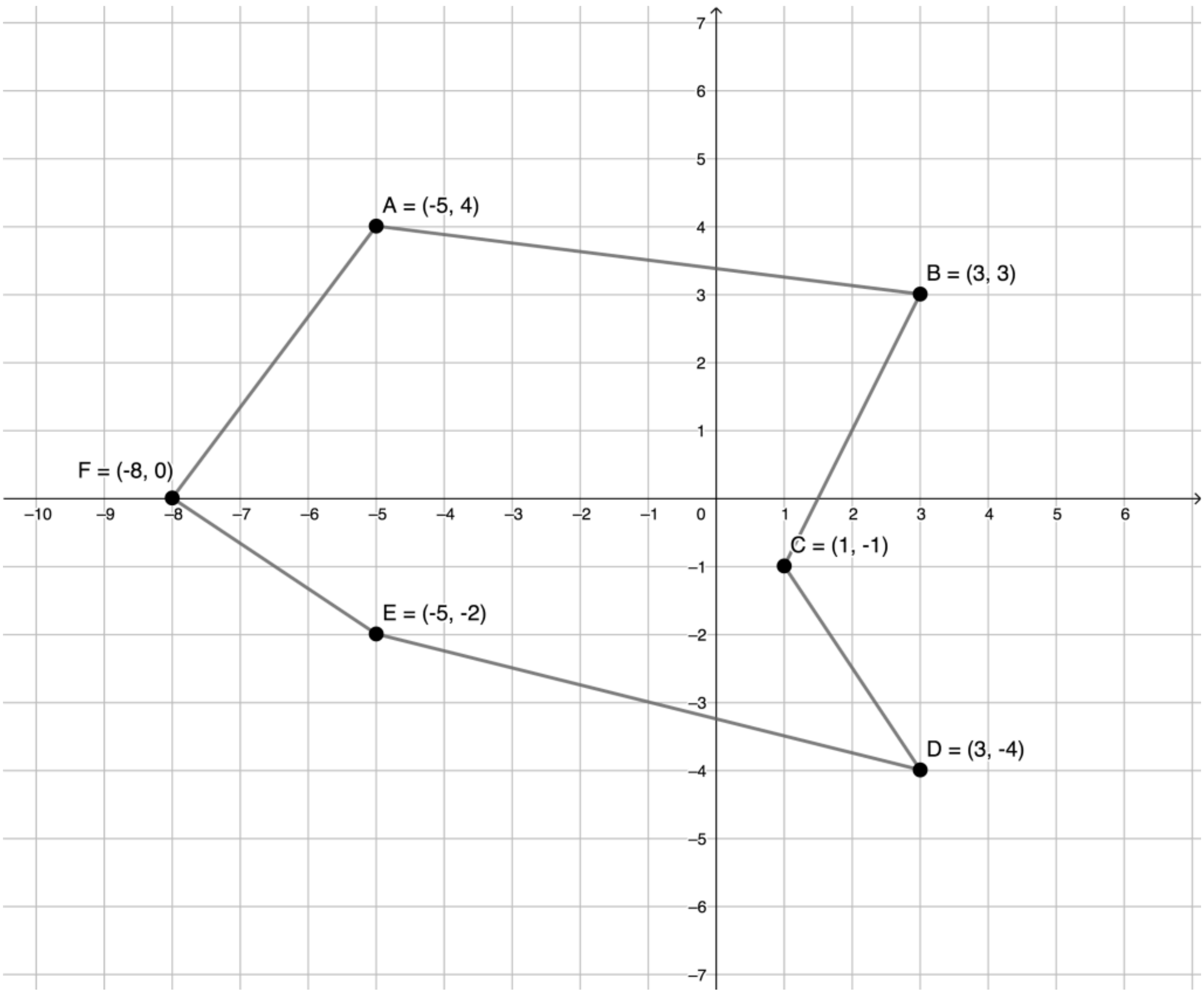
Have a look at OBCD that you drew in Activity 1.1 again. Did the order in which we joined the points with straight lines matter? In order for us to draw the rectangle, we had to go from point O to B to C and then to D. What other orders would result in the same rectangle? BCDO, CDOB and DOBC would all work as well.

However, can you see that if we went from point O to C to B to D we would not have ended up with the rectangle.

So, the order in which we join the points matters and this order is represented in the name we give to the polygon, for example, OBCD.

Note

When you draw polygons on the Cartesian plane, the order of the letters for naming the figure is important. It indicates the order in which the points must be joined.



For example here we have: A to B, B to C, C to D, D to E, E to F, and F back to A. The above quadrilateral can also be referred to as quadrilateral $ABCDEF$ or $CDEFAB$ or $DEFABC$.

However, it is conventional to write the letters in alphabetical order.

Exercise 1.2

1. What shapes do the following points create when you join them in the order in which they are given below? You will need to draw each shape on a Cartesian plane to check your answers.
   a. $A(-4, 2)$, $B(-4, -4)$, $C(6, -4)$
   b. $D(-4, 3)$, $E(3, 3)$, $F(3, -1)$, $G(-4, -1)$
   c. $H(0, -2)$, $I(2, 0)$, $J(0, 2)$, $K(-2, 0)$
2. Work out the coordinates of the point N so that $LMNO$ is a square if $L(-3, 2)$, $M(-1, 5)$ and O is the origin $(0, 0)$?

The full solutions are at the end of the unit.

Summary

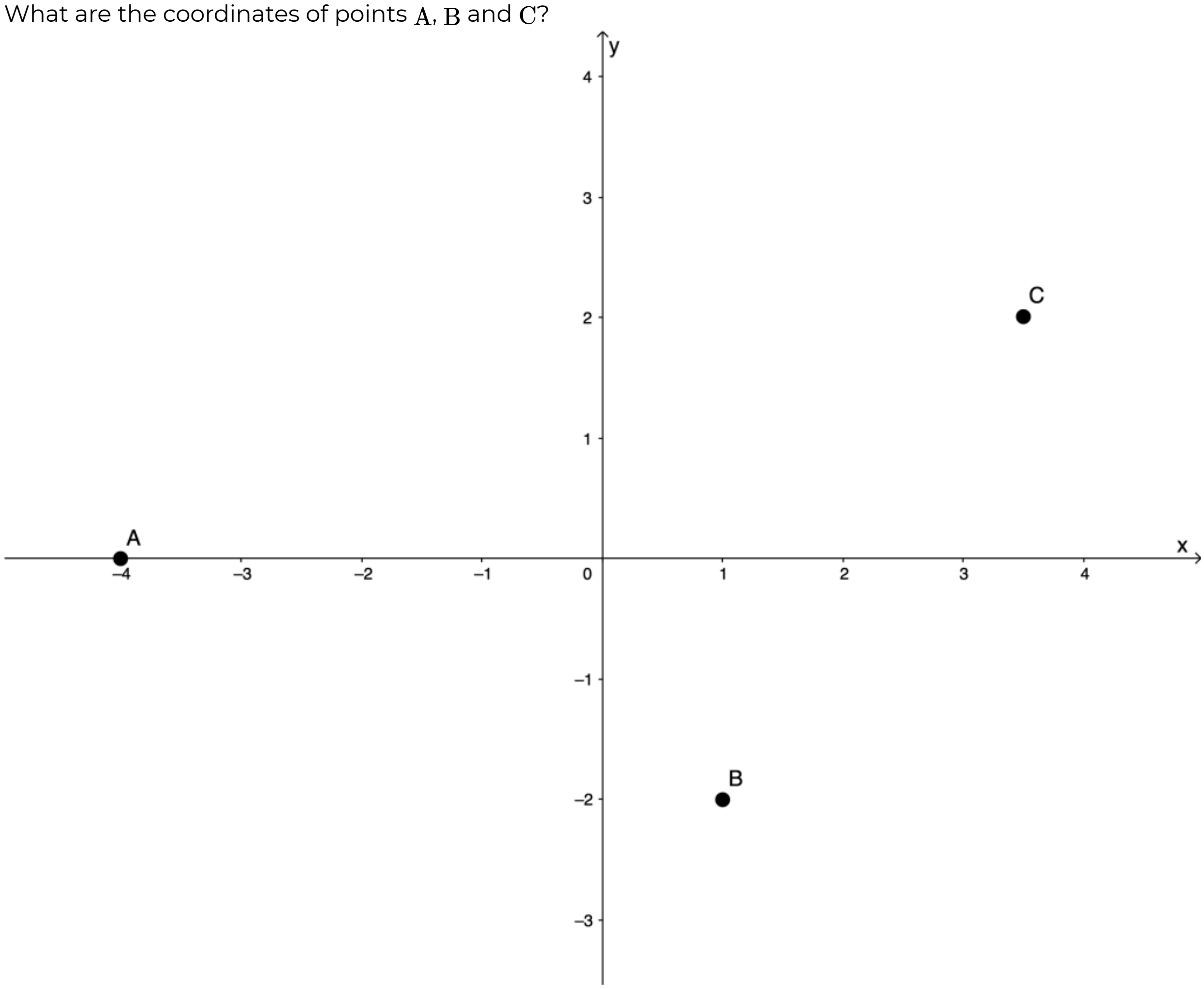
In this unit you have learnt the following:

- What the Cartesian plane is and what ordered pairs are.
- How to plot points on the Cartesian plane.
- How to identify the coordinates of points on the Cartesian plane.
- How to draw lines and polygons on the Cartesian plane.

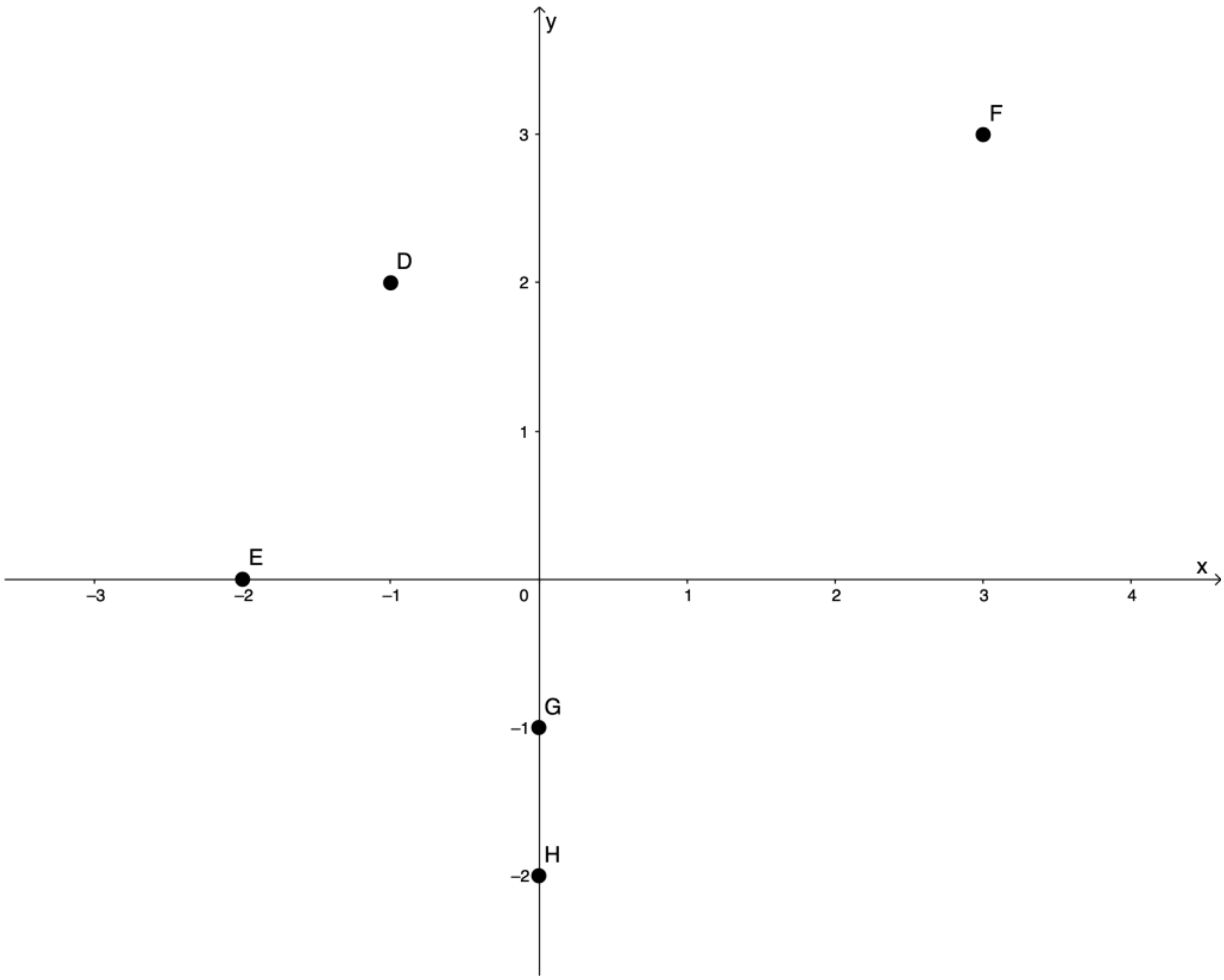
Unit 1: Assessment

Suggested time to complete: 15 minutes

1. What are the coordinates of points A, B and C?



2. Which point has coordinates $(0, -2)$?



3. ABCD is a rectangle on the Cartesian plane. What are the coordinates of D if $\mathrm{A}(-2,3)$, $\mathrm{B}(5,3)$ and $\mathrm{C}(5,-5)$?

Bonus question:

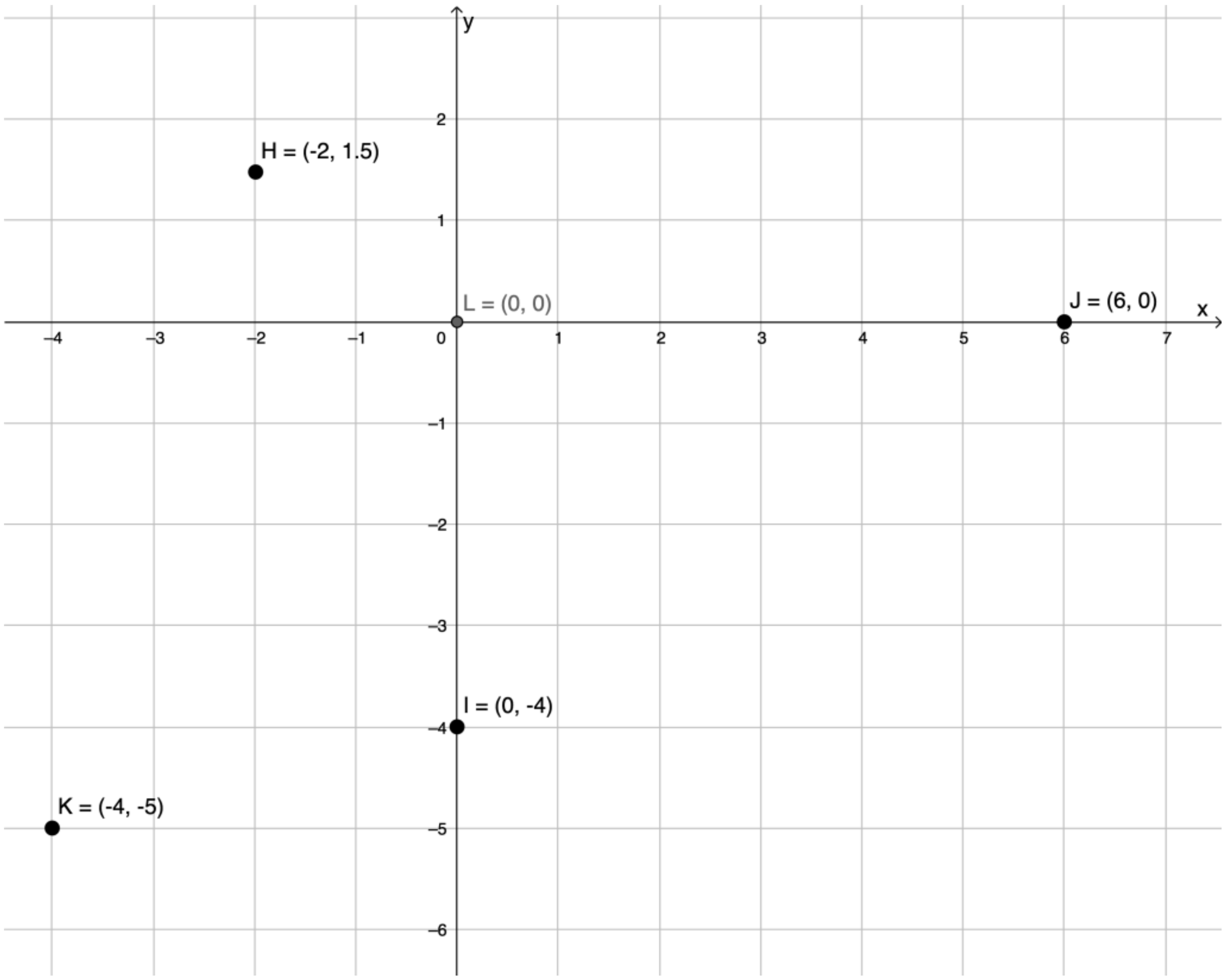
IJ and KL are line segments that are parallel and of equal length. If $\mathrm{I}(2,2)$, $\mathrm{J}(0,-3)$, and $\mathrm{L}(-1,-1)$, what are the two possible sets of coordinates for K?

The full solutions are at the end of the unit.

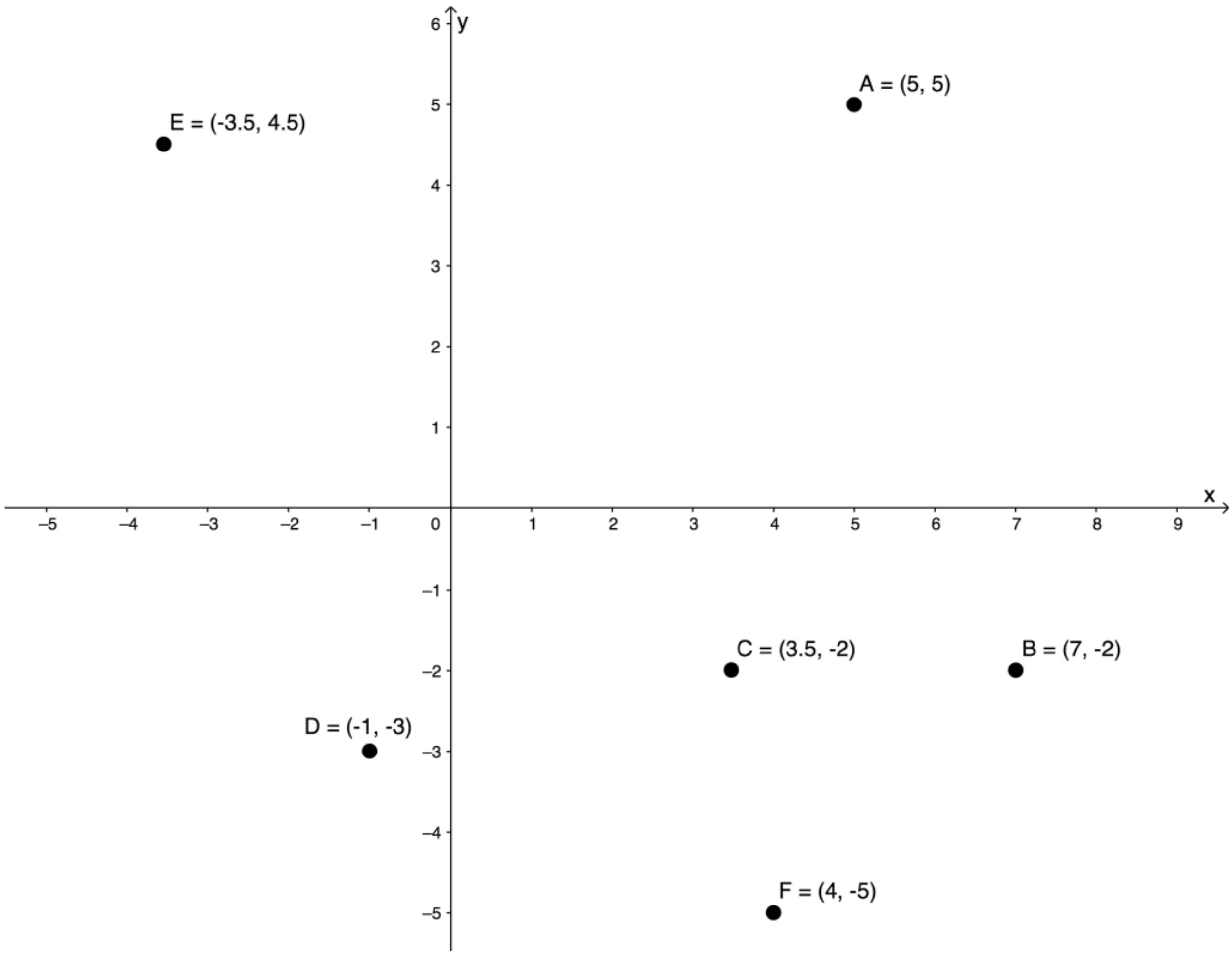
Unit 1: Solutions

Exercise 1.1

1.

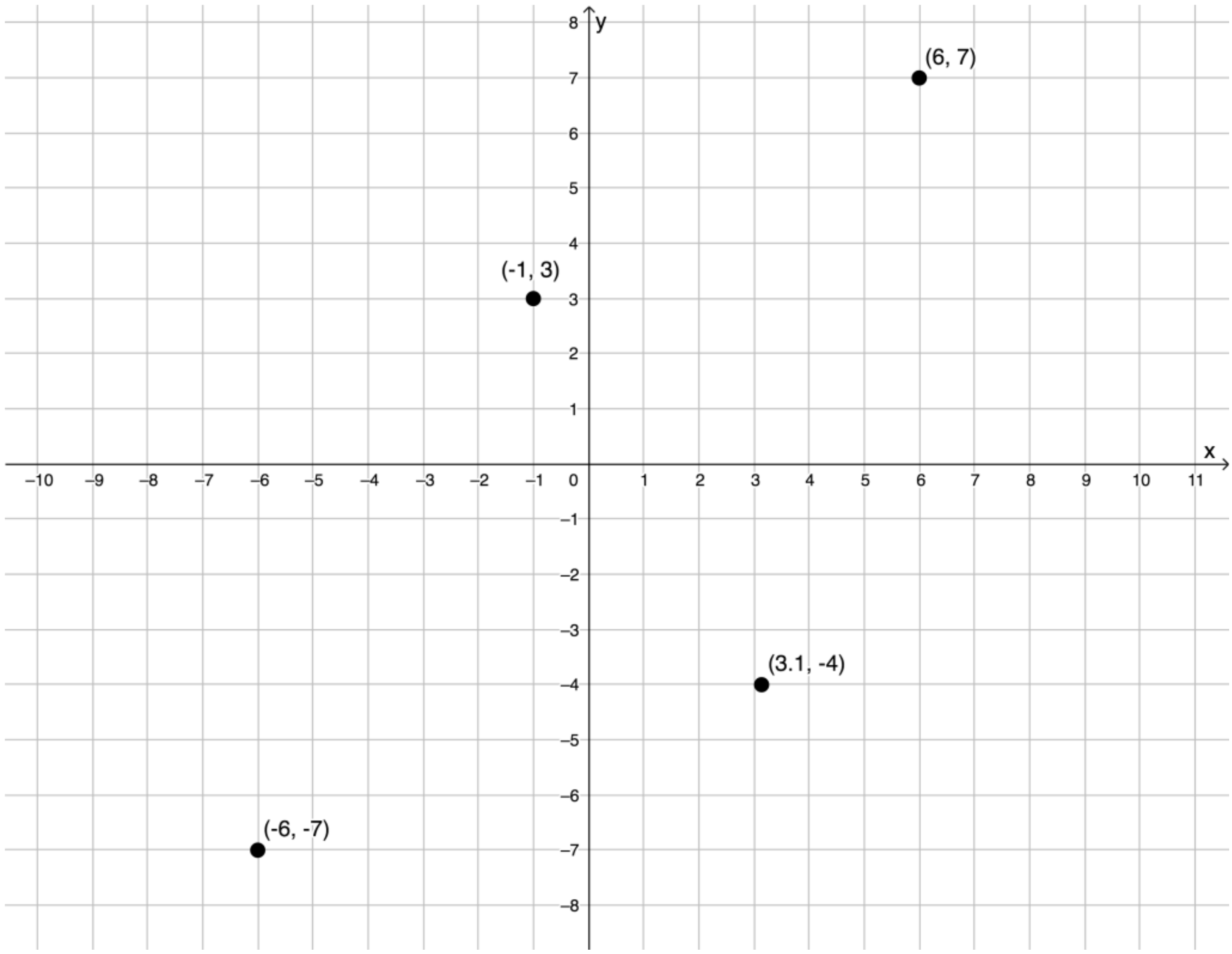


2.



3.

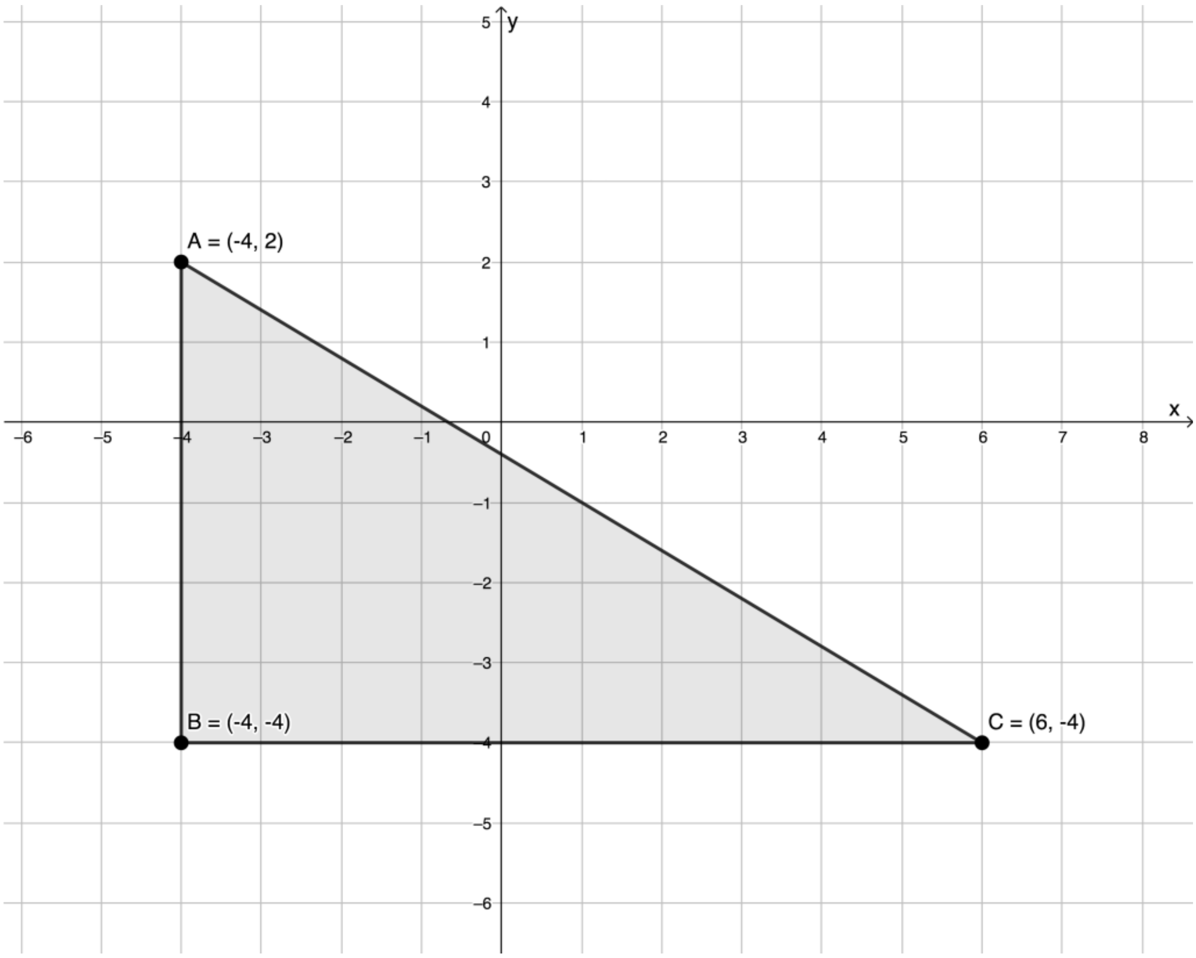
a. $(6, 7)$ is in quadrant 1.
b. $(-6, -7)$ is in quadrant 3.
c. $(-1, 3)$ is in quadrant 2.
d. $(\pi, -4)$ is in quadrant 4.



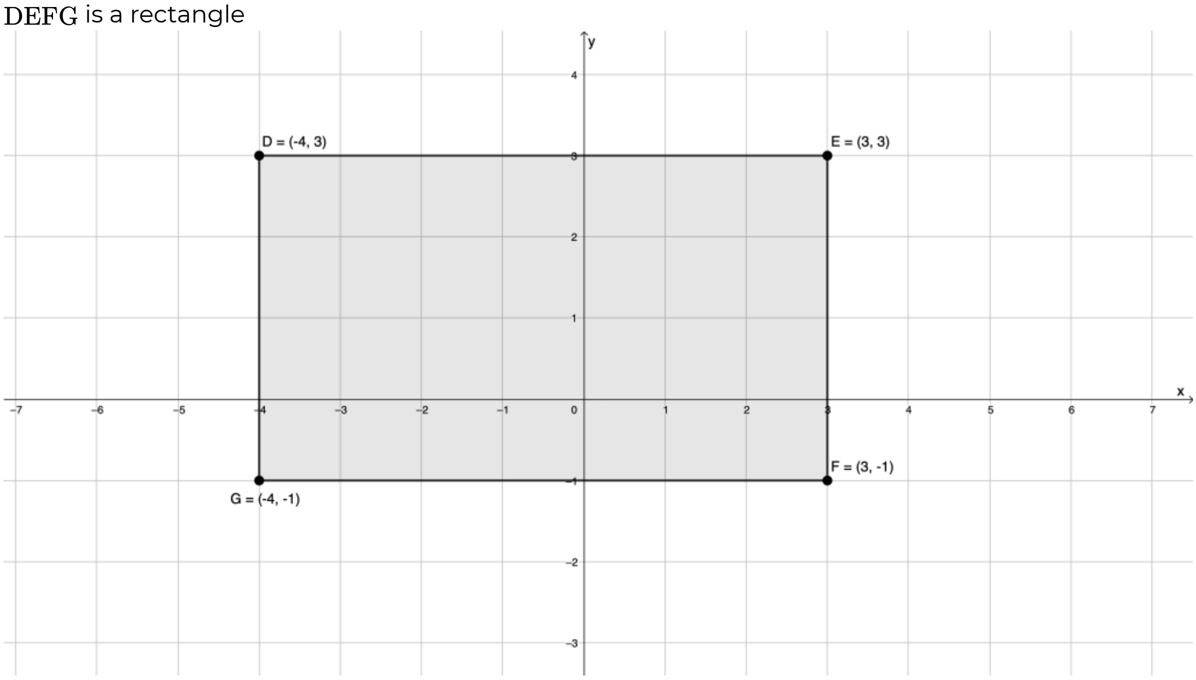
Back to Exercise 1.1

Exercise 1.2

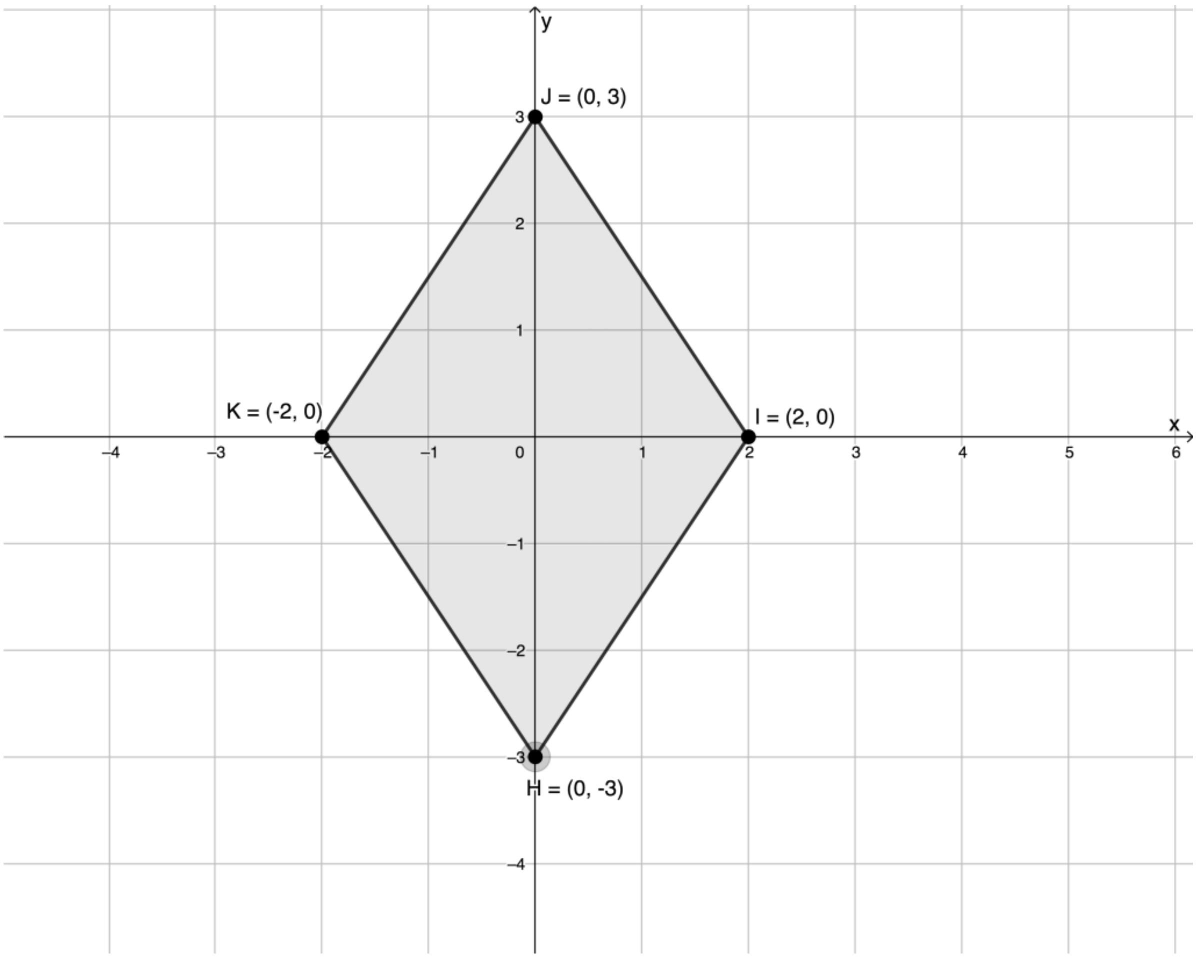
1.
   a. ABC is a right-angled triangle



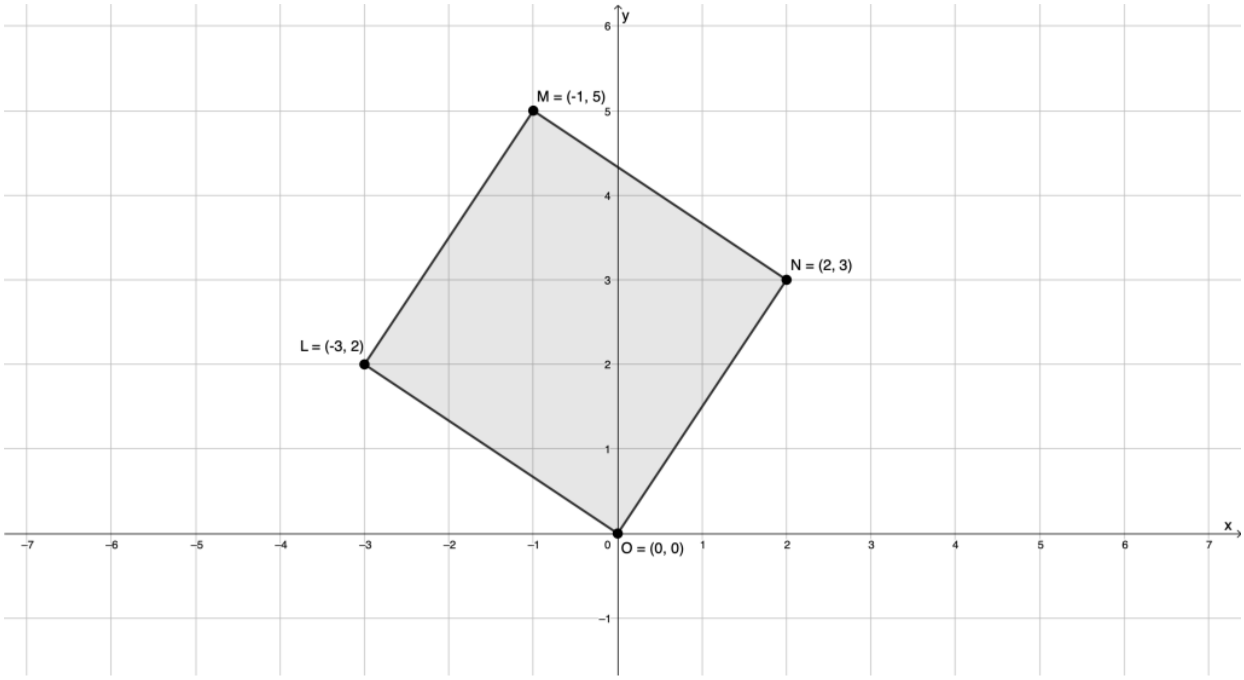
b. DEFG is a rectangle



c. HIJK is a square



2. N must be the point $(2, 3)$ in order for LMNO to be a square.

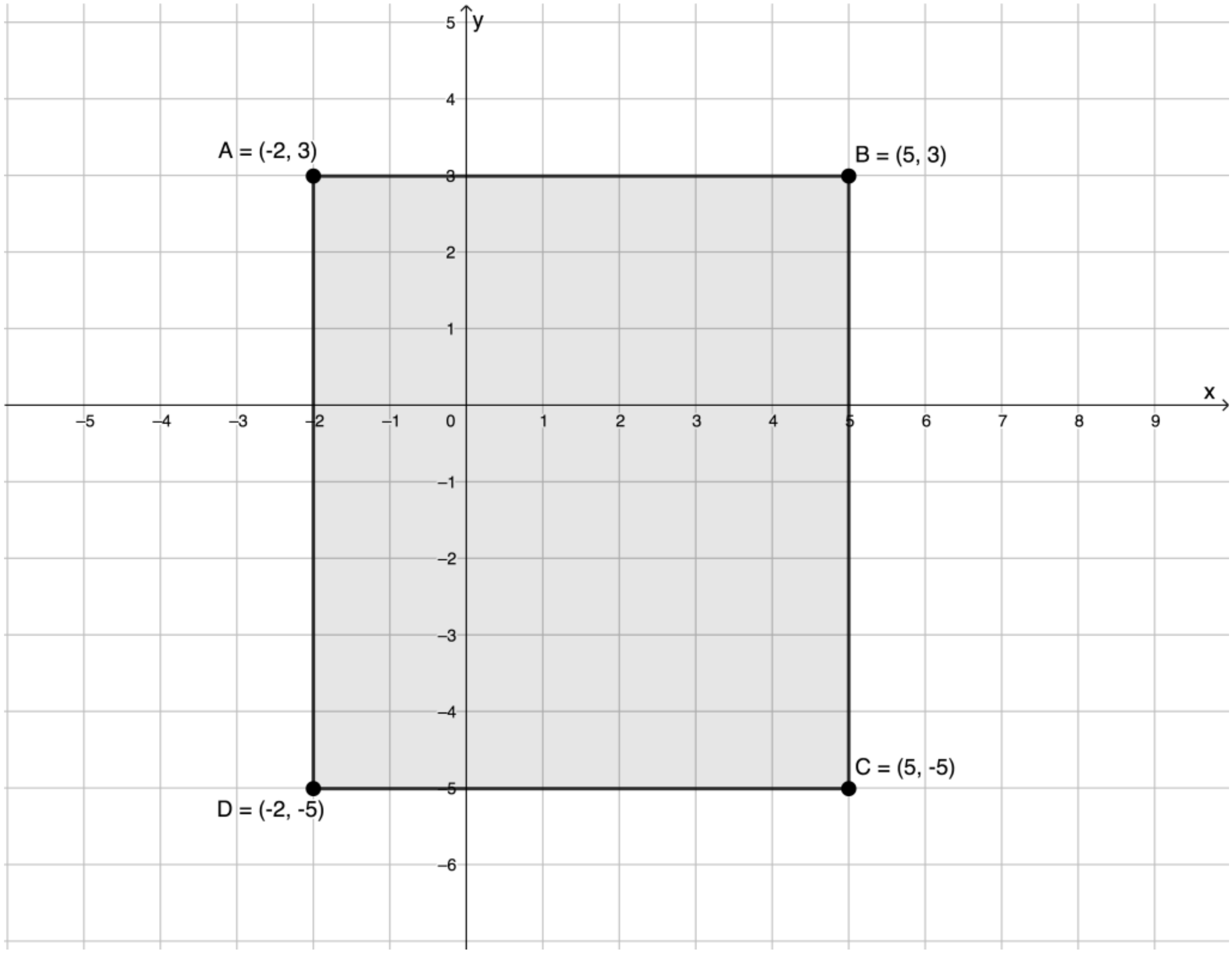


Back to Exercise 1.2

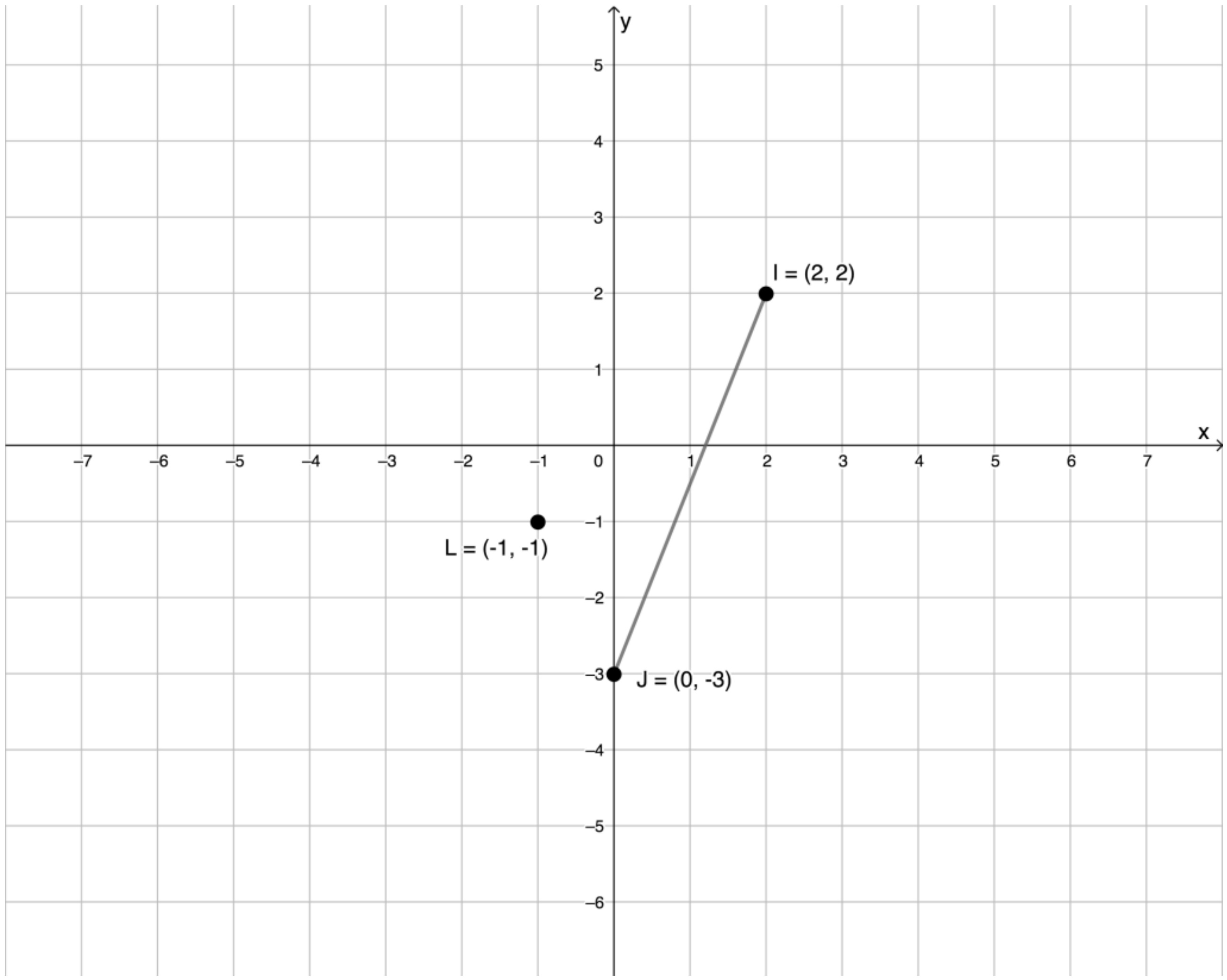
Unit 1: Assessment

1. $\mathrm{A}(-4, 0), \mathrm{B}(1, -2), \mathrm{C}(3.5, 2)$

2. $H(0, -2)$
3. If we plot points A, B and C, we can see that D must be the point $(-2, -5)$.

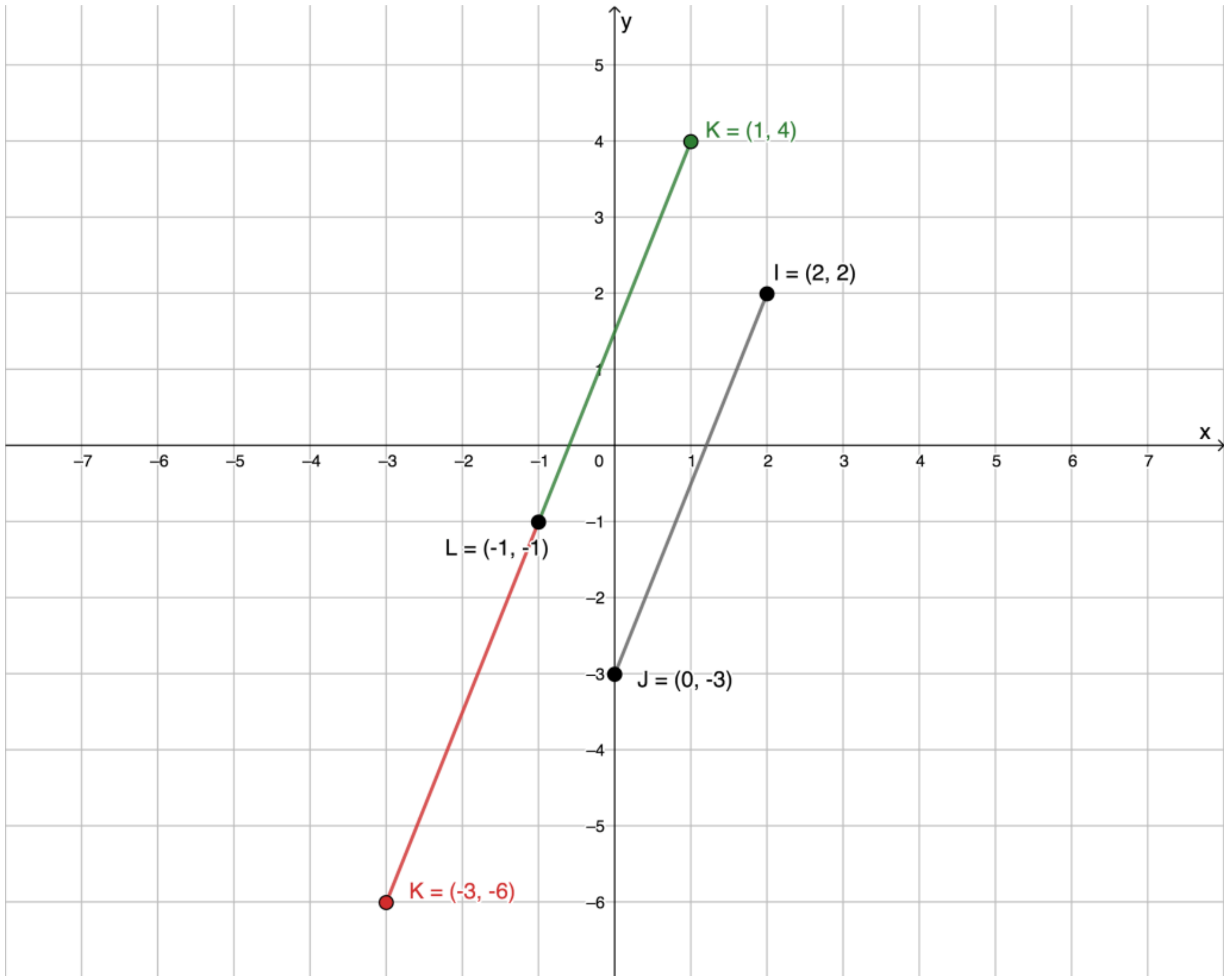


Bonus question:
First, we need to plot the information we have been given.



We can see that to get from I to J we need to move two units to the left and then five units down. Therefore, if KL is parallel to IJ and the same length, the same movements will be necessary to get from one point to the other.

If we start at L and move two units left and then five units down, we get to $(-3, -6)$. If we start at K it will need to be the point $(1, 4)$ so that when we move two units to the left and five units down we get to L. So, $\mathrm{K}(1, 4)$ or $\mathrm{K}(-3, -6)$.



[Back to Unit 1: Assessment]

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Unit 2: Distance, gradient and midpoints

DYLAN BUSA

Unit outcomes: Unit 2: Distance, gradient and midpoints

By the end of this unit you will be able to:

- Calculate the distance between two points on the Cartesian plane.
- Calculate the gradient between two points on a straight line.
- Work with the gradients of parallel and perpendicular lines.
- Calculate the midpoint between two points on the Cartesian plane.

What you should know

Before you start this unit, make sure you can

- Plot points on the Cartesian plane. Review Unit 1 of Subject outcome 3.3.
- Identify the coordinates of points on the Cartesian plane. Review Unit 1 of Subject outcome 3.3.
- Draw lines and polygons on the Cartesian plane. Review Unit 1 of Subject outcome 3.3.
- Use the theorem of Pythagoras to find missing lengths in right-angled triangles. Review Unit 1 of Subject outcomes 3.2 if you need help with this.
- Explain what the gradient of a straight line is. Review Unit 1 of Subject outcome 2.1.

Introduction

In Unit 1 of this Subject outcome we plotted points on the Cartesian plane to draw line segments and polygons. One type of polygon that we could draw is a square. But how can we prove that the shape we have drawn is really a square?

To prove that a polygon is a square, we need to prove that all the sides are the same length AND that the interior angles are all equal to 90°. We could take some measurements with a ruler and protractor, but this is not always accurate or practical.

We need better methods to analyse the lines and shapes we draw on the Cartesian plane. Let's start with the lengths of lines.

The distance between two points

Being able to calculate the distance between any two points on the Cartesian plane is very useful and surprisingly easy as you will discover in Activity 2.1.

Activity 2.1: Measure distance on the Cartesian plane

Time required: 15 minutes

What you need:

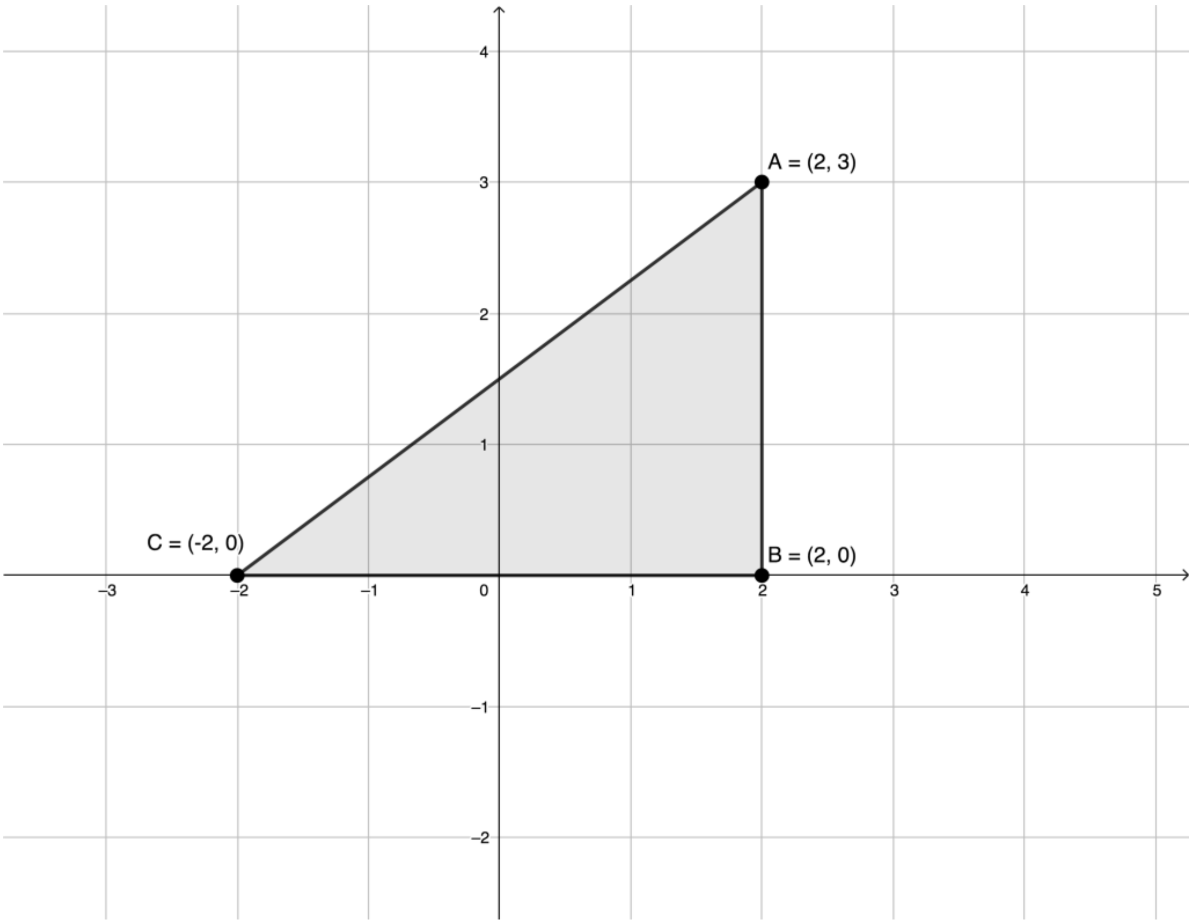
- a piece of paper (graph paper if possible)
- a pen or pencil

What to do:

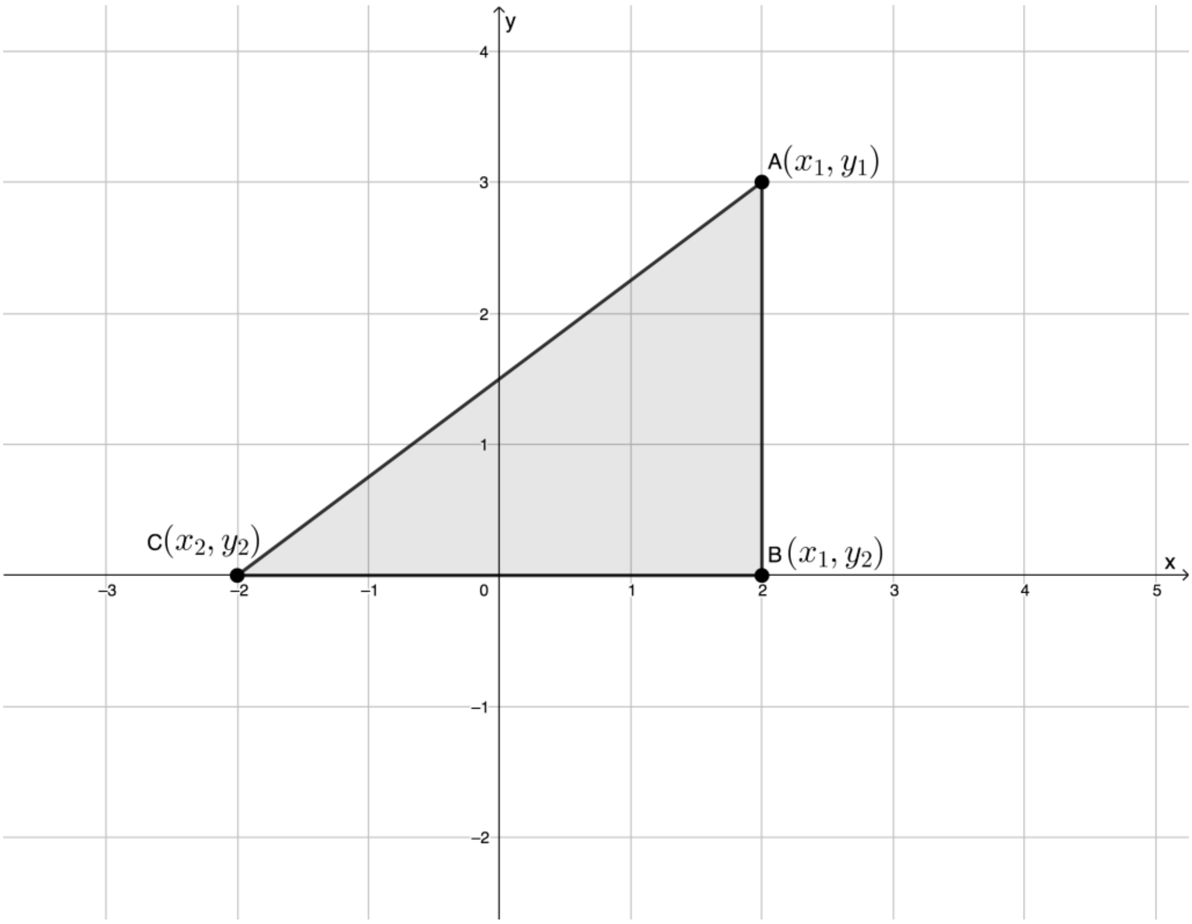
1. Draw your own Cartesian plane on a piece of paper. Use graph paper if you have some. Plot the following points: $A(2, 3)$, $B(2, 0)$ and $C(-2, 0)$. Now join the points A, B and C. What type of shape have you created?
2. How do you know that the angle at B is 90°?
3. What is the length of AB? How do you know?
4. What is the length of BC? How do you know?
5. How can we find the length of AC in this right-angled triangle?
6. Use the theorem of Pythagoras to find the length of AC.
7. If point A had coordinates (x_1, y_1), point B had coordinates (x_1, y_2), point and C had coordinates (x_2, y_2), write expressions for the lengths of AB and BC.
8. Now write an expression to find the length of AC.

What you found:

1. ABC is a triangle.



2. We know that the axes are perpendicular to each other. The line AB is parallel to the y-axis because both points have the same x-coordinate. The line BC is parallel to the x-axis because it lies on the x-axis. Therefore, we know that AB is perpendicular to BC.
3. Since A and B lie on the same vertical line, the length of AB is the difference in the y-coordinates. Therefore, the length of AB is $3-0=3$.
4. Since B and C lie on the same horizontal line, the length of BC is the difference in the x-coordinates. Therefore, the length of BC is $2-(-2)=2+2=4$.
5. In $\Delta\mathrm{ABC}$, AC lies opposite the 90° angle and is therefore the hypotenuse. We can use the theorem of Pythagoras to calculate the length of AC.
6. In $\Delta\mathrm{ABC}$:
$$\begin{aligned} \mathrm{AC}^2 &= \mathrm{AB}^2+\mathrm{BC}^2 \text{ Pythagoras} \\ \therefore \mathrm{AC}^2 &= 3^2+4^2 \\ \therefore \mathrm{AC}^2 &= 25 \\ \therefore \mathrm{AC} &= 5 \end{aligned}$$
7. The length of AB is y_1-y_2. The length of BC is x_1-x_2.



8. In ΔABC:

$$\text{AC}^2 = \text{AB}^2 + \text{BC}^2$$
$$\therefore \text{AC}^2 = (y_1 - y_2)^2 + (x_1 - x_2)^2$$
$$\therefore \text{AC} = \sqrt{(y_1 - y_2)^2 + (x_1 - x_2)^2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

We have just seen that to calculate the distance between any two points (x_1, y_1) and (x_2, y_2) on the Cartesian plane we can use the formula $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. This is called the **distance formula**.

Distance formula: $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

Note

Because $(x_1 - x_2)^2 = (x_2 - x_1)^2$ it does not matter which point we assign as point 1 and which we assign as point 2 in $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Example 2.1

If $\mathrm{H}(-3,6)$, $\mathrm{I}(8,4)$ and $\mathrm{J}(-3,-5)$, find the distance between H and I and between I and J.

Solution

Distance between H and I:
The first step is to assign each point to be either point 1 or point 2. Let H be (x_1, y_1) and I be (x_2, y_2).

The distance formula is $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. Substitute the given coordinates into the formula.

$$\begin{aligned}
d_{\mathrm{HI}} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
&= \sqrt{(-3-8)^2 + (6-4)^2} \\
&= \sqrt{(-11)^2 + (2)^2} \\
&= \sqrt{121 + 4} \\
&= \sqrt{125} \\
&= 11.2 \quad \text{rounded of to 1 decimal place}
\end{aligned}$$

The distance between $\mathrm{H}(-3,6)$ and $\mathrm{I}(8,4)$ is 11.2 units.

Distance between I and J:
The first step again is to assign each point to be either point 1 or point 2. Now let I be (x_1, y_1) and J be (x_2, y_2).

The distance formula is $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. Substitute the given coordinates into the formula.

$$\begin{aligned}
d_{\mathrm{IJ}} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
&= \sqrt{(8-(-3))^2 + (4-(-5))^2}
\end{aligned}$$

It is always a good idea to place negative values inside brackets (like we did with the coordinates of J) so that you don't make any mistakes with the signs.

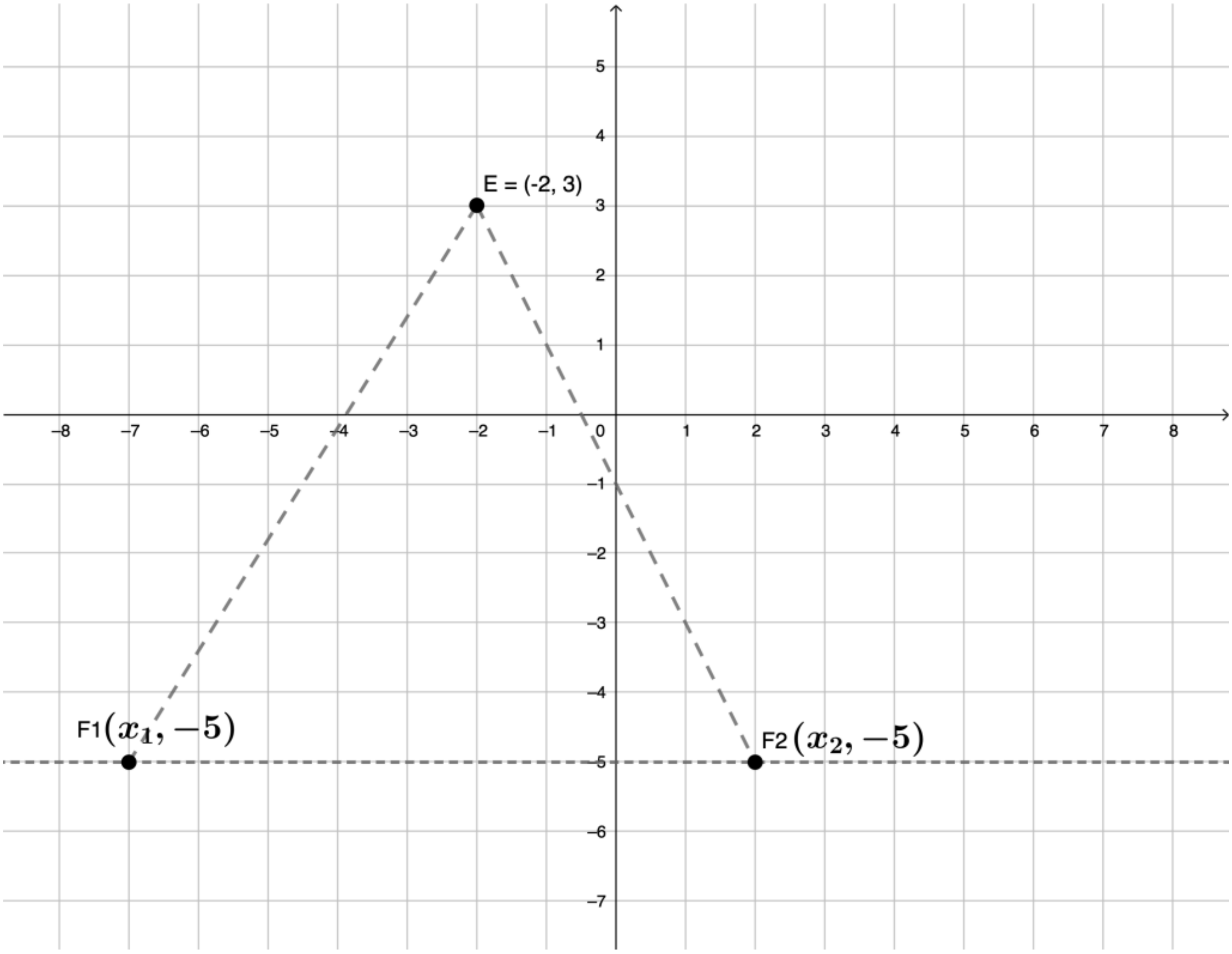
$$\begin{aligned}
d_{\mathrm{IJ}} &= \sqrt{(8-(-3))^2 + (4-(-5))^2} \\
&= \sqrt{(8+3)^2 + (4+5)^2} \\
&= \sqrt{11^2 + 9^2} \\
&= \sqrt{121 + 81} \\
&= \sqrt{202} \\
&= 14.2 \quad \text{rounded of to 1 decimal place}
\end{aligned}$$

Example 2.2

If $\mathrm{EF} = 4\sqrt{5}$, $\mathrm{E}(-2,3)$ and $\mathrm{F}(x,-5)$, find the value of x.

Solution

It is a good idea to make a sketch of the given information to help you think through the problem.



We know that the point F lies somewhere on the horizontal line that cuts the y-axis at -5 but there are two possible places where it could be (F_1 or F_2).

We need to assign each point to be either point 1 or point 2. Let E be (x_1, y_1) and F be (x_2, y_2).

The distance formula is $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. Substitute the given coordinates into the formula.

$$\begin{aligned} d_{\text{EF}} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(-2 - x)^2 + (3 - (-5))^2} \\ &= \sqrt{(-2 - x)^2 + (8)^2} \\ &= \sqrt{(-2 - x)^2 + 64} \end{aligned}$$

But we know that $d_{\text{EF}} = 4\sqrt{5}$.

$$\begin{aligned} 4\sqrt{5} &= \sqrt{(-2 - x)^2 + 64} \\ \therefore \left(4\sqrt{5}\right)^2 &= (-2 - x)^2 + 64 \quad \text{Expand } (-2 - x)^2 \\ \therefore 80 &= 4 + 4x + x^2 + 64 \quad \text{Solve the quadratic equation} \\ \therefore x^2 + 4x - 12 &= 0 \\ \therefore (x + 6)(x - 2) &= 0 \\ \therefore x = -6 &\text{ or } x = 2 \end{aligned}$$

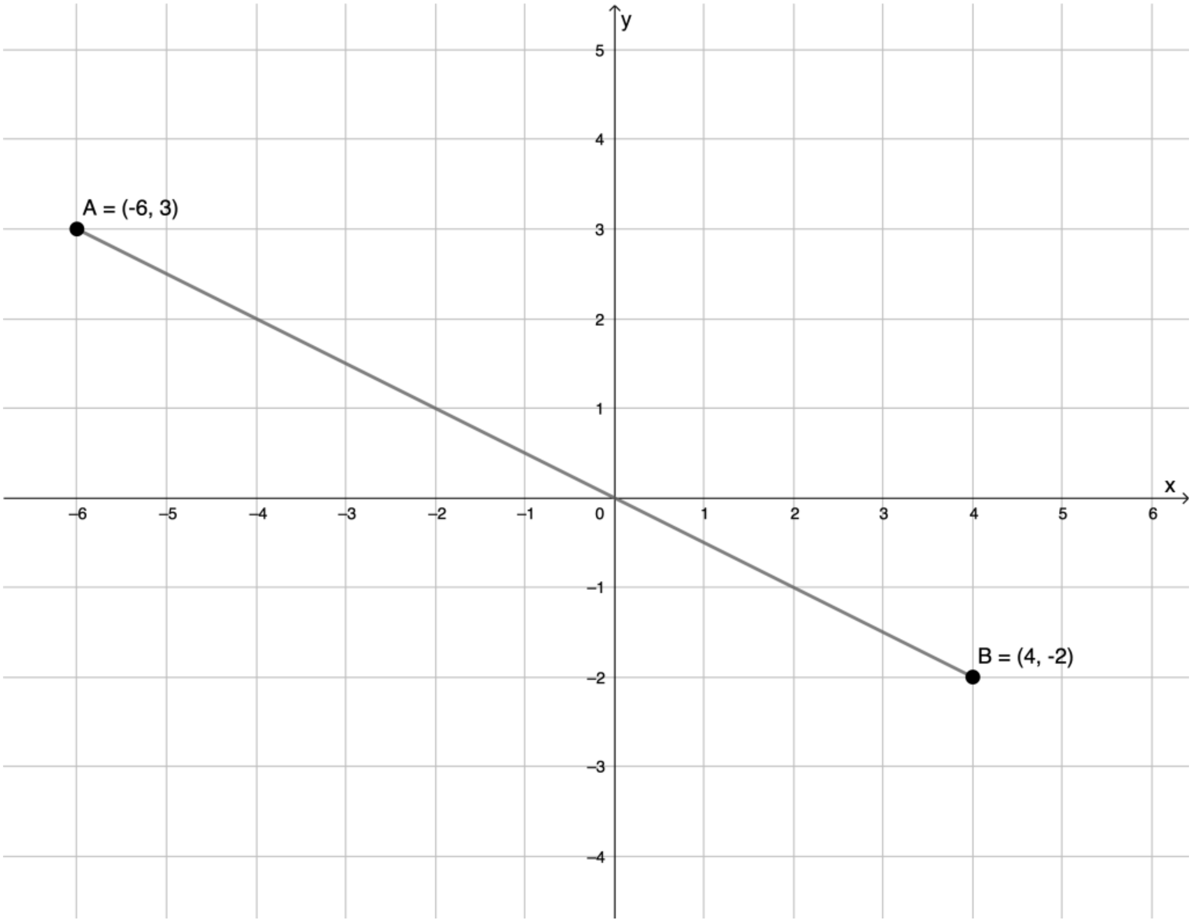
As we expected, we get two possible answers.

Note

Drawing a sketch of the situation helps you to make sense of the information given as well as to check the feasibility of your answers.

Exercise 2.1

1. Calculate the length of AB. Round your answer to one decimal place.



2. Find the distance between $\mathrm{C}(0,-9)$ and $\mathrm{D}(2,3)$. Leave your answer in surd form.
3. Calculate d_{QT} if $\mathrm{Q}(x, y)$ and $\mathrm{T}(x+4, y-7)$. Leave your answer in surd form.
4. If the distance between $\mathrm{S}(0,-3)$ and $\mathrm{F}(8, a)$ is 10 units, find the possible values of a.

The full solutions are at the end of the unit.

Note

If you would like more practise using the distance formula to find the length of a line on the Cartesian plane, try the interactive activity called Practice the Distance Formula.

Gradients of lines

We learnt about gradient (m) in Subject outcome 2.1 Unit 1 on linear functions. Gradient is a measure of the steepness of a line. In Figure 1, OA has a steeper gradient than OB. While OA and OB have positive gradients, OC and OD have negative gradients.

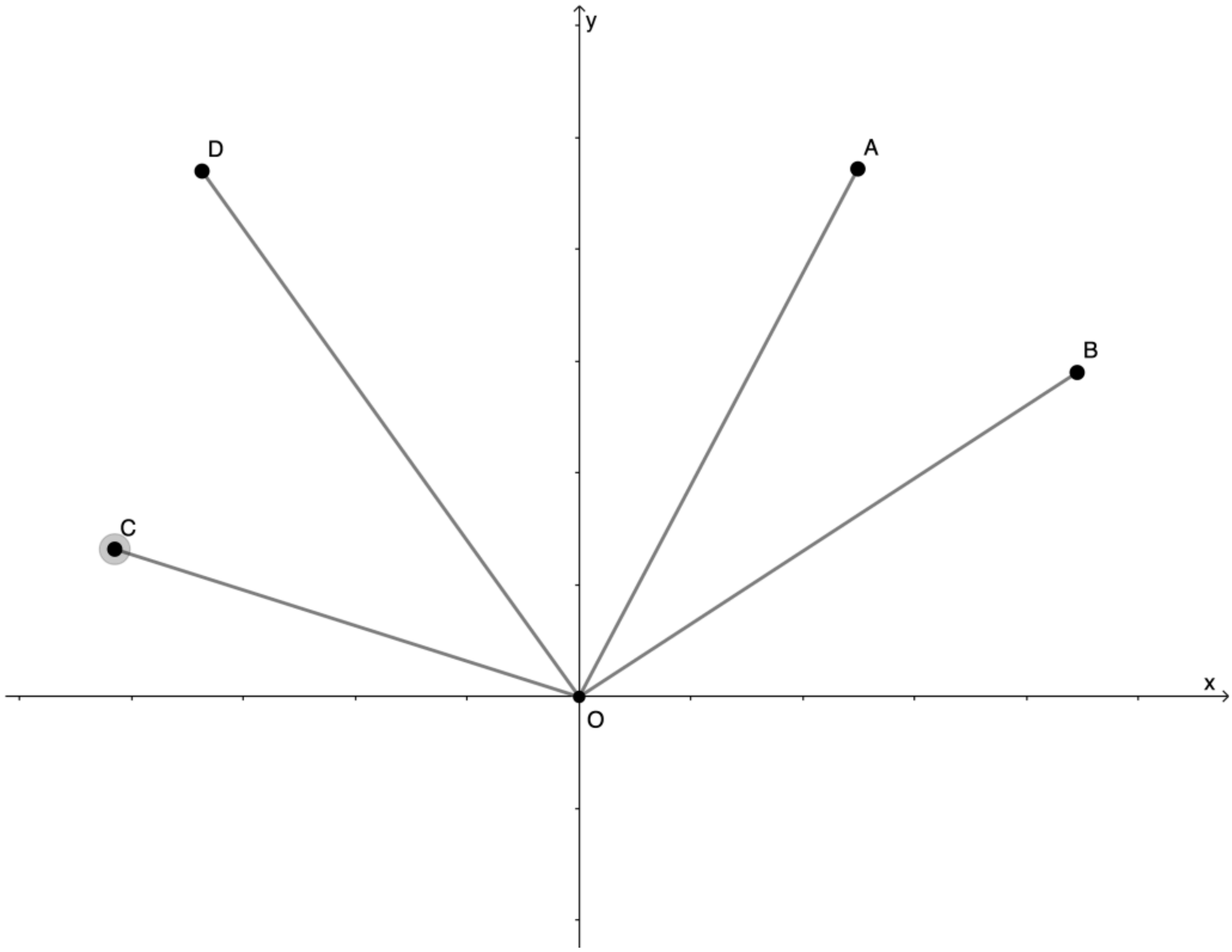


Figure 1: Different gradients

We define the gradient of a line as the ratio of the change in y values between any two points on the line (the rise or the vertical movement up or down) to the change in x values between those same two points (the run or the horizontal movement left or right). We can write this ratio as a fraction like this:

$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\text{rise}}{\text{run}}$.

To work out the change in y or rise between any two points on the Cartesian plane, we simply need to find the difference between their y-coordinates. We find the difference between their x-coordinates to work out the change in x or run. Therefore, the gradient between any two points on the Cartesian plane can be expressed as follows:

$m = \frac{y_2 - y_1}{x_2 - x_1}$ or $m = \frac{y_1 - y_2}{x_1 - x_2}$

We call this the **gradient formula**.

Gradient formula: $m = \frac{y_2 - y_1}{x_2 - x_1}$ or $m = \frac{y_1 - y_2}{x_1 - x_2}$

Note

The order of the points does not matter but you must be consistent.

$m = \frac{y_2 - y_1}{x_2 - x_1}$ or $m = \frac{y_1 - y_2}{x_1 - x_2}$

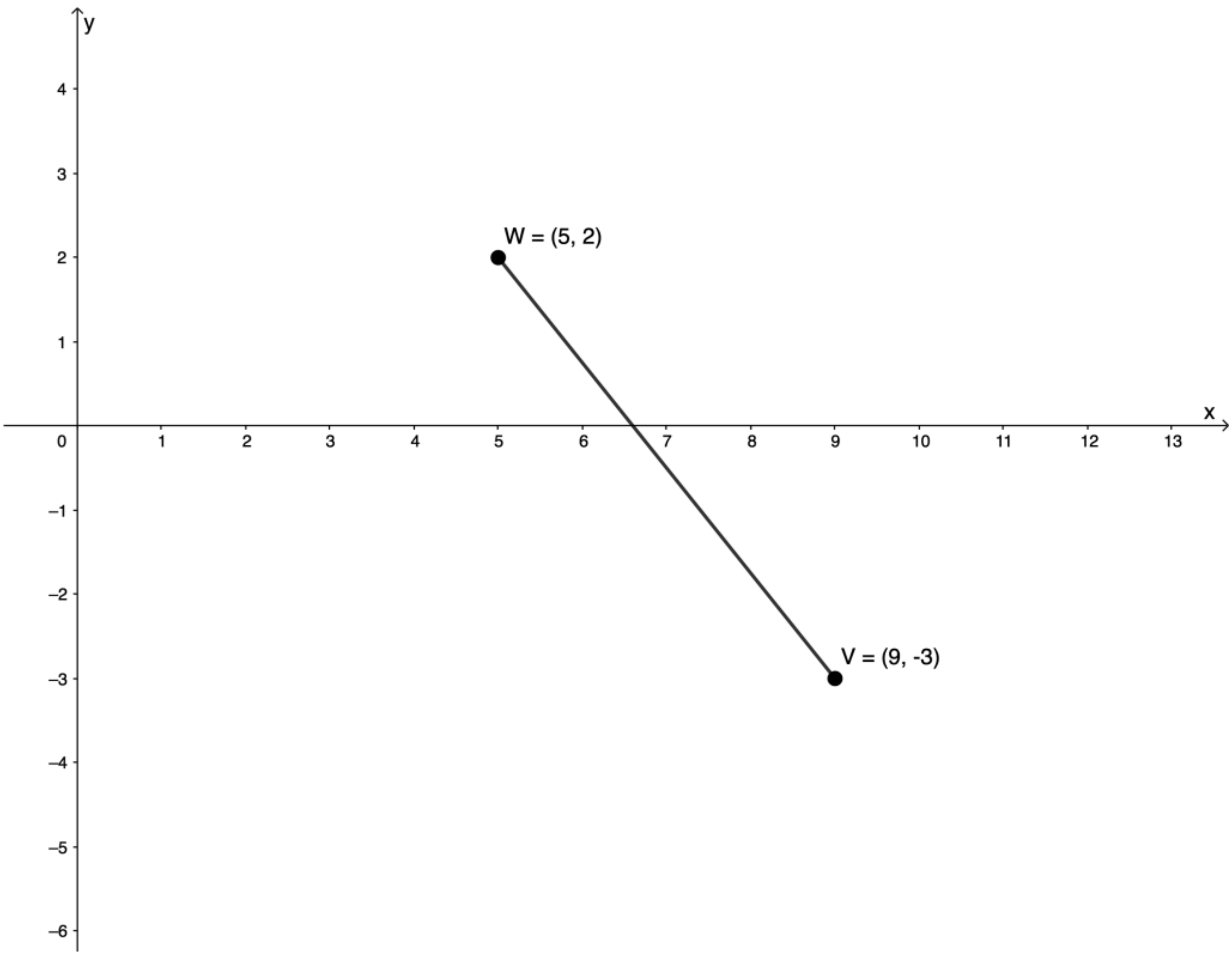
$m \neq \frac{y_2 - y_1}{x_1 - x_2}$ and $m \neq \frac{y_1 - y_2}{x_2 - x_1}$

Example 2.3

Find the gradient between $\text{W}(5, 2)$ and $\text{V}(9, -3)$.

Solution

If you want to, you can draw a sketch of the given information. This can help you to see if your answer makes sense.



We can see that the line slopes from left to right so we expect to get a negative answer for the gradient.

Let $\mathrm{W}(5, 2)$ be (x_1, y_1) and $\mathrm{V}(9, -3)$ be (x_2, y_2).

$$\begin{aligned} m_{\mathrm{WV}} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-3 - 2}{9 - 5} \\ &= \frac{-5}{4} \\ &= -\frac{5}{4} \end{aligned}$$

Therefore $m_{\mathrm{WV}} = -\frac{5}{4}$. We get a negative answer as expected.

Note

If you would like more practise finding the gradient between two points, visit the interactive simulation called Calculating gradient.

Lines that run **horizontally** (parallel with the x-axis) have a gradient of zero. Since the y-values are the same everywhere on the line, the change in y (the rise) of the line is zero.

$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\text{rise}}{\text{run}} = \frac{0}{\text{run}} = 0$. In this case $y_2 - y_1 = 0$.

Lines that run **vertically** (parallel with the y-axis) have an undefined gradient. Here the change in x (the run) is zero since the x-values are the same everywhere on the line but we are never allowed to divide by zero.

$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\text{rise}}{\text{run}} = \frac{\text{rise}}{0}$ which is undefined.

$m_{\text{horizontal line}} = 0$

$m_{\text{vertical line}}$ is undefined

Example 2.4

$m_{\text{MN}} = \frac{5}{3}$ and $\text{M}(-2, 4)$. Find a if $\text{N}\left(\frac{3}{2}, a\right)$.

Solution

We know that $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5}{3}$. Let $\text{M}(-2, 4)$ be (x_1, y_1) and $\text{N}\left(\frac{3}{2}, a\right)$ be (x_2, y_2).

$$\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} = \frac{5}{3} \\
\therefore \frac{5}{3} &= \frac{a - 4}{\frac{3}{2} - (-2)} \\
&= \frac{a - 4}{\frac{3 + 4}{2}} \quad \text{Add the fractions in the denominator} \\
&= (a - 4) \times \frac{2}{7} \quad \text{Simplify by muliplying by the inverse of the denominator} \\
\therefore \frac{5}{3} &= \frac{2a - 8}{7} \\
\therefore 5 \times 7 &= 3(2a - 8) \\
\therefore 35 &= 6a - 24 \\
\therefore 6a &= 59 \\
\therefore a &= \frac{59}{6} = 9\frac{5}{6}
\end{aligned}$$

Exercise 2.2

1. Find the gradient of GH if:
   a. $\mathrm{G}(3,-5)$ and $\mathrm{H}(-2,7)$.
   b. $\mathrm{G}\left(\frac{3}{2},0\right)$ and $\mathrm{H}\left(3,-\frac{3}{4}\right)$.
   c. $\mathrm{G}(x-3,y)$ and $\mathrm{H}(x,y-4)$.
2. If the gradient of $m_{\mathrm{ST}}=\frac{2}{3}$, find q if:
   a. $\mathrm{S}(8,q)$ and $\mathrm{T}(16,2)$.
   b. $\mathrm{S}(3,2q)$ and $\mathrm{T}(9,14)$.

The full solutions are at the end of the unit.

Parallel and perpendicular lines

We know that parallel lines are always the same distance apart. This means that on the Cartesian plane, parallel lines have the same gradient.

In Figure 2, we have two parallel lines. If we measure the gradient of one line between points A and B, we can see that the gradient is $\frac{2}{4}=\frac{1}{2}$. If we measure the gradient of the other line between points C and D, we can see that that gradient is $\frac{4}{8}=\frac{1}{2}$.

The gradients of these two lines are the same. This makes sense. If parallel lines never meet, they must have exactly the same slope, in other words, the same gradient.

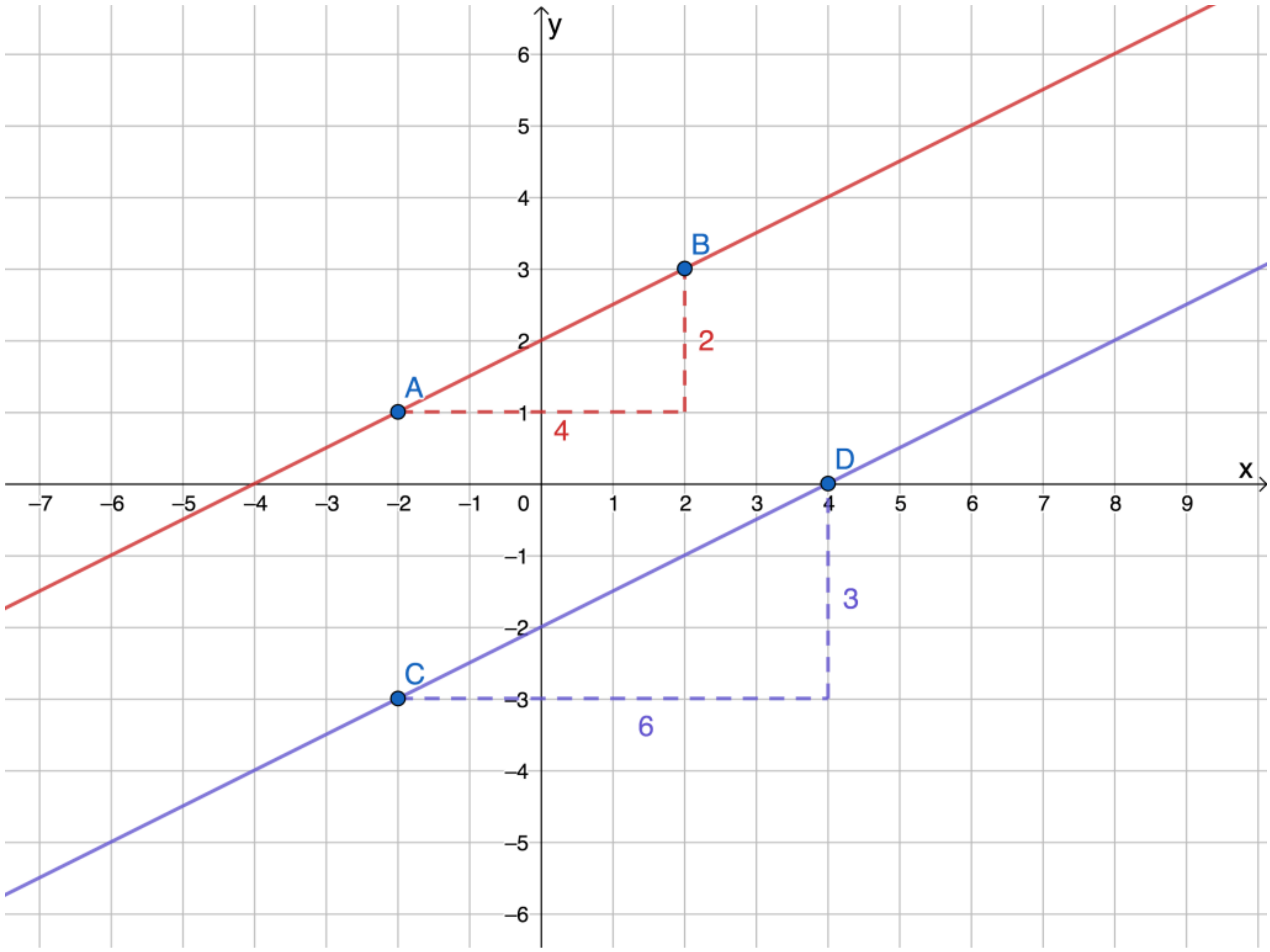


Figure 2: Parallel lines have the same gradient

We also know that perpendicular lines are at 90° to each other. That is what perpendicular means. But what is the relationship between the gradients of perpendicular lines? We will discover this in Activity 2.2.

Activity 2.2: The gradients of perpendicular lines

Time required: 10 minutes

What you need:

- a piece of paper (graph paper if possible)
- a pen or pencil

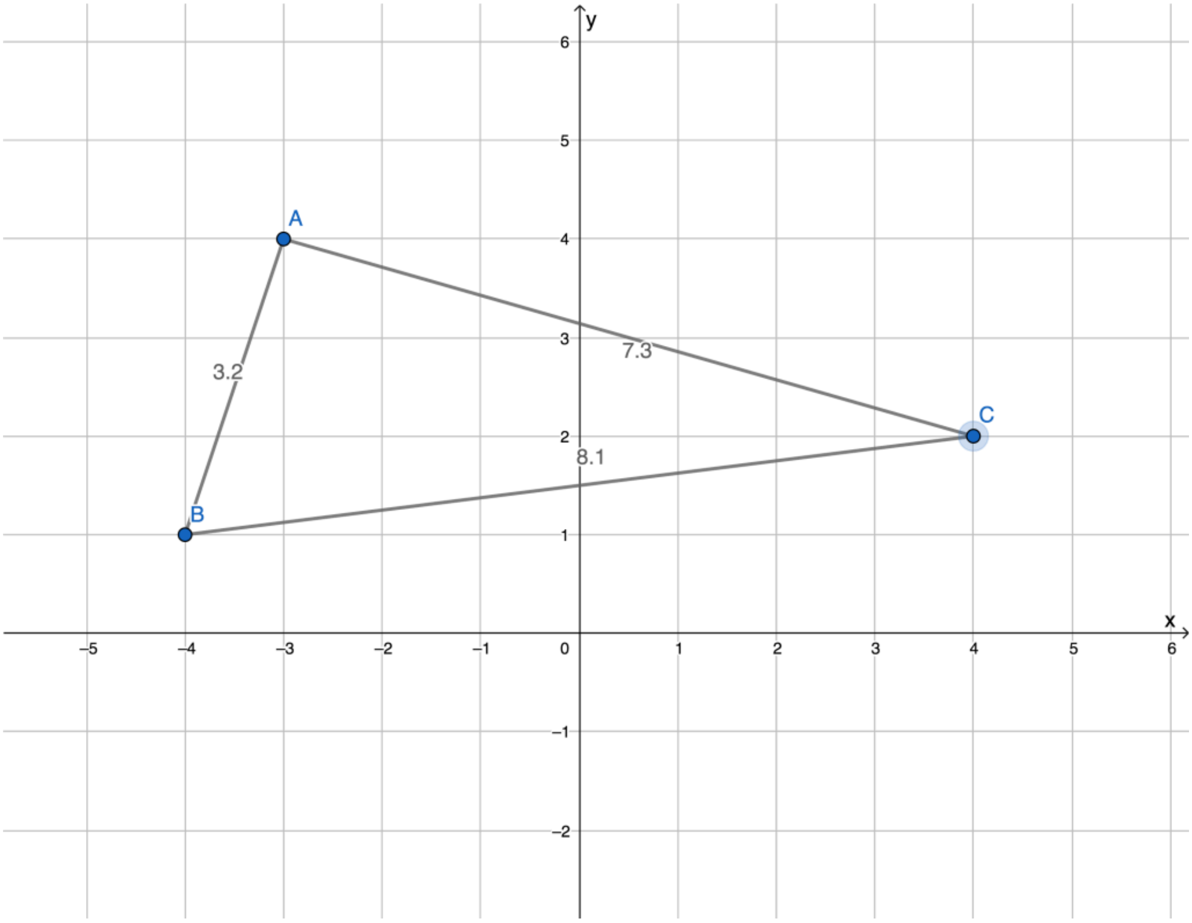
What to do:

1. On your own Cartesian plane, draw ABC such that $A(-3, 4)$, $B(-4, 1)$ and $C(3, 2)$.
2. Calculate the lengths of AB, AC and BC.
3. Use the lengths of the sides and the theorem of Pythagoras to prove that the angle at A is 90°.
4. Calculate the gradients of AB and AC. What do you notice about the gradients of AB and AC?

5. What can we say about the gradients of perpendicular lines?

What you found:

1. Here is a sketch of $\Delta\mathrm{ABC}$.



2.

$$\begin{aligned} d_{\mathrm{AB}} &= \sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \\ &= \sqrt{(-3-(-4))^2+(4-1)^2} \\ &= \sqrt{(1)^2+(3)^2} \\ &= \sqrt{10} \end{aligned}$$

$$\begin{aligned} d_{\mathrm{AC}} &= \sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \\ &= \sqrt{(-3-3)^2+(4-2)^2} \\ &= \sqrt{(-6)^2+(2)^2} \\ &= \sqrt{40} \end{aligned}$$

$$\begin{aligned} d_{\mathrm{BC}} &= \sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \\ &= \sqrt{(-4-3)^2+(1-2)^2} \\ &= \sqrt{(-7)^2+(-1)^2} \\ &= \sqrt{50} \end{aligned}$$

3. If the angle at $\hat{\mathrm{A}} = 90^{\circ}$ then $\mathrm{AB}^2 + \mathrm{AC}^2 = \mathrm{BC}^2$.

$$\begin{aligned} \mathrm{AB}^2 + \mathrm{AC}^2 &= 10 + 40 \\ &= 50 \\ &= \mathrm{BC}^2 \end{aligned}$$

Therefore $\hat{\mathrm{A}} = 90^{\circ}$.

4.

$$\begin{aligned} m_{\text{AB}} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{4-1}{-3-(-4)} \\ &= \frac{3}{1} \\ &= 3 \\ m_{\text{AC}} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{4-2}{-3-3} \\ &= \frac{2}{-6} \\ &= -\frac{1}{3} \end{aligned}$$

We can see that m_{AC} is the negative inverse of m_{AB}. This means that $m_{\text{AB}} \times m_{\text{AC}} = -1$.

5. The product of the gradients of perpendicular lines is -1.

Activity 2.2 showed us that if two lines are perpendicular to each other then, when you multiply their gradients together, the answer is -1.

If $\text{AB} \parallel \text{CD}$ then $m_{\text{AB}} = m_{\text{CD}}$

If $\text{AB} \perp \text{CD}$ then $m_{\text{AB}} \times m_{\text{CD}} = -1$

Example 2.5

Prove that line AB with $\text{A}(0, 2)$ and $\text{B}(2, 6)$ is parallel to line CD with equation $2x = y - 7$.

Solution

First we need to work out the gradient of AB.

$$\begin{aligned} m_{\text{AB}} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{6-2}{2-0} \\ &= \frac{4}{2} \\ &= 2 \end{aligned}$$

Next we need to work out the gradient of the line CD based on the equation of the function given.

$$\begin{aligned} 2x &= y - 7 \\ \therefore y &= 2x + 7 \end{aligned}$$

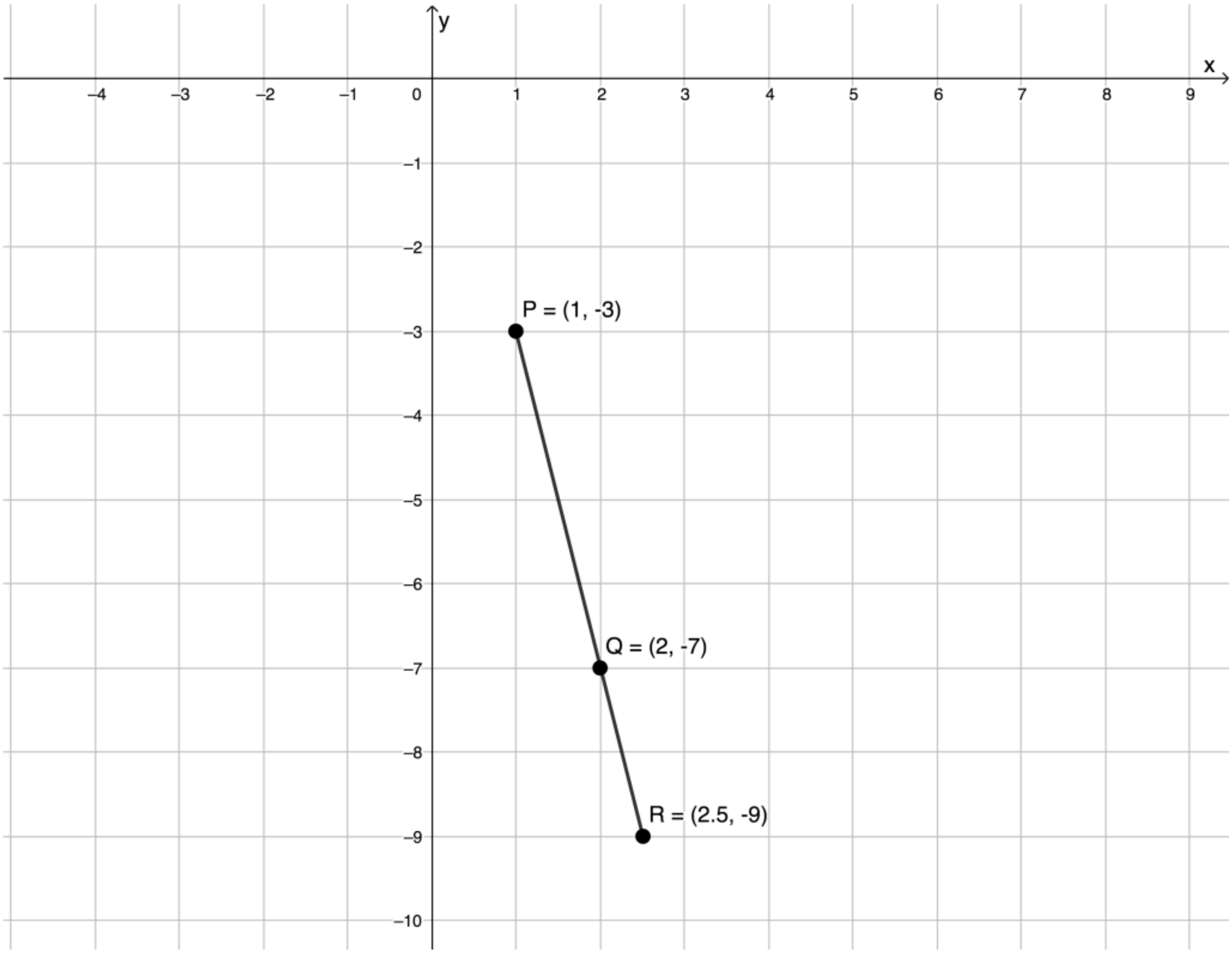
Therefore the gradient of CD is 2. So $m_{\text{AB}} = m_{\text{CD}}$. Therefore the lines are parallel.

Example 2.6

Prove that $\mathrm{Q}(2, -7)$ lies on the line PR where $\mathrm{P}(1, -3)$ and $\mathrm{R}\left(2\frac{1}{2}, -9\right)$.

Solution

If Q lies on PR, then $m_{\mathrm{PQ}} = m_{\mathrm{QR}} = m_{\mathrm{PR}}$.



$$
\begin{aligned}
m_{\mathrm{PQ}} &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{-3 - (-7)}{1 - 2} \\
&= \frac{-3 + 7}{-1} \\
&= -4
\end{aligned}
$$

$$
\begin{aligned}
m_{\mathrm{QR}} &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{-7-(-9)}{2-2\frac{1}{2}} \\
&= \frac{-7+9}{-\frac{1}{2}} \\
&= \frac{2}{1} \times \left(-\frac{2}{1}\right) \\
&= -4
\end{aligned}
$$

$$
\begin{aligned}
m_{\mathrm{PR}} &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{-3-(-9)}{1-2\frac{1}{2}} \\
&= \frac{-3+9}{-\frac{3}{2}} \\
&= \frac{6}{1} \times \left(-\frac{2}{3}\right) \\
&= -\frac{12}{3} = -4
\end{aligned}
$$

Therefore $m_{\mathrm{PQ}} = m_{\mathrm{QR}} = m_{\mathrm{PR}}$ and so Q lies on PR. We say that the points P, Q, and R are **collinear** because they all lie on the same straight line.

Note

Points are collinear if they lie on the same straight line.

Example 2.7

Line KL is perpendicular to line BC. Find a if $\mathrm{B}(2,-3)$, $\mathrm{C}(-2,6)$, $\mathrm{K}(4,3)$ and $\mathrm{L}(7,a)$.

Solution

We are told that line KL is perpendicular to line BC. Therefore we know that $m_{\mathrm{KL}} \times m_{\mathrm{BC}} = -1$.

First we need to calculate m_{BC}.

$$
\begin{aligned}
m_{\mathrm{BC}} &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{-3-6}{2-(-2)} \\
&= \frac{-9}{4}
\end{aligned}
$$

Therefore, we know that $m_{\text{KL}} = \frac{4}{9}$. We can use this to solve for a.

$$
\begin{aligned}
m_{\text{BC}} &= \frac{y_2 - y_1}{x_2 - x_1} \\
\therefore \frac{4}{9} &= \frac{3-a}{4-7} \\
\therefore \frac{4}{9} &= \frac{3-a}{-3} \\
\therefore -12 &= 9(3-a) \\
\therefore -12 &= 27 - 9a \\
\therefore 9a &= 27 + 12 = 39 \\
\therefore a &= \frac{39}{9} = 4\frac{3}{9} = 4\frac{1}{3}
\end{aligned}
$$

Exercise 2.3

1. Determine whether AB and CD are parallel, perpendicular or neither.
   a. $\text{A}(-1,-1), \text{B}(0,-4), \text{C}(3,-4), \text{D}(5,2)$
   b. $\text{A}(-1,3), \text{B}(-2,2), \text{C}(3,-4), \text{D}(5,2)$
2. Do $\text{K}(-6,2)$, $\text{L}(-3,1)$ and $\text{M}(1,-1)$ lie on the same straight line?
3. JK is perpendicular to the line given by $4x - 3y = 9$. Find q if $\text{J}(-3,1)$ and $\text{K}(q,-3)$.

The full solutions are at the end of the unit.

The midpoint of a line

Sometimes it is necessary to find the middle or midpoint of a line segment on the Cartesian plane. Work through Activity 2.3 to find out how to do this.

Activity 2.3: Find the midpoint of a line segment

Time required: 10 minutes

What you need:

- a piece of paper (graph paper if possible)
- a pen or pencil

What to do:

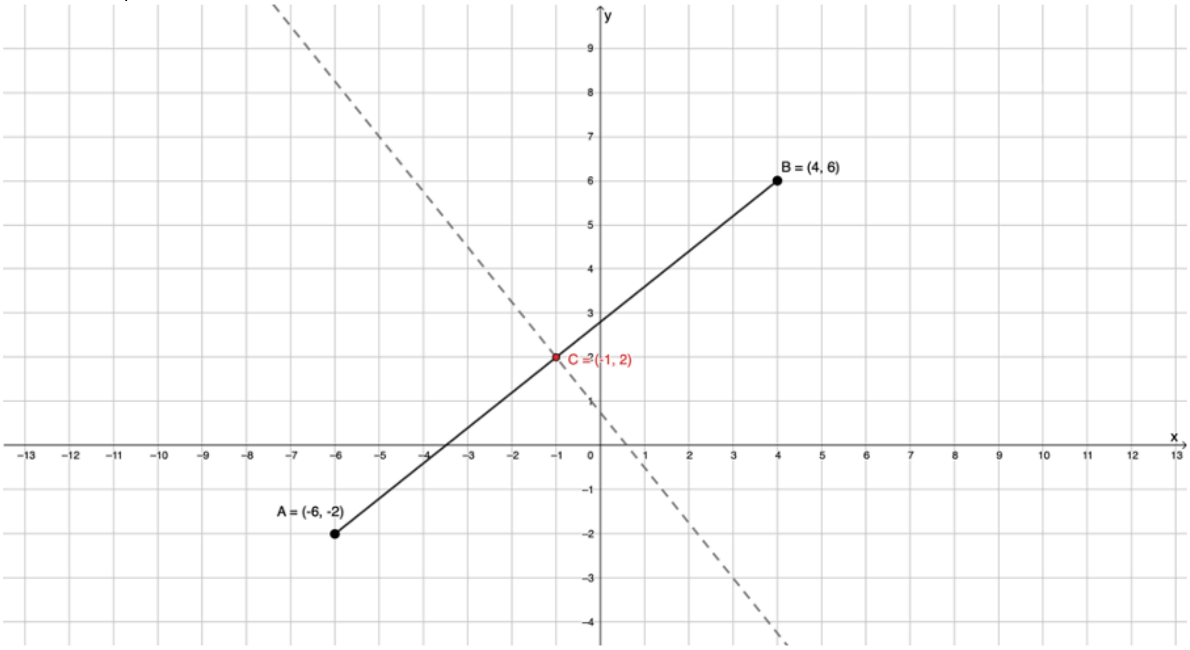
1. On your own Cartesina plane plot the points $\text{A}(-6,-2)$ and $\text{B}(5,7)$ and draw the line AB. Fold your

piece of paper so that points A and B are exactly on top of each other. What are the coordinates of point C, the point where the fold line cuts line AB?

2. What can you say about the position of C in relation to line AB?
3. Write the coordinates of point C in terms of the coordinates of points A and B.
4. Write a general expression for finding the coordinates of the midpoint of a line segment with $A(x_1, y_1)$ and $B(x_2, y_2)$.

What you found:

1. The coordinates of C are $(-1, 2)$. The dotted line in the following graph shows its location on the Cartesian plane.



2. Point C is the middle of the line segment AB.
3. The x-coordinate of C is the sum of the x-coordinates of A and B divided by 2: $\dfrac{-6+4}{2} = -1$

   The y-coordinate of C is the sum of the y-coordinates of A and B divided by 2: $\dfrac{-2+6}{2} = 2$
4. $C\left(\dfrac{x_A + x_B}{2}, \dfrac{y_A + y_B}{2}\right)$
5. The midpoint of AB is given by $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$

The midpoint of a line segemnt is $M(x, y) = \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$. It does not matter which point is made point 1 or point 2.

Example 2.8

Find the midpoint $\mathrm{M}(x, y)$ of ST where $\mathrm{S}(4,0)$ and $\mathrm{T}(-5,6)$. In which quadrant does the midpoint lie?

Solution

$\mathrm{M}(x, y)=\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$. Let S be point 1.

$$\begin{aligned} \mathrm{M}(x, y) &=\left(\frac{4-5}{2}, \frac{0+6}{2}\right) \\ &=\left(-\frac{1}{2}, 3\right) \end{aligned}$$

The midpoint lies in quadrant II.

Example

CD has a midpoint $\mathrm{M}(1,-3)$. Find the point C if $\mathrm{D}(6,5)$.

Solution

In this case we are given the midpoint $\mathrm{M}(x, y)$ and we know that $\mathrm{M}(x, y)=\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$. We can calculate the x- and y-coordinates of C seperately. Let $\mathrm{D}(6,5)$ be point 1.

The x-coordinate of C:

$$\begin{aligned} 1 &=\frac{x_1+x_2}{2} \\ \therefore 1 &=\frac{6+x_2}{2} \\ \therefore 2 &=6+x_2 \\ \therefore x_2 &=-4 \end{aligned}$$

The y-coordinate of C:

$$\begin{aligned} -3 &=\frac{y_1+y_2}{2} \\ \therefore -3 &=\frac{5+y_2}{2} \\ \therefore -6 &=6+y_2 \\ \therefore y_2 &=-12 \end{aligned}$$

Exercise 2.4

1. Find the midpoint of AB where $\mathrm{A}(-4,7)$ and $\mathrm{B}(2,5)$.
2. Find the midpoint of GH where $\mathrm{G}(x+3, y-2)$ and $\mathrm{H}(x-5, y-4)$.
3. The midpoint M of PQ is $(3,9)$. Find Q if $\mathrm{P}(1,1)$.

The full solutions are at the end of the unit.

Summary

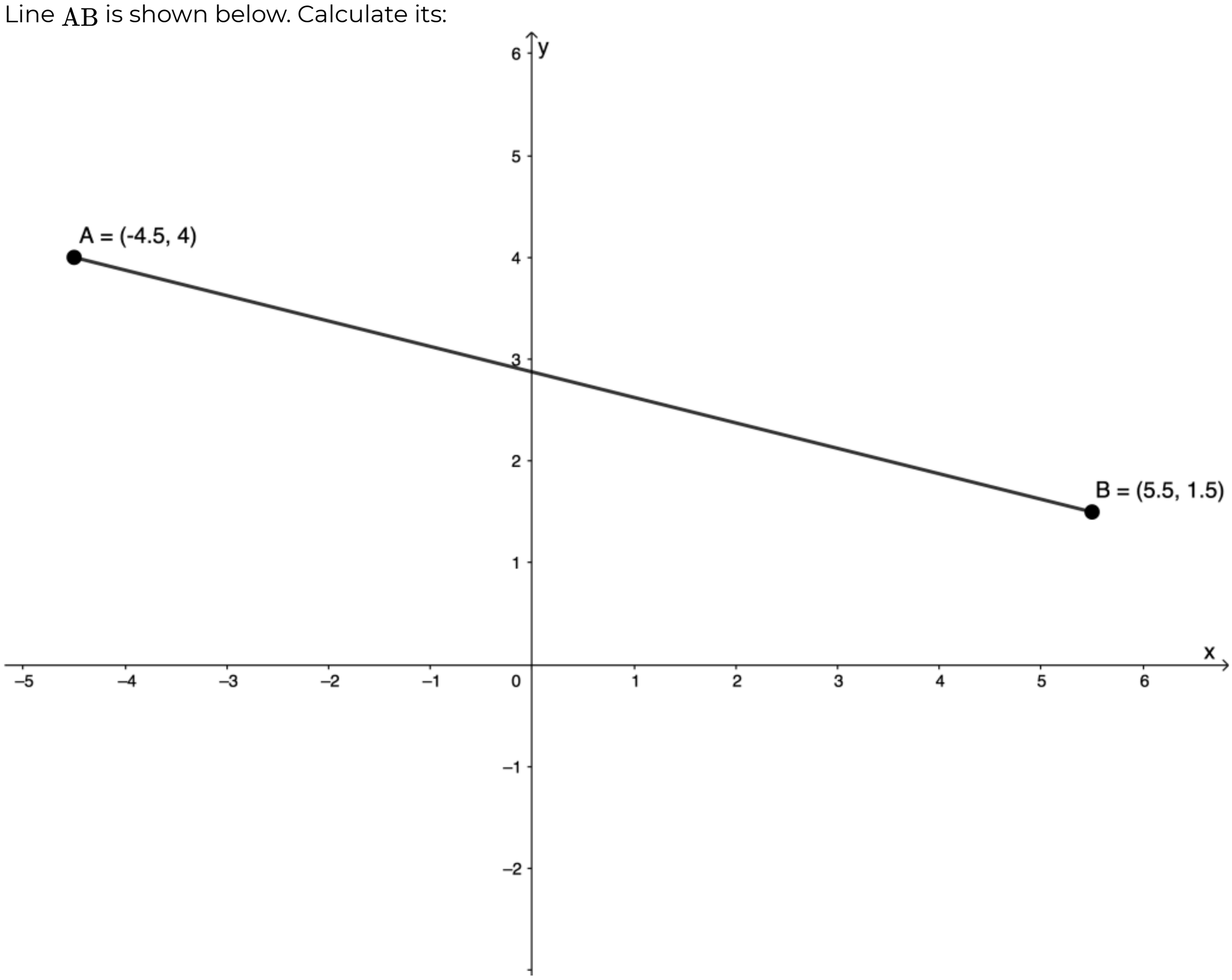
In this unit you have learnt the following:

- How to calculate the distance between two points on the Cartesian plane using the distance formula.
- How to calculate the gradient between any two points on a straight line.
- How to work with the gradients of parallel and perpendicular lines.
- That the gradients of parallel lines are the same.
- That the product of the gradients of perpendicular lines is equal to -1.
- How to calculate the midpoint between two points on the Cartesian plane.

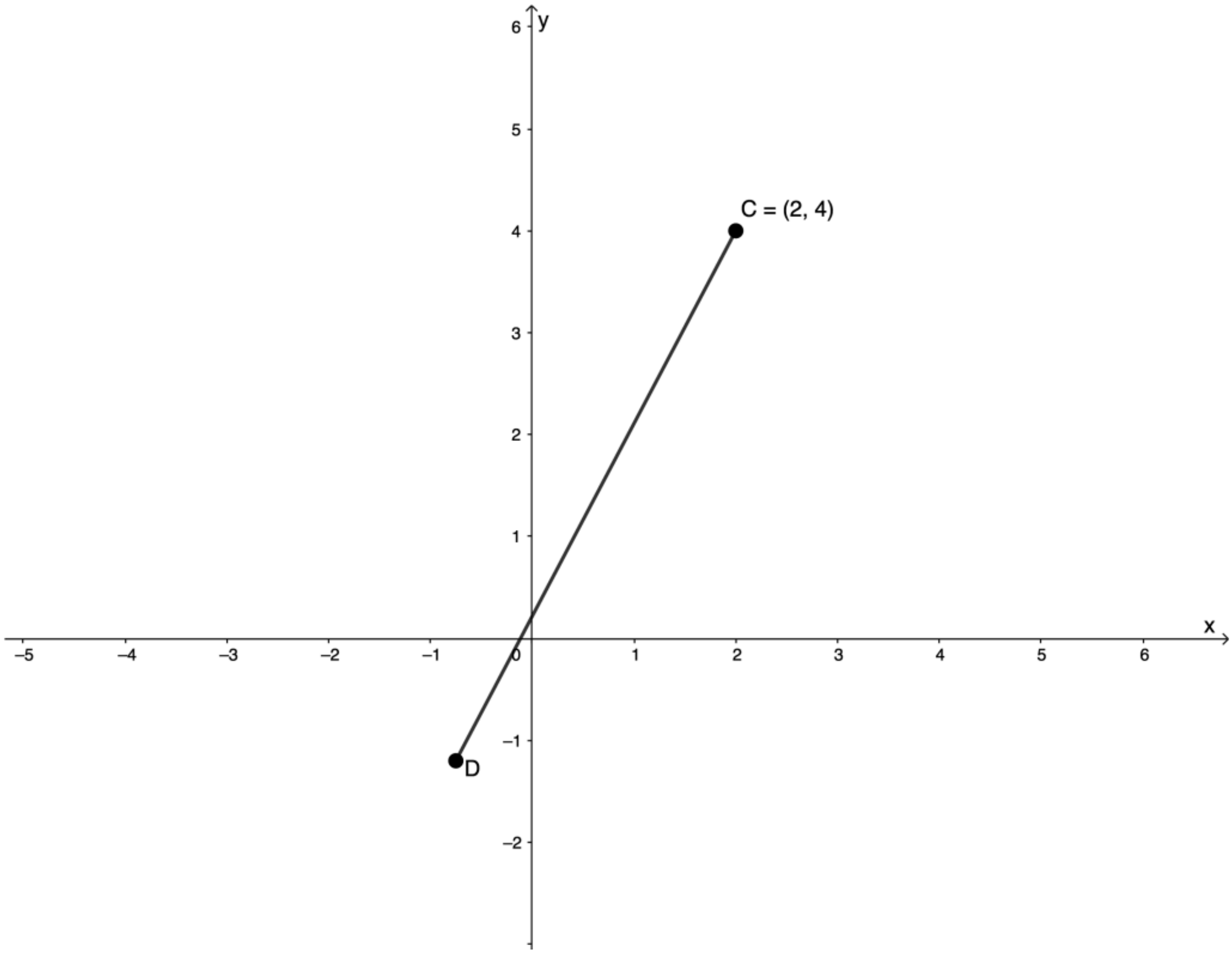
Unit 2: Assessment

Suggested time to complete: 45 minutes

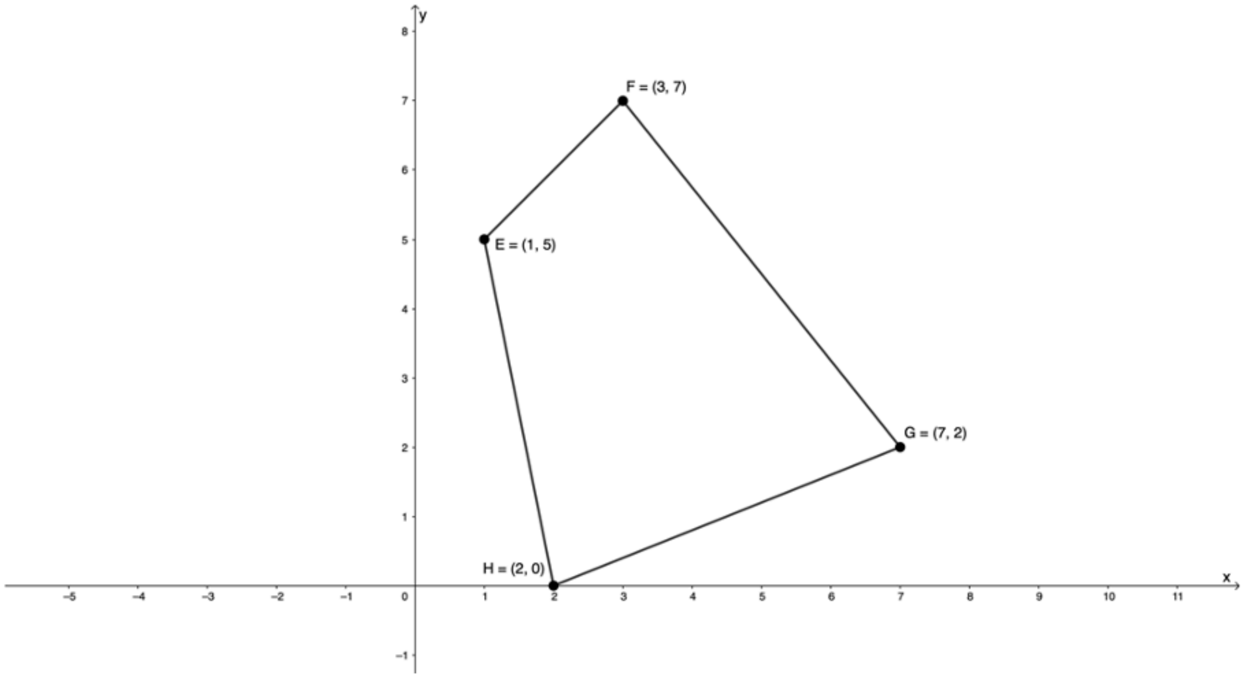
1. Line AB is shown below. Calculate its:



   a. Length (correct to one decimal place)
   b. Gradient
   c. Midpoint
2. If $\mathrm{CD} = 5$, find a if $\mathrm{D}\,(-1, a)$.



3. ABCD is a parellogram where $\mathrm{A}(5,3)$, $\mathrm{B}(2,1)$ and $\mathrm{C}(7,-3)$. Find D.
   HINT: The diagonals of a parallelogram bisect each other (they cut each other in half). Start by drawing a diagram of the information you have been given.
4. The vertices of EFGH are shown below.



   a. Calculate the lengths of the sides of EFGH.
   b. Are the opposite sides of EFGH parallel?
   c. Do the diagonals of EFGH bisect each other?

d. What kind of quadrilateral is EFGH? Give reasons for your answer.
(Question adapted from Everything Maths Grade 10 Exercise 8-6 Question 19)

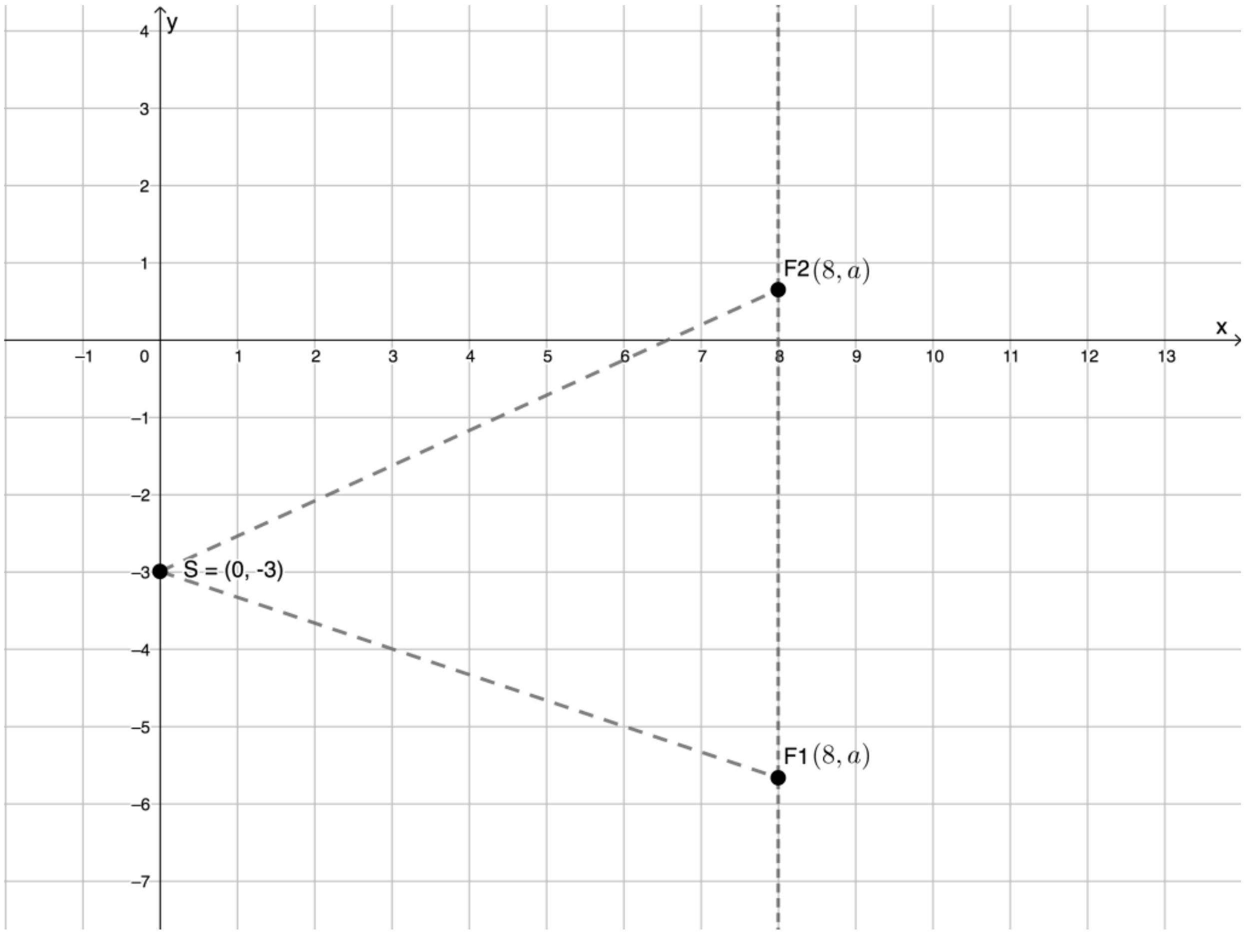
5. *Question 5 adapted from Everything Maths Grade 10 Exercise 8-6 Question 37*
$\mathrm{A}(-2,4)$, $\mathrm{B}(-4,-2)$and $\mathrm{C}(4,0)$ are the vertices of $\Delta\mathrm{ABC}$. D and $\mathrm{E}(1,2)$ are the midpoints of AB and AC respectively.
   a. Find the gradient of BC.
   b. Show that D is the point $(-3,1)$.
   c. Find the length of DE.
   d. Find the gradient of DE. Make a conjecture regarding lines BC and DE.
   e. Determine the equation of BC.

The full solutions are at the end of the unit.

Unit 2: Solutions

Exercise 2.1

1. Let A be (x_1,y_1) and B be (x_2,y_2)
$$\begin{aligned} d_{\mathrm{AB}} &= \sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \\ &= \sqrt{(-6-4)^2+(3-(-2))^2} \\ &= \sqrt{(-10)^2+(5)^2} \\ &= \sqrt{100+25} \\ &= \sqrt{125} \\ &= 11.2 \end{aligned}$$
2. $\mathrm{C}(0,-9)$ be (x_1,y_1) and $\mathrm{D}(2,3)$ be (x_2,y_2)
$$\begin{aligned} d_{\mathrm{CD}} &= \sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \\ &= \sqrt{(0-2)^2+(-9-3)^2} \\ &= \sqrt{(-2)^2+(-12)^2} \\ &= \sqrt{4+144} \\ &= \sqrt{148} = \sqrt{4\times 37} = 2\sqrt{37} \end{aligned}$$
3. Let $\mathrm{Q}(x,y)$ be (x_1,y_1) and $\mathrm{T}(x+4,y-7)$ be (x_2,y_2)
$$\begin{aligned} d_{\mathrm{QT}} &= \sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \\ &= \sqrt{(x-(x+4))^2+(y-(y-7))^2} \\ &= \sqrt{(4)^2+(7)^2} \\ &= \sqrt{16+49} \\ &= \sqrt{65} \end{aligned}$$
4. The distance between $\mathrm{S}(0,-3)$ and $\mathrm{F}(8,a)$ is 10 units.



Let S be (x_1, y_1) and F be (x_2, y_2)

$$\begin{aligned} d_{SF} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(0-8)^2 + (-3-a)^2} \\ &= \sqrt{(-8)^2 + (-3-a)^2} \\ &= \sqrt{64 + (-3-a)^2} \end{aligned}$$

But $d_{SF} = 10$

$$10 = \sqrt{64 + (-3-a)^2}$$

$$\therefore (10)^2 = 64 + (-3-a)^2 \quad \text{Expand } (-3-a)^2$$

$$\therefore 100 = 64 + 9 + 6a + a^2 \quad \text{Solve the quadratic equation}$$

$$\therefore a^2 + 6a - 27 = 0$$

$$\therefore (a+9)(a-3) = 0$$

$$\therefore a = -9 \text{ or } a = 3$$

Back to Exercise 2.1

Exercise 2.2

1.

   a. $G(3, -5)$ and $H(-2, 7)$.

$$\begin{aligned} m_{\mathrm{GH}} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-5-7}{3-(-2)} \\ &= \frac{-12}{5} \end{aligned}$$

b. $\mathrm{G}\left(\frac{3}{2}, 0\right)$ and $\mathrm{H}\left(3, -\frac{3}{4}\right)$.

$$\begin{aligned} m_{\mathrm{GH}} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{0 - \left(-\frac{3}{4}\right)}{\frac{3}{2} - 3} \\ &= \frac{\frac{3}{4}}{-\frac{3}{2}} \\ &= \frac{3}{4} \times \left(-\frac{2}{3}\right) \\ &= -\frac{1}{2} \end{aligned}$$

c. $\mathrm{G}(x-3, y)$ and $\mathrm{H}(x, y-4)$.

$$\begin{aligned} m_{\mathrm{GH}} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{y-(y-4)}{(x-3)-x} \\ &= \frac{-4}{-3} \\ &= \frac{4}{3} \end{aligned}$$

2. $m_{\mathrm{ST}} = \frac{2}{3}$

a. $\mathrm{S}(8, q)$ and $\mathrm{T}(16, 2)$.

$$\begin{aligned} m_{\mathrm{ST}} &= \frac{y_2 - y_1}{x_2 - x_1} \\ \therefore \frac{2}{3} &= \frac{q-2}{8-16} \\ \therefore \frac{2}{3} &= \frac{q-2}{-8} \\ \therefore -16 &= 3(q-2) \\ \therefore -16 &= 3q - 6 \\ \therefore 3q &= -10 \\ \therefore q &= \frac{-10}{3} = -3\frac{1}{3} \end{aligned}$$

b. $\mathrm{S}(3, 2q)$ and $\mathrm{T}(9, 14)$.

$$
\begin{aligned}
m_{\mathrm{ST}} &= \frac{y_2 - y_1}{x_2 - x_1}\\
\therefore \frac{2}{3} &= \frac{2q - 14}{3 - 9}\\
\therefore \frac{2}{3} &= \frac{2q - 14}{-6}\\
\therefore -12 &= 3(2q - 14)\\
\therefore -12 &= 6q - 42\\
\therefore 6q &= 30\\
\therefore q &= \frac{30}{6} = 5
\end{aligned}
$$

Back to Exercise 2.2

Exercise 2.3

1.

a. $\mathrm{A}(-1,-1), \mathrm{B}(0,-4), \mathrm{C}(3,-4), \mathrm{D}(5,2)$

$$
\begin{aligned}
m_{\mathrm{AB}} &= \frac{y_2 - y_1}{x_2 - x_1}\\
&= \frac{-1 - (-4)}{-1 - 0}\\
&= \frac{-3}{-1}\\
&= 3\\
m_{\mathrm{CD}} &= \frac{y_2 - y_1}{x_2 - x_1}\\
&= \frac{-4 - 2}{3 - 5}\\
&= \frac{-6}{-2}\\
&= 3
\end{aligned}
$$

Therefore AB and CD are parallel.

b. $\mathrm{A}(-1,3), \mathrm{B}(-2,2), \mathrm{C}(3,-4), \mathrm{D}(5,2)$

$$
\begin{aligned}
m_{\mathrm{AB}} &= \frac{y_2 - y_1}{x_2 - x_1}\\
&= \frac{3 - 2}{-1 - (-2)}\\
&= \frac{1}{1}\\
&= 1\\
m_{\mathrm{CD}} &= \frac{y_2 - y_1}{x_2 - x_1}\\
&= \frac{-4 - 2}{3 - 5}\\
&= \frac{-6}{-2}\\
&= 3
\end{aligned}
$$

Therefore AB and CD are neither parallel nor perpendicular.

2. $\mathrm{K}(-6,2), \mathrm{L}(-3,1)$ and $\mathrm{M}(1,-1)$

$$
\begin{aligned}
m_{\mathrm{KL}} &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{2-1}{-6-(-3)} \\
&= \frac{1}{-3} \\
&= -\frac{1}{3} \\
m_{\mathrm{LM}} &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{1-(-1)}{-3-1} \\
&= \frac{2}{-4} \\
&= -\frac{1}{2}
\end{aligned}
$$

$m_{\mathrm{KL}} \neq m_{\mathrm{LM}}$. Therefore $\mathrm{K}(-6,2)$, $\mathrm{L}(-3,1)$ and $\mathrm{M}(1,-1)$ do not lie on the same straight line.

3. Gradient of the line $4x - 3y = 9$:

$$
\begin{aligned}
4x - 3y &= 9 \\
\therefore -3y &= -4x - 9 \\
\therefore y &= \frac{4}{3}x + 3
\end{aligned}
$$

Therefore $m_{\mathrm{JK}} = -\frac{3}{4}$

$$
\begin{aligned}
m_{\mathrm{JK}} &= -\frac{3}{4} = \frac{y_2 - y_1}{x_2 - x_1} \\
\therefore -\frac{3}{4} &= \frac{1-(-3)}{-3-q} \\
&= \frac{4}{-3-q} \\
\therefore -\frac{3}{4} &= \frac{4}{-3-q} \\
\therefore -3(-3-q) &= 16 \\
\therefore 9 + 3q &= 16 \\
\therefore 3q &= 7 \\
\therefore q &= \frac{7}{3}
\end{aligned}
$$

Back to Exercise 2.3

Exercise 2.4

1. $\mathrm{A}(-4,7)$ and $\mathrm{B}(2,5)$

$$
\begin{aligned}
\mathrm{M}(x,y) &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\
&= \left(\frac{-4+2}{2}, \frac{7+5}{2}\right) \\
&= \left(\frac{-2}{2}, \frac{12}{2}\right) \\
&= (-1,6)
\end{aligned}
$$

2. $\mathrm{G}(x+3, y-2)$ and $\mathrm{H}(x-5, y-4)$

$$\begin{aligned}
M(x,y) &= \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) \\
&= \left(\frac{(x+3)+(x-5)}{2}, \frac{(y-2)+(y-4)}{2}\right) \\
&= \left(\frac{2x-2}{2}, \frac{2y-6}{2}\right) \\
&= \left(\frac{2(x-1)}{2}, \frac{2(y-3)}{2}\right) \\
&= (x-1, y-3)
\end{aligned}$$

3. The midpoint M of PQ is $(3,9)$. Find Q if $P(1,1)$.
   Let $P(1,1)$ be point 1.
   x-coordinate:

$$\begin{aligned}
3 &= \frac{x_1+x_2}{2} \\
\therefore 3 &= \frac{1+x_2}{2} \\
\therefore 6 &= 1+x_2 \\
\therefore x_2 &= 5
\end{aligned}$$

   y-coordinate:

$$\begin{aligned}
9 &= \frac{y_1+y_2}{2} \\
\therefore 9 &= \frac{1+y_2}{2} \\
\therefore 18 &= 1+y_2 \\
\therefore y_2 &= 17
\end{aligned}$$

   Q is the point $(5,17)$.

Back to Exercise 2.4

Unit 2: Assessment

1. $A(-4.5,4), B(5.5,1.5)$
   a.

$$\begin{aligned}
d_{AB} &= \sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \\
&= \sqrt{(-4.5-5.5)^2+(4-1.5)^2} \\
&= \sqrt{(-10)^2+(2.5)^2} \\
&= \sqrt{100+6.25} \\
&= \sqrt{106.25} \\
&= 10.3
\end{aligned}$$

   b.

$$\begin{aligned}
m_{AB} &= \frac{y_2-y_1}{x_1-x_1} \\
&= \frac{1.5-4}{5.5-4.5} \\
&= \frac{-2.5}{1} \\
&= -2\frac{1}{2} = -\frac{5}{2}
\end{aligned}$$

c.

$$\begin{aligned}\mathrm{M}(x,y) &= \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)\\ &= \left(\frac{-4.5+5.5}{2}, \frac{4+1.5}{2}\right)\\ &= \left(\frac{1}{2}, \frac{5.5}{2}\right)\\ &= \left(\frac{1}{2}, \frac{\frac{11}{2}}{2}\right)\\ &= \left(\frac{1}{2}, \frac{11}{4}\right)\end{aligned}$$

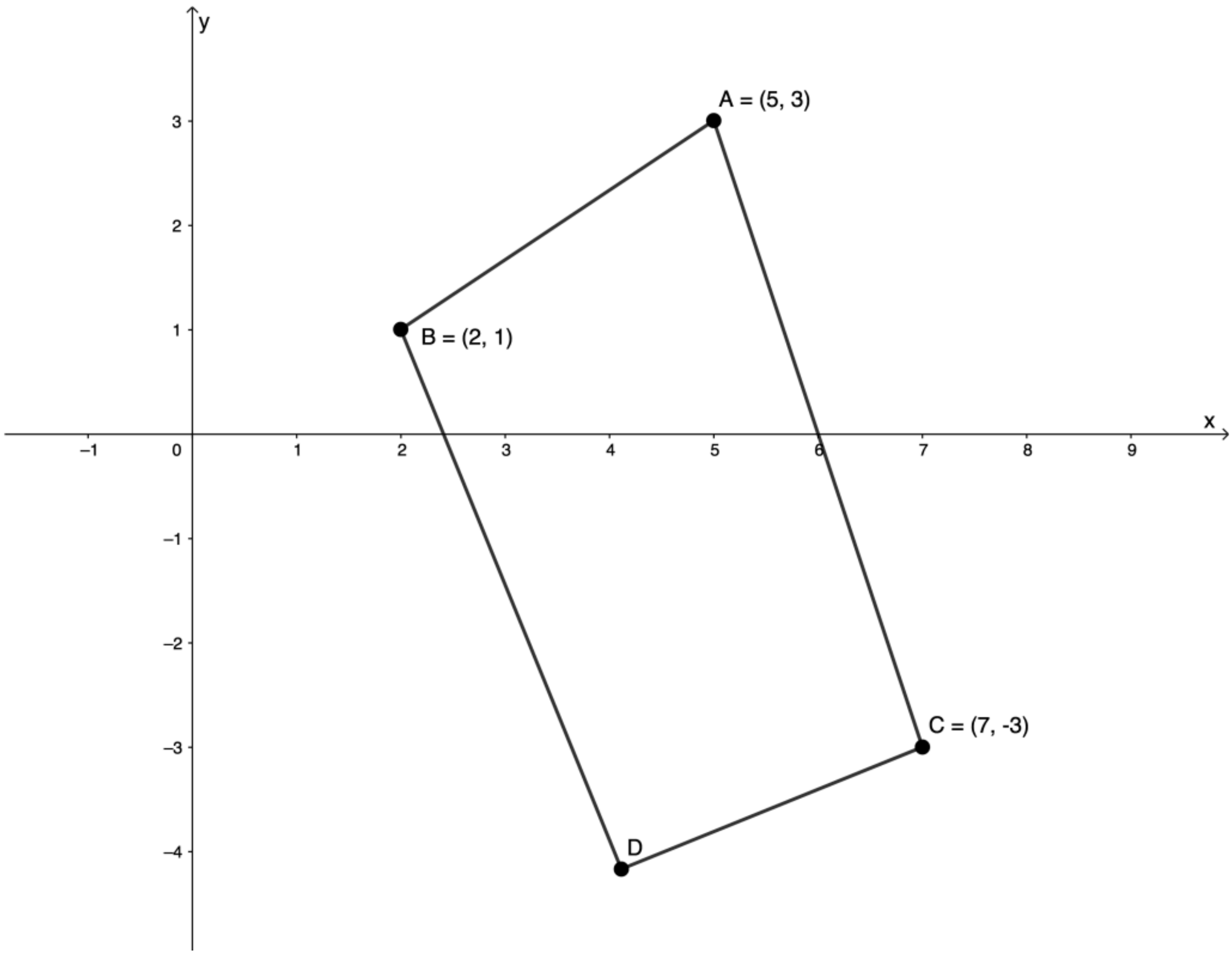
2. $\mathrm{CD} = 5$, $\mathrm{C}(2,4)$, $\mathrm{D}\,(-1,a)$.

$$\begin{aligned} d_{\mathrm{CD}} &= \sqrt{(x_1-x_2)^2+(y_1-y_2)^2} = 5\\ \therefore 5 &= \sqrt{(2-(-1))^2+(4-a)^2}\\ &= \sqrt{3^2+(4-a)^2}\\ &= \sqrt{9+(4-a)^2}\\ \therefore 25 &= 9+(4-a)^2\\ \therefore 25 &= 9+16-8a+a^2\end{aligned}$$

$$\therefore a^2-8a=0$$
$$\therefore a(a-8)=0$$
$$\therefore a=0 \text{ or } a=8$$

We are told that D has an x-coordinate of -1. Therefore $a=0$. $a \neq 8$ in this situation as this would mean that D has an x-coordinate greater than 2.

3. ABCD is a parellogram where $\mathrm{A}(5,3)$, $\mathrm{B}(2,1)$ and $\mathrm{C}(7,-3)$.
Here is a diagram of the information given.



We know that the diagonals of a parallelogram bisect each other. Therefore, the midpoint of AD will be the same point as the midpoint of BC.Midpoint BC

$$\begin{aligned}\mathrm{M_{BC}} &= \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)\\ &= \left(\frac{2+7}{2}, \frac{1-3}{2}\right)\\ &= \left(\frac{9}{2}, \frac{-2}{2}\right)\\ &= (4.5, -1)\end{aligned}$$

$\mathrm{M_{AD}} = (4.5, -1)$x-coordinate of $\mathrm{D}(x, y)$:

$$\begin{aligned}4.5 &= \frac{5+x}{2}\\ \therefore 9 &= 5+x\\ \therefore x &= 4\end{aligned}$$

y-coordinate of $\mathrm{D}(x, y)$:

$$\begin{aligned}-1 &= \frac{3+y}{2}\\ \therefore -2 &= 3+y\\ \therefore y &= -5\end{aligned}$$

The coordinates of $\mathrm{D}(x, y)$ are $(4, -5)$.

4.

a.

$$
\begin{aligned}
d_{\mathrm{EF}} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
&= \sqrt{(1-3)^2 + (5-7)^2} \\
&= \sqrt{(-2)^2 + (-2)^2} \\
&= \sqrt{4+4} \\
&= \sqrt{8} \\
&= 2\sqrt{2} \\
d_{\mathrm{FG}} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
&= \sqrt{(3-7)^2 + (7-2)^2} \\
&= \sqrt{(-4)^2 + (-5)^2} \\
&= \sqrt{16+25} \\
&= \sqrt{41} \\
d_{\mathrm{GH}} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
&= \sqrt{(7-2)^2 + (2-0)^2} \\
&= \sqrt{(5)^2 + (2)^2} \\
&= \sqrt{25+4} \\
&= \sqrt{29} \\
d_{\mathrm{HE}} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
&= \sqrt{(2-1)^2 + (0-5)^2} \\
&= \sqrt{(1)^2 + (-5)^2} \\
&= \sqrt{1+25} \\
&= \sqrt{26}
\end{aligned}
$$

b.

$$
\begin{aligned}
m_{\mathrm{EF}} &= \frac{y_2 - y_1}{x_1 - x_1} \\
&= \frac{7-5}{3-1} \\
&= \frac{2}{2} \\
&= 1 \\
m_{\mathrm{FG}} &= \frac{y_2 - y_1}{x_1 - x_1} \\
&= \frac{2-7}{7-3} \\
&= \frac{-5}{4} \\
m_{\mathrm{GH}} &= \frac{y_2 - y_1}{x_1 - x_1} \\
&= \frac{0-2}{2-7} \\
&= \frac{-2}{-5} \\
&= \frac{2}{5}
\end{aligned}
$$

$$\begin{aligned} m_{\mathrm{HE}} &= \frac{y_2 - y_1}{x_1 - x_1} \\ &= \frac{5-0}{1-2} \\ &= \frac{5}{-1} \\ &= -5 \end{aligned}$$

c.

$$\begin{aligned} \mathrm{M}_{\mathrm{EG}} &= \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) \\ &= \left(\frac{1+7}{2}, \frac{5+2}{2}\right) \\ &= \left(\frac{8}{2}, \frac{7}{2}\right) \\ &= (4, 3.5) \\ \mathrm{M}_{\mathrm{FH}} &= \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) \\ &= \left(\frac{3+2}{2}, \frac{7+0}{2}\right) \\ &= \left(\frac{5}{2}, \frac{7}{2}\right) \\ &= (2.5, 3.5) \end{aligned}$$

The diagonals do not bisect each other.

d. EFGH is a quadrilateral. It does not have any equal or parallel sides nor do its diagonals bisect each other.

5. $\mathrm{A}(-2,4)$, $\mathrm{B}(-4,-2)$and $\mathrm{C}(4,0)$ are the vertices of $\Delta\mathrm{ABC}$. D and $\mathrm{E}(1,2)$ are the midpoints of AB and AC respectively.

a.

$$\begin{aligned} m_{\mathrm{BC}} &= \frac{y_2 - y_1}{x_1 - x_1} \\ &= \frac{0-(-2)}{4-(-4)} \\ &= \frac{2}{8} \\ &= \frac{1}{4} \end{aligned}$$

b.

$$\begin{aligned} \mathrm{M}_{\mathrm{AB}} &= \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) \\ &= \left(\frac{-2-4}{2}, \frac{4-2}{2}\right) \\ &= \left(\frac{-6}{2}, \frac{2}{2}\right) \\ &= (-3, 1) \end{aligned}$$

D is the midpoint of AB. Therefore $\mathrm{D}(-3,1)$.

c.

$$\begin{aligned} d_{\mathrm{DE}} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(-3-1)^2 + (1-2)^2} \\ &= \sqrt{(-4)^2 + (-1)^2} \\ &= \sqrt{16+1} \\ &= \sqrt{17} \end{aligned}$$

d.

$$\begin{aligned} m_{\mathrm{DE}} &= \frac{y_2 - y_1}{x_1 - x_1} \\ &= \frac{2-1}{1-(-3)} \\ &= \frac{1}{4} \end{aligned}$$

$m_{\mathrm{BC}} = m_{\mathrm{DE}}$ and D and E are the midpoints of AB and AC respectively. Therefore, the length of DE is half the length of AB.

e.

$m_{\mathrm{BC}} = \frac{1}{4}$. Therefore, the equation of BC is of the form $y = \frac{1}{4}x + c$. But $\mathrm{C}(4,0)$ is a point on BC. Substitute these coordinates into the equation to solve for c.

$$\begin{aligned} 0 &= \frac{1}{4}(4) + c \\ \therefore 0 &= 1 + c \\ \therefore c &= -1 \end{aligned}$$

Therefore, the equation of BC is $y = \frac{1}{4}x - 1$.

Back to Unit 2: Assessment

Media Attributions

SUBJECT OUTCOME IX

SPACE, SHAPE AND MEASUREMENT: USE AND APPLY TRANSFORMATIONS TO PLOT CO-ORDINATES

Subject outcome 3.4

Use the Cartesian co-ordinate system to derive and apply equations

Learning outcomes

- Use transformation geometry to translate p units horizontally and q units vertically.
- Use transformation geometry to reflect graphs about the x-axis, the y-axis and about the lines $y = x$ and $y = -x$.

Unit outcomes: Unit 1: Transformation geometry

By the end of this unit you will be able to:

- Find the coordinates of the point (x, y) after it is translated p units horizontally and q units vertically.
- Find the coordinates of the point (x, y) after it is reflected about the x-axis, the y-axis, and the lines $y = x$ and $y = -x$.

Unit 1: Transformation geometry

DYLAN BUSA

Unit outcomes: Unit 1: Transformation geometry

By the end of this unit you will be able to:

- Find the coordinates of the point (x, y) after it is translated p units horizontally and q units vertically.
- Find the coordinates of the point (x, y) after it is reflected about the x-axis, the y-axis, and the lines $y = x$and $y = -x$.

What you should know

Before you start this unit, make sure you can:

- Plot points on the Cartesian plane. Review Unit 1 of Subject outcome 3.3 if you need help with this.

Introduction

In Unit 1 of Subject outcome 3.3 we learnt how to plot points and draw polygons on the Cartesian plane. The same process of plotting points on a plane very similar to the Cartesian plane is used by computer animators to create computer generated imagery (CGI) animated movies. Everything we see on screen is actually just thousands of carefully drawn polygons (mostly triangles).

Figure 1: An example of a computer generated image (CGI) used in animation

But, these are movies. The polygons are not static. They move around the screen. Instead of drawing the polygons required for each frame, animators instead write rules for how a computer must change or **transform** each shape from frame to frame.

Although these are sophisticated transformations, they all fall into four broad categories:

- **Translation** – moving a point or a shape from one place to another.
- **Reflection** – creating a mirror image of a point or a shape about a given 'mirror' line or line of symmetry.
- **Scaling** – making a shape bigger (or smaller) without changing its shape.
- **Rotation** – turning a shape around a centre point by a given number of degrees.

In this unit, we are only going to explore translations and reflections. But before we do, spend some time exploring all four different types of transformations. To do this, you will need an internet connection. Visit the interactive simulation called Types of Transformations.

Here you will be able to explore rotations, reflections, enlargements and translations. In each case, click and drag the blue points to alter the shape being transformed and see how the shape is being transformed.

Note

To see each of these transformations in action, watch the video called "Rigid transformation intro".

Rigid transformation intro (Duration: 07.22)

Translations

Translations are probably the simplest type of transformation we can do. We can translate (move) individual points or entire polygons or functions. When we translate a polygon or function, we are really just moving each point that makes up or defines the polygon or function.

Look at Figure 2. Here we have ΔABC and the **image** of this triangle after translation which we call $\Delta A'B'C'$ (A dash, B dash, C dash). We can see that the whole triangle has been translated **vertically** 1 unit **up**. We can also say that each of the vertices of the triangle has been translated vertically 1 unit **up**. The y-coordinate of each point has changed. We have **added** 1 to each y-coordinate.

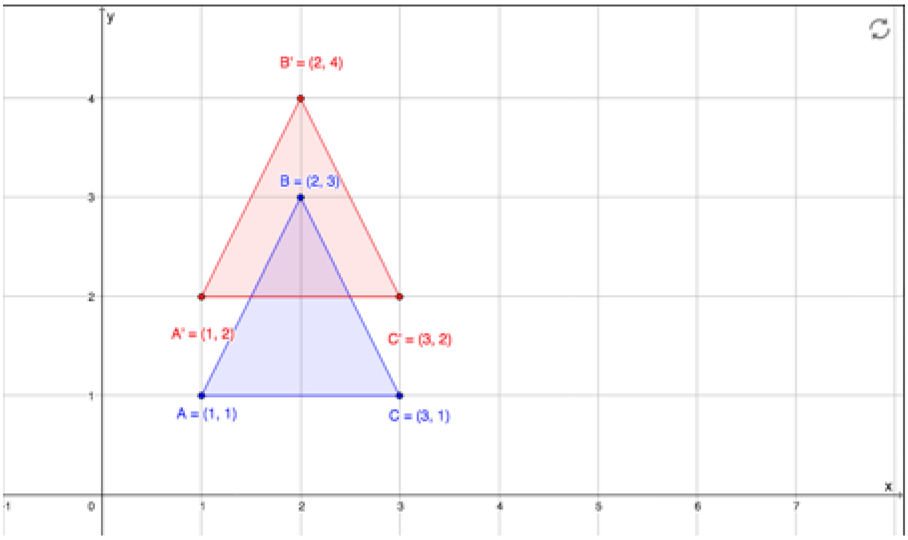


Figure 2: Translation of ΔABC 1 *unit up*

We can also translate ΔABC **horizontally**, for example 2 units to the left, as show in Figure 3.

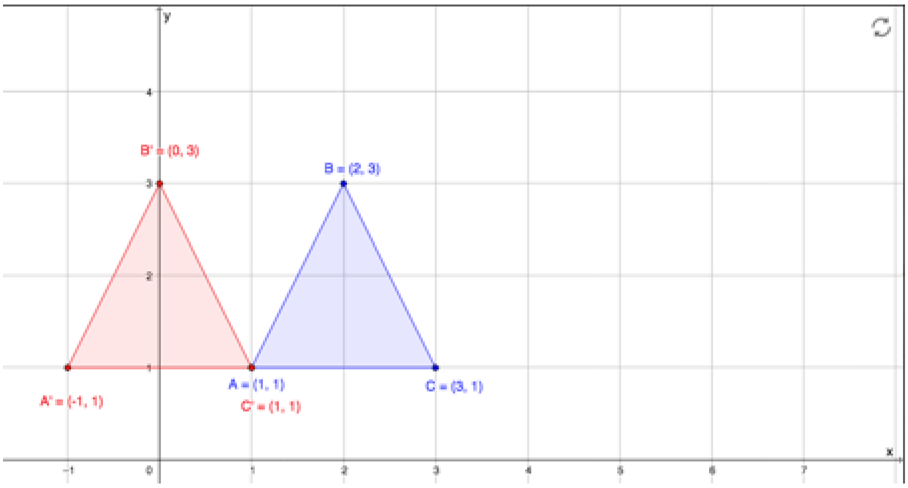


Figure 3: Translation of ΔABC 2 *units to the left*

In this case, we have subtracted 2 from the x-coordinate of each vertex.

Naturally, we can translate a point, shape or function both vertically and horizontally at the same time. How has ΔABC been translated in Figure 4?

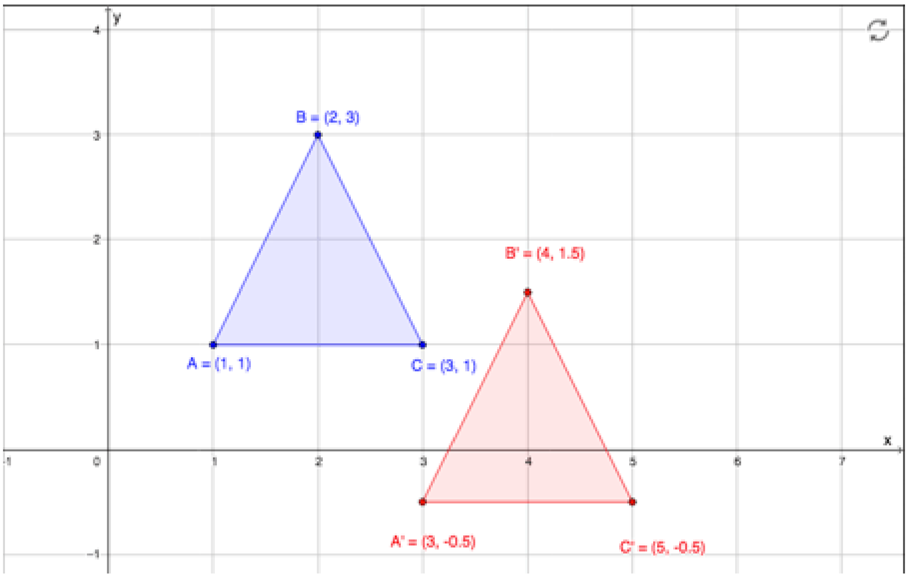


Figure 4: Translation of $\Delta\mathrm{ABC}$ 2 *units to the right and* 1.5 *units down*

If you have an internet connection, visit the interactive simulation called Translations to practise translating a triangle around the Cartesian plane.

You can click and drag each vertex of the blue triangle to change its shape.

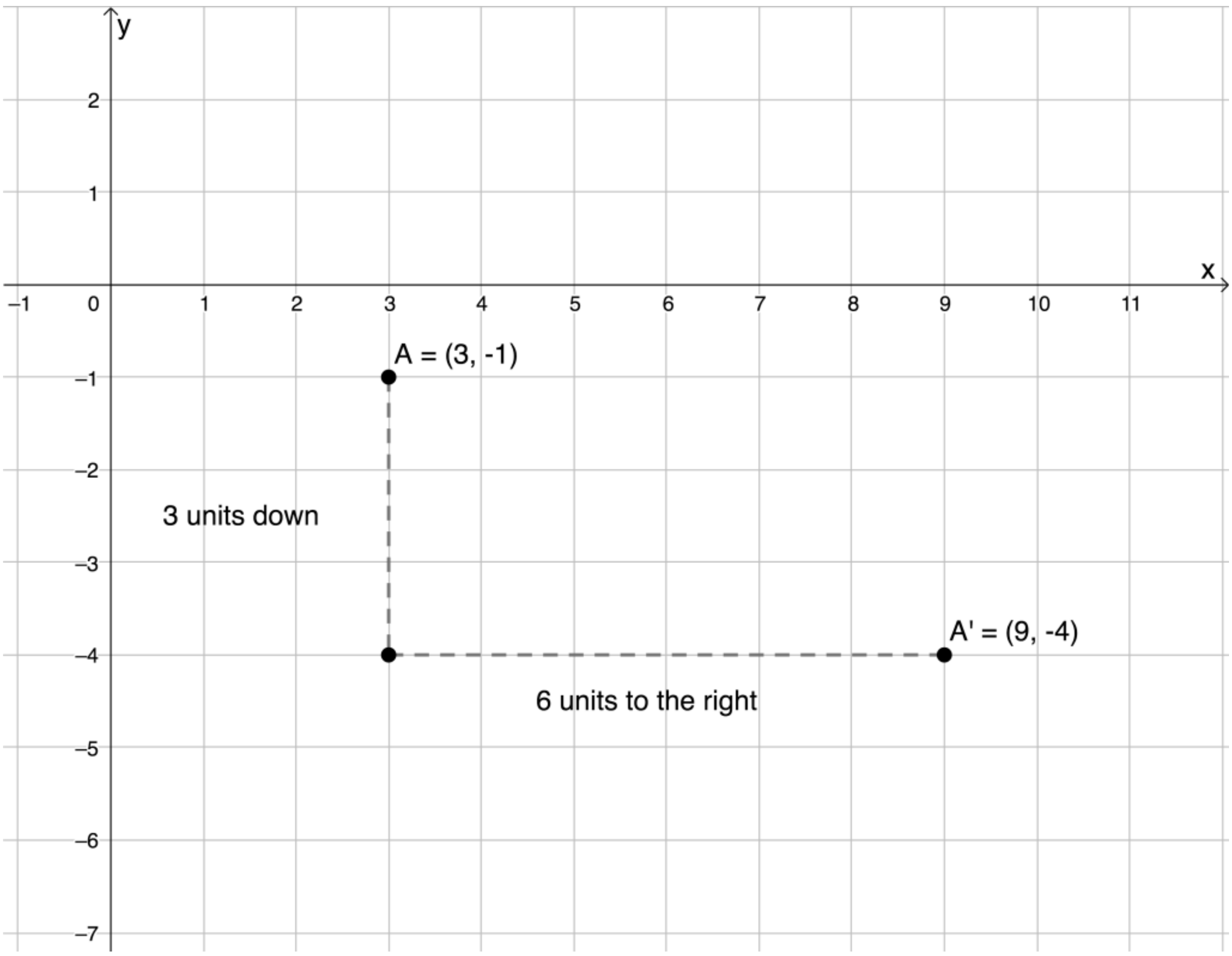
Example 1.1

What are the coordinates of A' if $\mathrm{A}(3, -1)$ is translated 3 units down and 6 units to the right?

Solution

If $\mathrm{A}(3, -1)$ is translated 3 units **down**, we need to **subtract** 3 from the y-coordinate of A. This coordinate will become $-1 - 3 = -4$.

If $\mathrm{A}(3, -1)$ is translated 6 units to the **right**, we need to **add** 6 to the x-coordinate of A. This coordinate will become $3 + 6 = 9$.



Therefore A' will have coordinates $(9, -4)$.

Note

Any vertical (up or down) movement of a point is reflected as a change in the point's y-coordinate. As a point is moved up, its y-coordinate gets bigger.

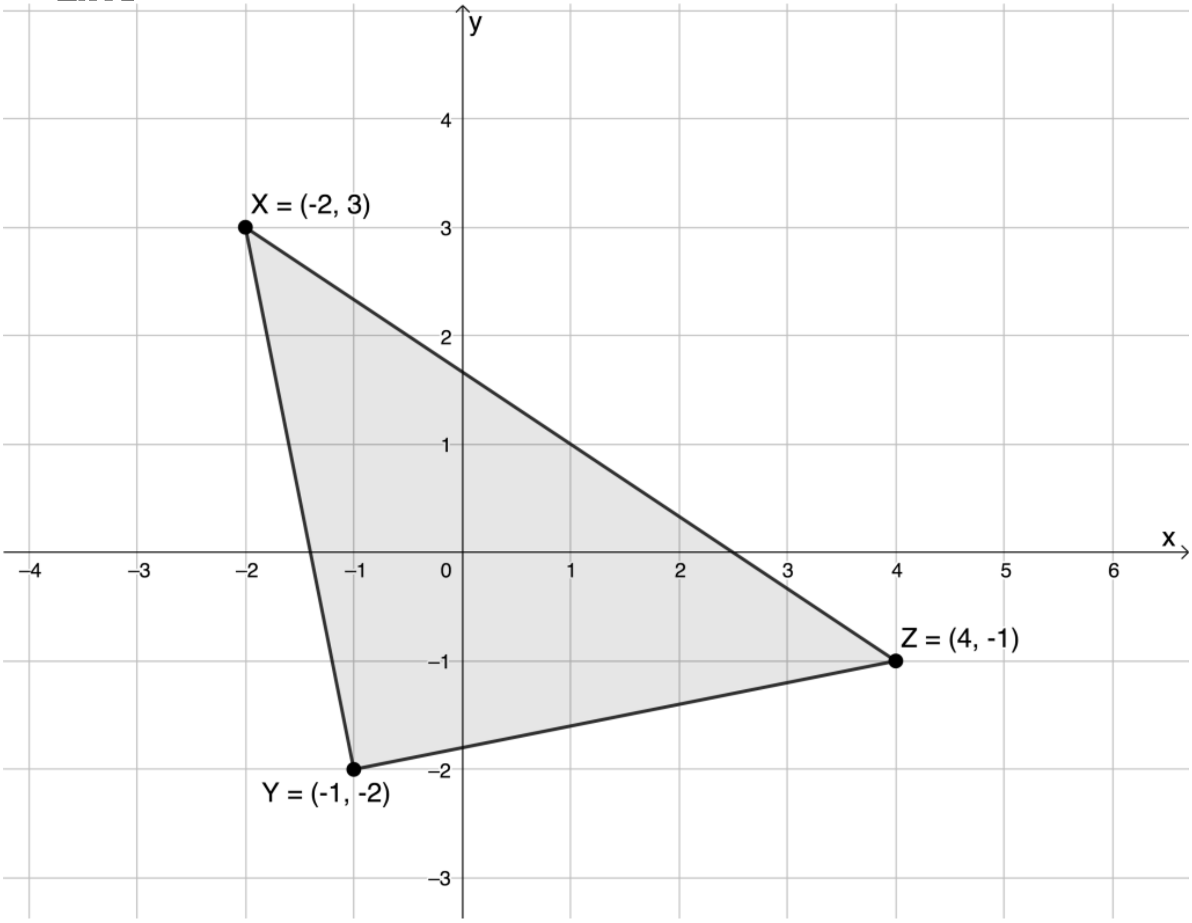
Any horizontal (left or right) movement of a point is reflected as a change in the point's x-coordinate. As a point moves to the right, its x-coordinate gets bigger.

Exercise 1.1

1. Write down the coordinates of Q' if $Q(-1.5, 5)$ is translated:
   1. 2 units to the left.
   2. 8 units up and 5 units to the right.
   3. 3 units to the right and 3 units down.

4. p units to the left and q units up.

2. In the following diagram $\Delta\mathrm{XYZ}$ has vertices as indicated. On the same set of axes, sketch $\Delta\mathrm{XYZ}$ and $\Delta\mathrm{X'Y'Z'}$ if $\Delta\mathrm{XYZ}$ is translated 2 units to the right and 6 units down.



3. If $\mathrm{G}'(-6,3)$, what are the coordinates of G if G was translated 3 units down and 2 units to the right.

The full solutions are at the end of the unit.

- To translate a point p units to the right, add p to the x-coordinate.
- To translate a point p units to the left, subtract p from the x-coordinate.
- To translate a point q units up, add q to the y-coordinate.
- To translate a point p units down, subtract q from the y-coordinate.

Reflections

We all know what a reflection is. We see one every time we look in a mirror. We can use the same principle to reflect points, shapes and functions across different lines (or mirrors) on the Cartesian plane.

In this unit we will only look at reflections made across the following lines:

- The y-axis (the line $x = 0$)
- The x-axis (the line $y = 0$)

- The line $y = x$
- The line $y = -x$

Activity 1.1: Reflect across the axes

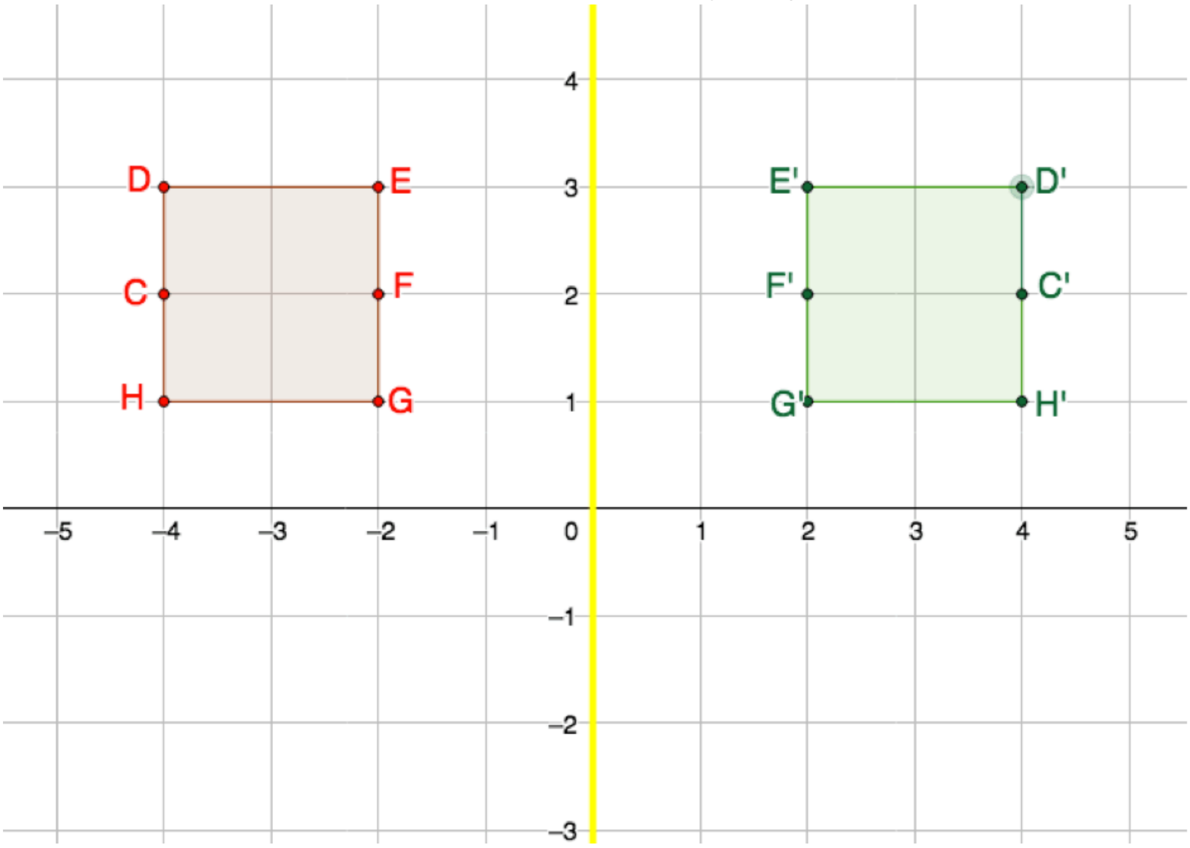
Time required: 10 minutes

What you need:

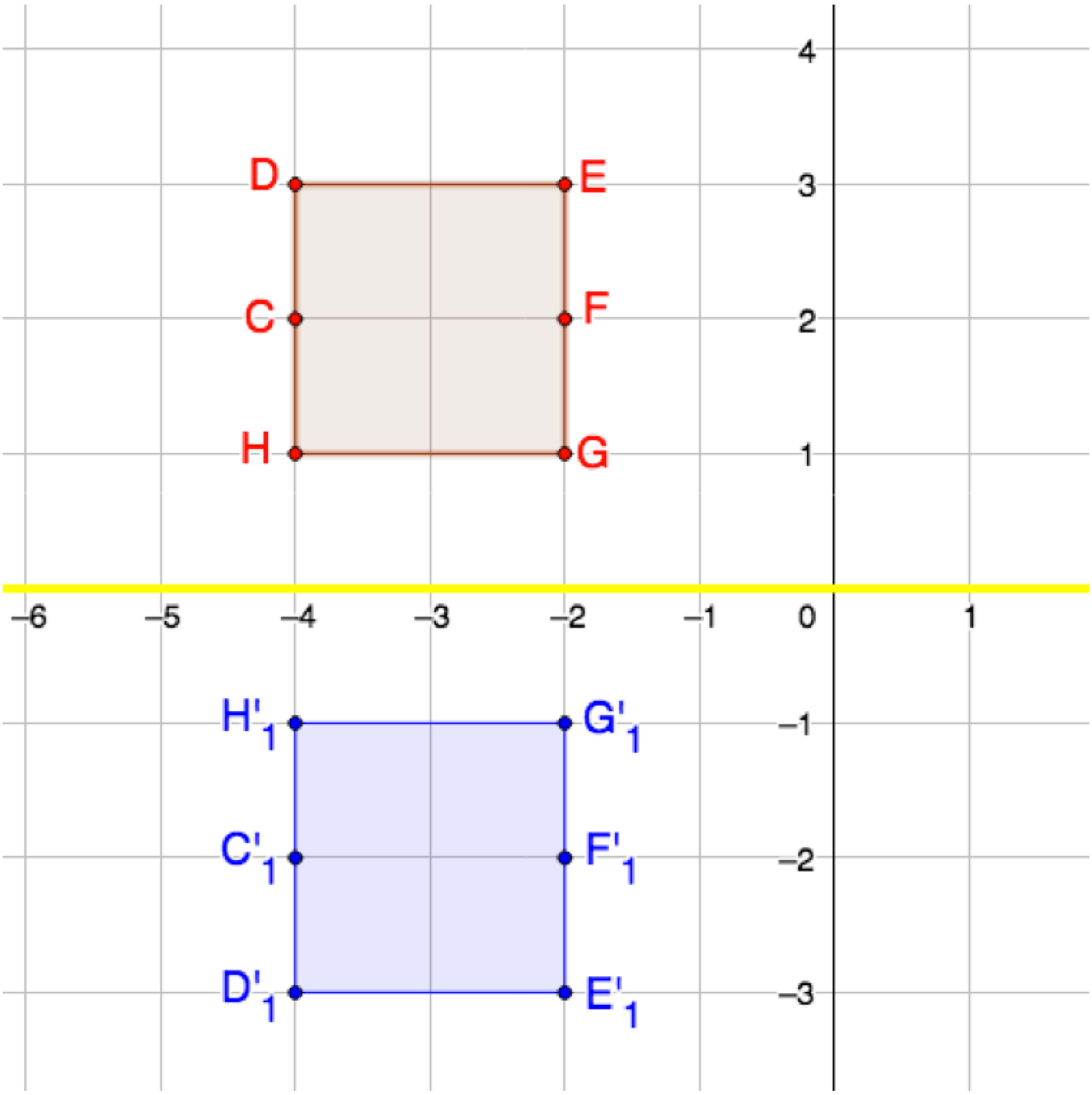
- a piece of paper
- a pen or pencil

What to do:

1. The figure below shows CDEFGH reflected across the y-axis to form the image $\mathrm{C'D'E'F'G'H'}$. Write down the transformation of each vertex from CDEFGH to $\mathrm{C'D'E'F'G'H'}$.



2. Describe the 'rule' that we can use to transform the coordinates of a point when reflected about the y-axis or the line $x = 0$.
3. The following figure at shows CDEFGH reflected across the x-axis to form the image $\mathrm{C'_1D'_1E'_1F'_1G'_1H'_1}$. Write down the transformation of each vertex from CDEFGH to $\mathrm{C'_1D'_1E'_1F'_1G'_1H'_1}$.



4. Describe the 'rule' that we can use to transform the coordinates of a point when reflected about the x-axis or the line $y = 0$.
5. If you have an internet connection, visit the interactive simulation called Reflections.

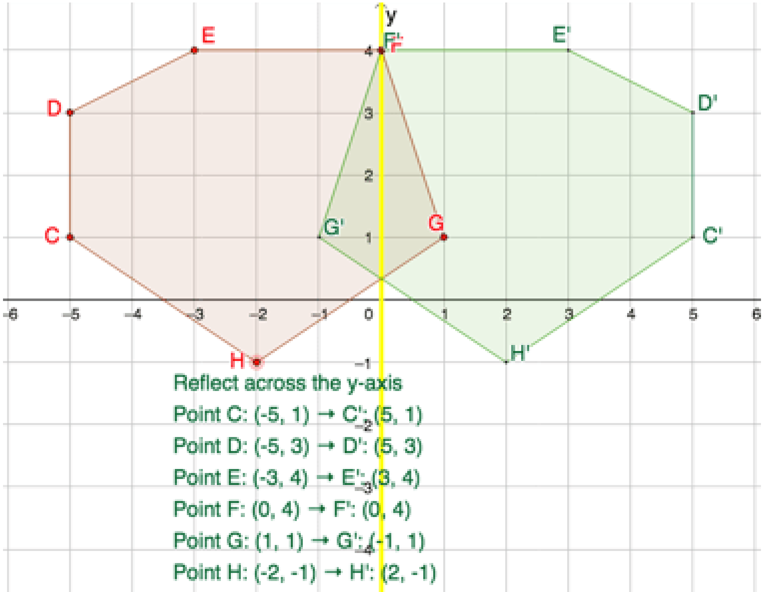
   Click and drag any/all of the vertices of CDEFGH to change the shape of the polygon. Do your rules for transforming these points when the shape is reflected about the y-axis and the x-axis hold?

What did you find:

1. This is how each of the vertices was transformed:
   $\mathrm{C}(-3, 2) \rightarrow \mathrm{C}'(3, 2)$

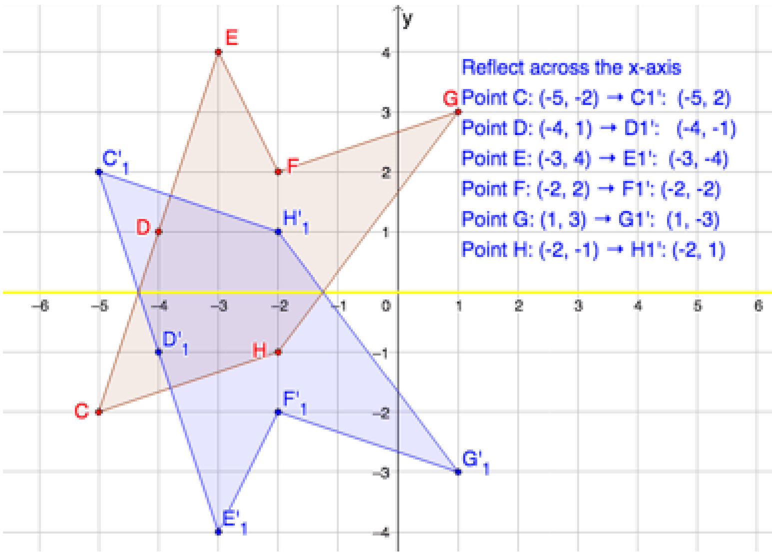
$D(-3,3) \rightarrow D'(3,3)$
$E(-1,3) \rightarrow E'(1,3)$
$F(-1,2) \rightarrow F'(1,2)$
$G(-1,1) \rightarrow G'(1,1)$
$H(-3,1) \rightarrow H'(3,1)$

2. In each case, the x-coordinate was multiplied by -1 while the y-coordinate remained the same.
3. This is how each of the vertices was transformed:
$C(-3,2) \rightarrow C'_1(-3,-2)$
$D(-3,3) \rightarrow D'_1(-3,-3)$
$E(-1,3) \rightarrow E'_1(-1,-3)$
$F(-1,2) \rightarrow F'_1(-1,-2)$
$G(-1,1) \rightarrow G'_1(-1,-1)$
$H(-3,1) \rightarrow H'_1(-3,-1)$
4. In each case, the y-coordinate was multiplied by -1 while the x-coordinate remained the same.
5. Here is a screenshot of the interactive simulation where the coordinates of all the vertices of $CDEFGH$ have been changed and the shape reflected across the y-axis.



We can see that the rule we discovered of simply multiplying the x-coordinate of each point by -1 still holds.

Here is another screenshot of the same simulation where the vertices of $CDEFGH$ have been changed again and the shape reflected about the x-axis. Once again, we can see that our rule of simply multiplying the y-coordinates of each point by -1 also still holds.



We discovered in Activity 1.1 that when we reflect a point across the y-axis (the line $x = 0$), all we need to do is change the sign of the x-coordinate (multiply it by -1) and leave the y-coordinate the same.

When we reflect a point across the x-axis (the line $y = 0$), all we need to do is change the sign of the y-coordinate (multiply it by -1) and leave the x-coordinate the same.

When reflecting a point across the y-axis / the line $x = 0$: $\text{A}(x, y) \rightarrow \text{A}'(-x, y)$

When reflecting a point across the x-axis / the line $y = 0$: $\text{A}(x, y) \rightarrow \text{A}'(x, -y)$

Activity 1.2: Reflect across the line y= x

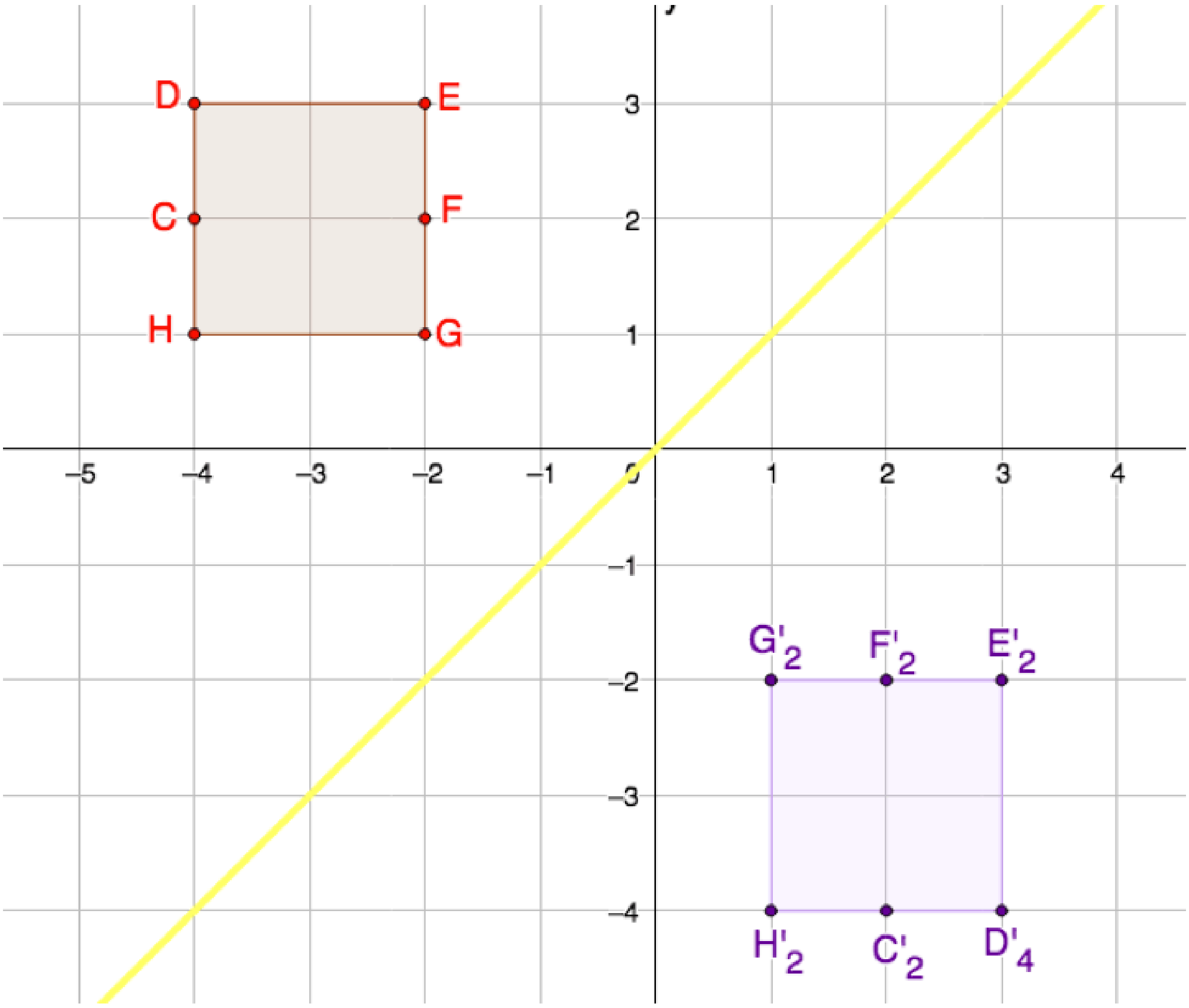
Time required: 10 minutes

What you need:

- a piece of paper
- a pen or pencil

What to do:

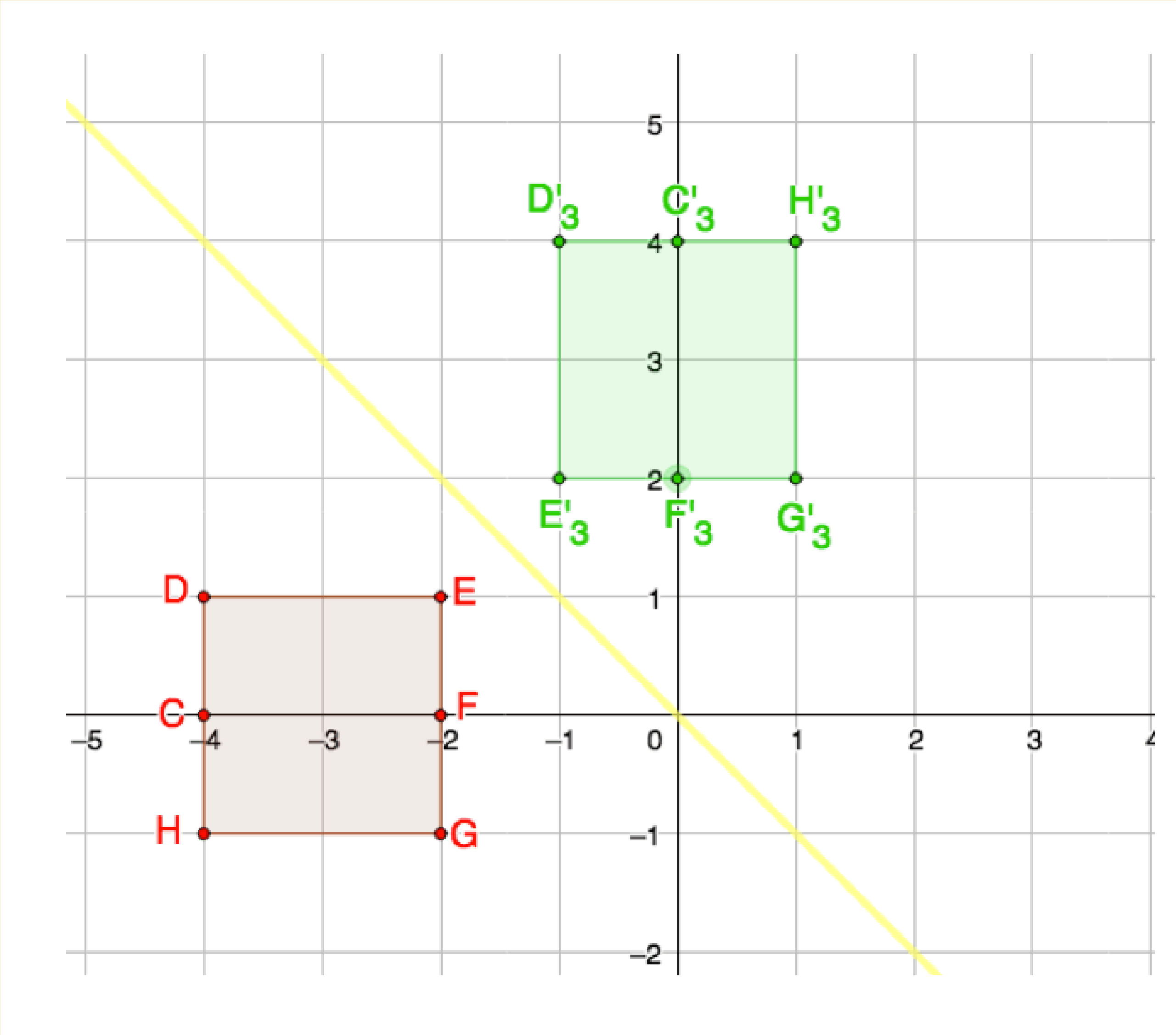
1. Below is CDEFGH reflected across the line $y = x$ to form the image $\text{C}'_2\text{D}'_2\text{E}'_2\text{F}'_2\text{G}'_2\text{H}'_2$. Write down the transformation of each vertex from CDEFGH to $\text{C}'_2\text{D}'_2\text{E}'_2\text{F}'_2\text{G}'_2\text{H}'_2$.



2. Describe the 'rule' that we can use to transform the coordinates of a point when reflected about the line $y = x$.
3. What do you think the 'rule' would be when reflecting a point about the line $y = -x$?

What did you find:

1. This is how each of the vertices was transformed:
   $\text{C}(-3,2) \to \text{C}'_2(2,-3)$
   $\text{D}(-3,3) \to \text{D}'_2(3,-3)$
   $\text{E}(-1,3) \to \text{E}'_2(3,-1)$
   $\text{F}(-1,2) \to \text{F}'_2(2,-1)$
   $\text{G}(-1,1) \to \text{G}'_2(1,-1)$
   $\text{H}(-3,1) \to \text{H}'_2(1,-3)$
2. In each case, the x- and y-coordinates were swopped around. In other words, the point (x, y) was transformed into the point (y, x). This makes sense given that we are reflecting across the line $y = x$. The new y-coordinate is equal to the old x-coordinate. The new x-coordinate is equal to the old y-coordinate.
3. If we think about reflecting a point across the line $y = -x$ we will need to swop the coordinates around and change their signs (in other words multiply them by -1). So, the point (x, y) will become the point $(-y, -x)$. The figure below shows what the image of CDEFGH looks like when reflected across the line $y = -x$.



We discovered in Activity 1.2 that when we reflect a point across the line $y = x$, we need to swop the coordinates around to find the reflected point.

When we reflect a point across the line $y = -x$, we need to swop the coordinates and multiply them by -1 to find the reflected point.

If you have an internet connection, spend some time playing with the interactive simulation called Reflections.

Click and drag any/all of the vertices of CDEFGH to change the shape of the polygon and see how these points are transformed when reflected across the lines $y = x$ and $y = -x$.

When reflecting a point across the line $y = x$: $\mathrm{A}(x, y) \rightarrow \mathrm{A}'(y, x)$

When reflecting a point across the line $y = -x$: $\mathrm{A}(x, y) \rightarrow \mathrm{A}'(-y, -x)$

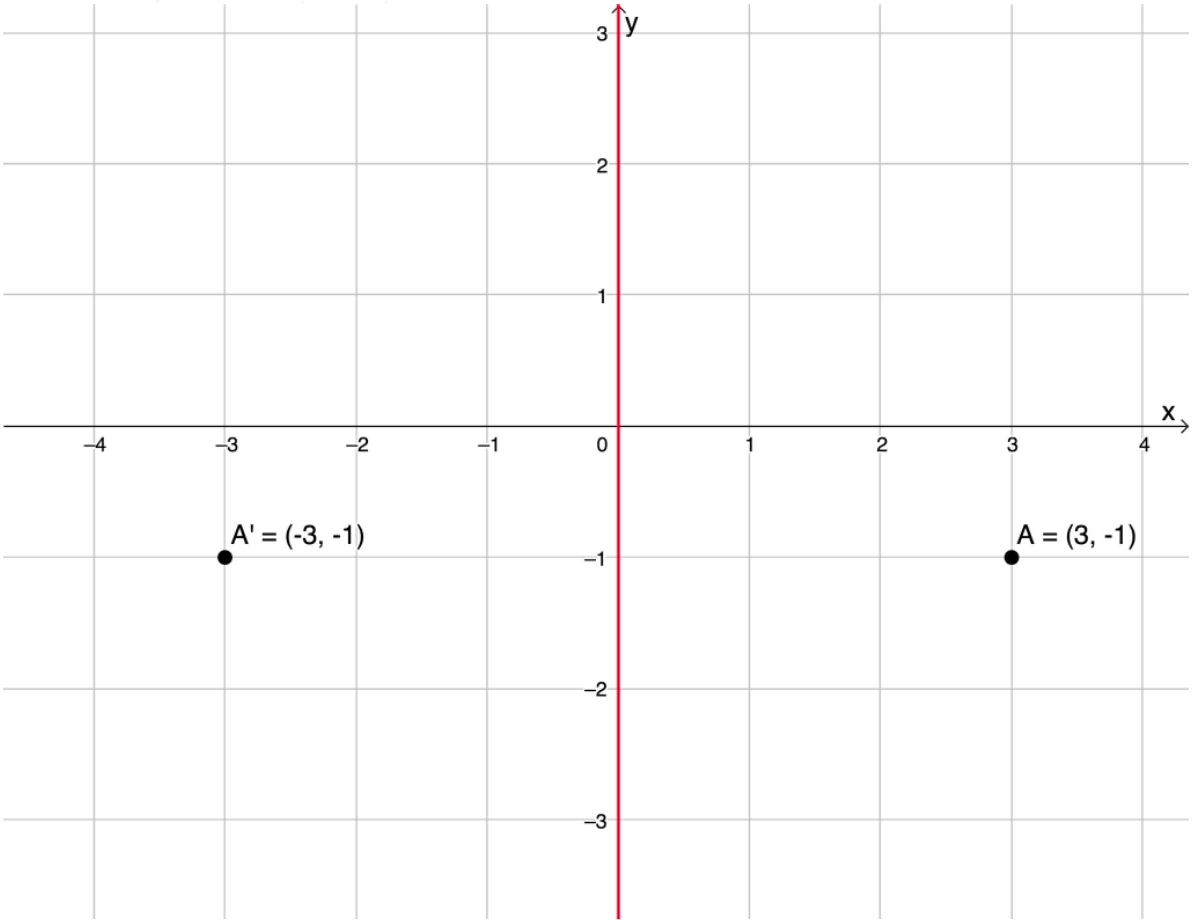
Example 1.2

What are the coordinates of A' if $\mathrm{A}(3, -1)$ is reflected across:

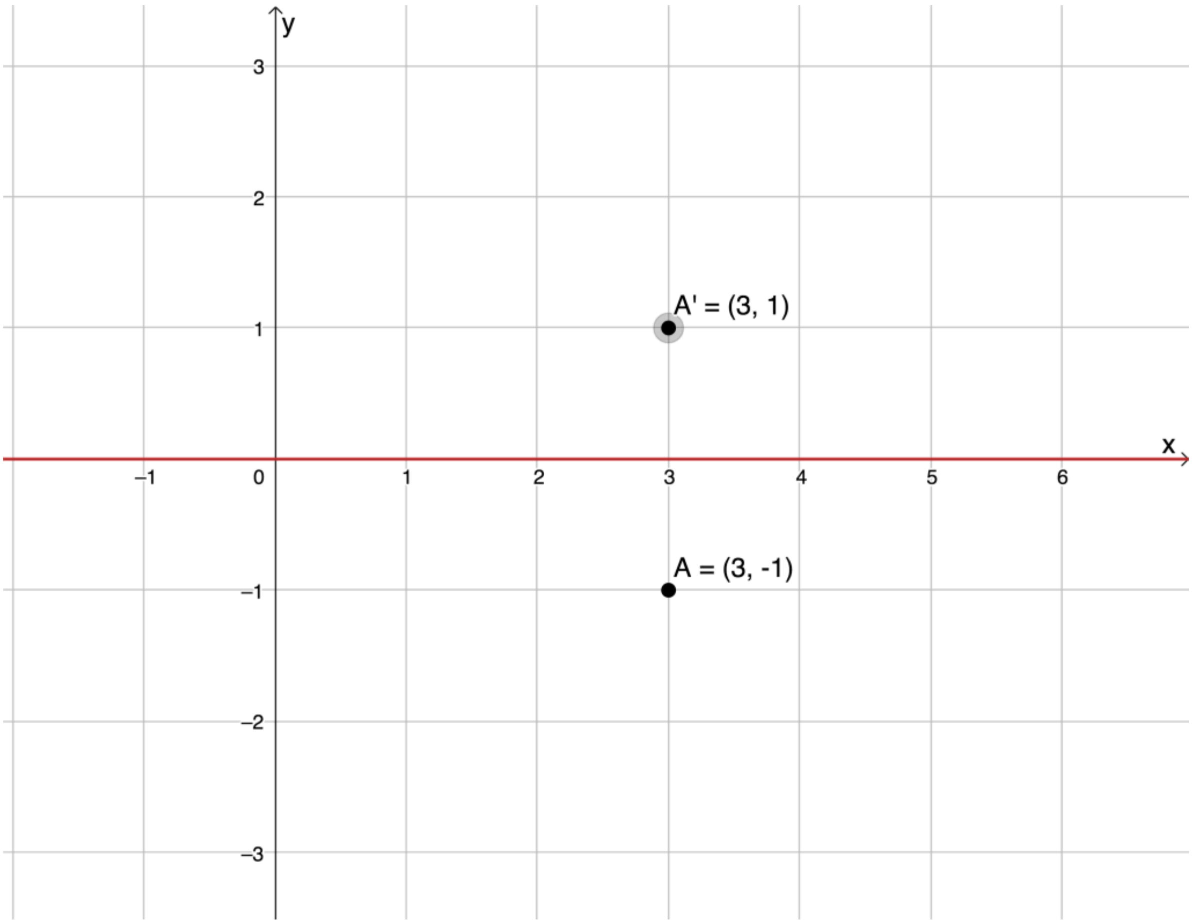
1. the y-axis
2. the x-axis
3. the line $y = x$
4. the line $y = -x$

Solutions

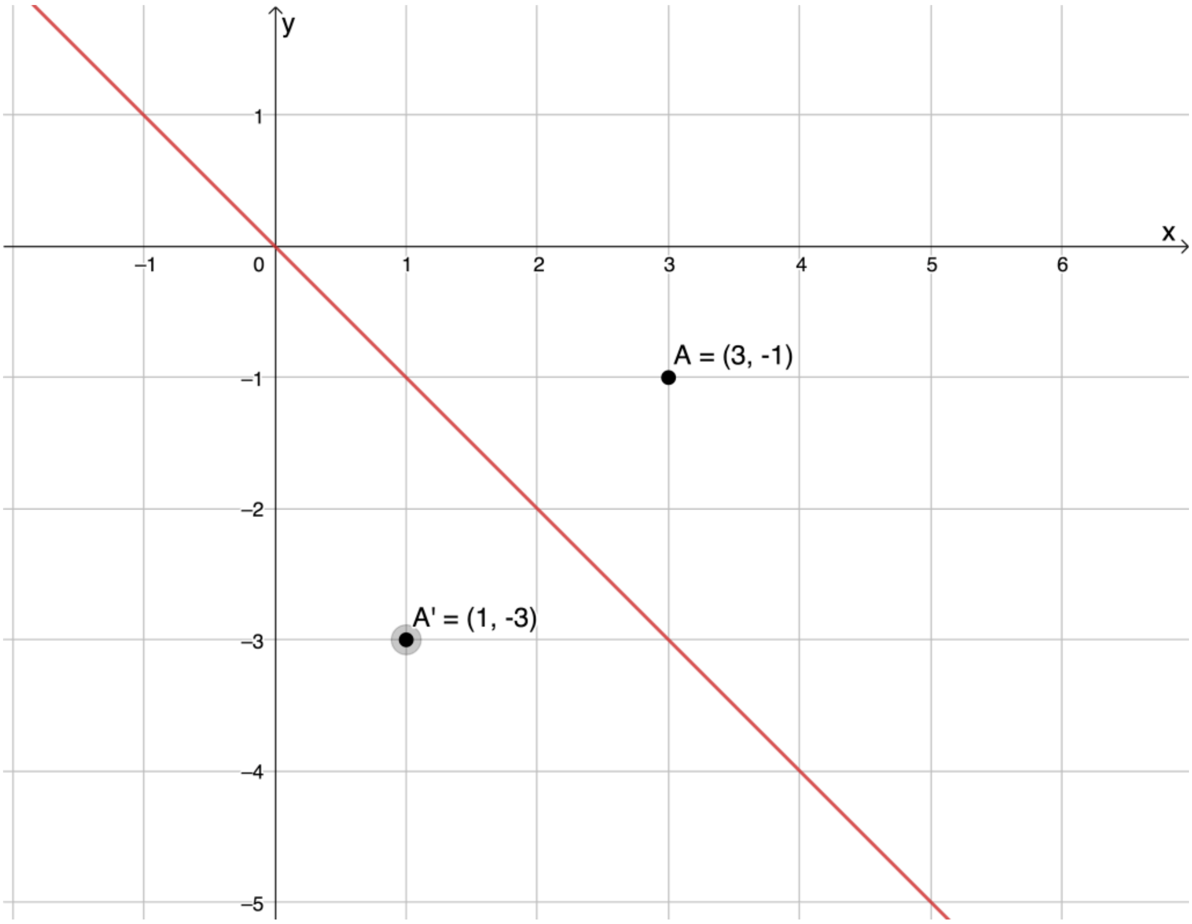
1. When reflecting a point across the y-axis, we transform the point as follows: $\mathrm{A}(x, y) \rightarrow \mathrm{A}'(-x, y)$. Therefore, $\mathrm{A}(3, -1) \rightarrow \mathrm{A}'(-3, -1)$.



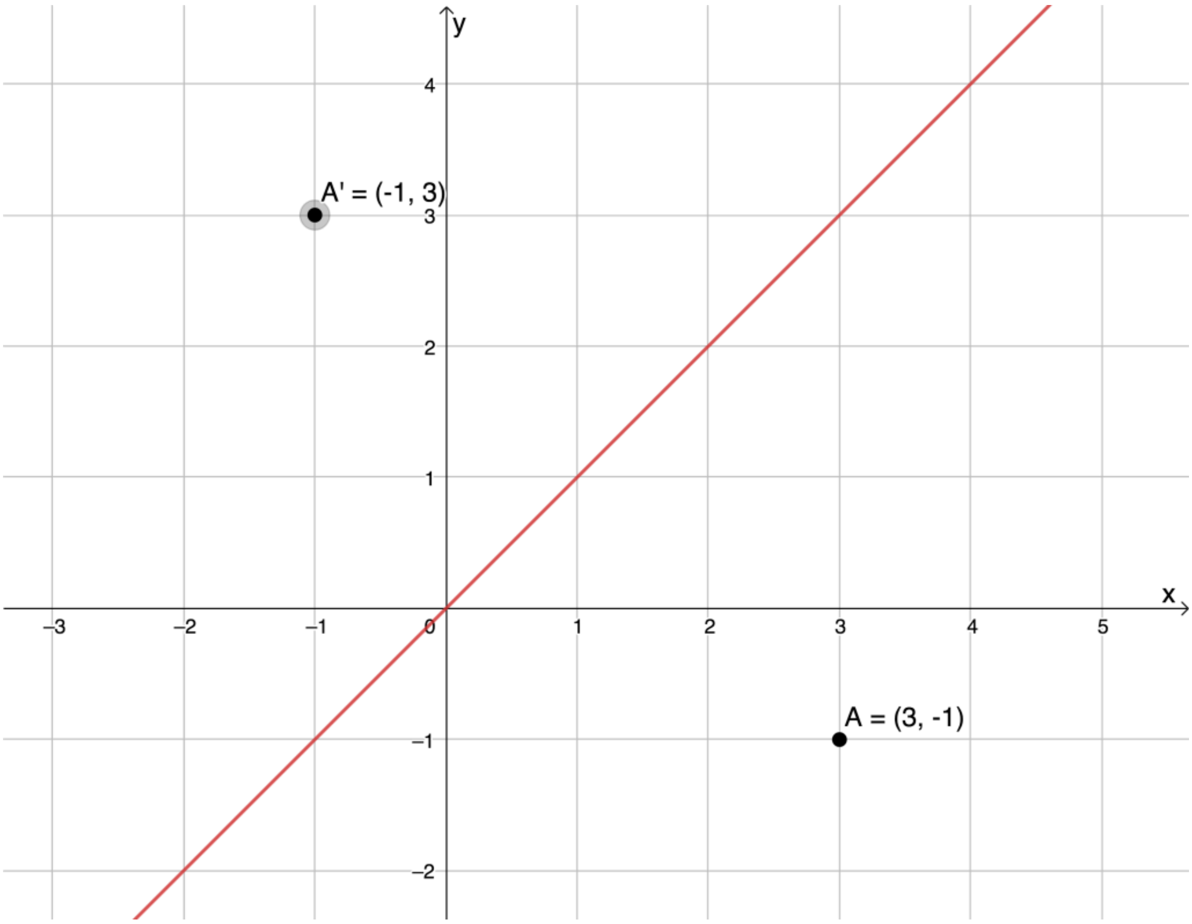
2. When reflecting a point across the x-axis, we transform the point as follows: $\mathrm{A}(x, y) \rightarrow \mathrm{A}'(x, -y)$. Therefore, $\mathrm{A}(3, -1) \rightarrow \mathrm{A}'(3, 1)$.



3. When reflecting a point across the line $y = x$, we transform the point as follows: $\mathrm{A}(x, y) \rightarrow \mathrm{A}'(y, x)$. Therefore, $\mathrm{A}(3, -1) \rightarrow \mathrm{A}'(-1, 3)$.

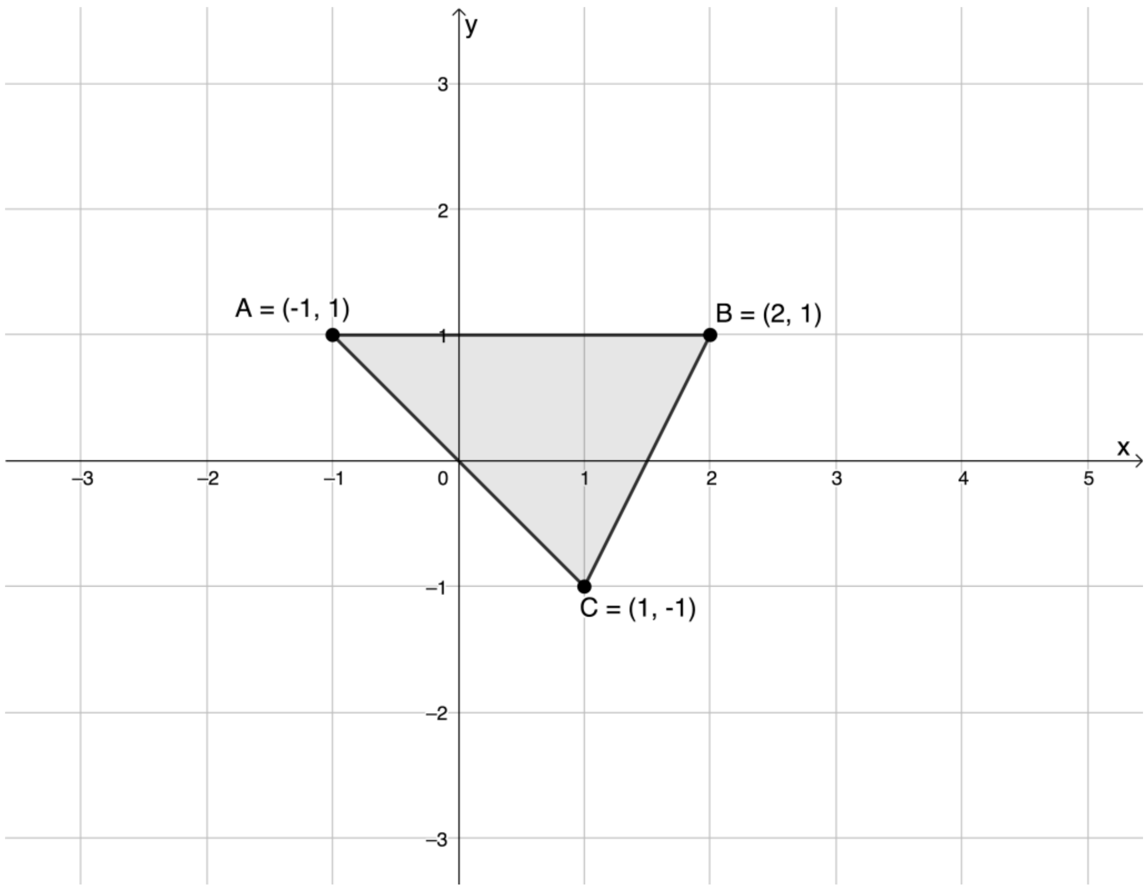


4. When reflecting a point across the line $y = -x$, we transform the point as follows: $\mathrm{A}(x, y) \rightarrow \mathrm{A}'(-y, -x)$. Therefore, $\mathrm{A}(3, -1) \rightarrow \mathrm{A}'(1, -3)$.



Exercise 1.2

1. $\Delta \mathrm{ABC}$ on the Cartesian plane is shown in the diagram below. On the same set of axes, sketch $\Delta \mathrm{ABC}$ and the image $\Delta \mathrm{A'B'C'}$ if $\Delta \mathrm{ABC}$ is reflected across:
   a. the x-axis.
   b. the line $y = -x$.



2. If $D'(-6,-2)$ is a reflection of D across the line $y=x$, what are the coordinates of D?

The full solutions are at the end of the unit.

Summary

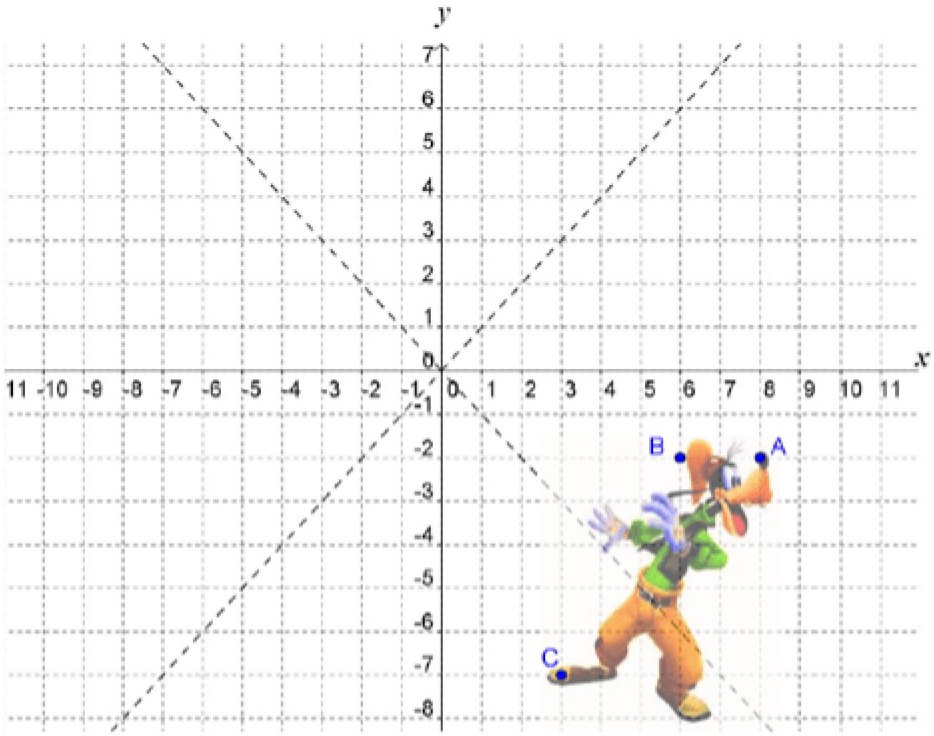
In this unit you have learnt the following:

- How to find the coordinates of the point (x,y) after it is translated p units horizontally and q units vertically.
- How to find the coordinates of the point (x,y) after it is reflected about the x-axis, the y-axis, and the lines $y=x$and $y=-x$.

Unit 1: Assessment

Suggested time to complete: 10 minutes

1. The following diagram shows an image of a cartoon character. Three points A, B and C are found on the diagram. Use the sketch given below and answer the questions that follow.



a. What would be the co-ordinates of point A if it were translated 3 units up and 2 units to the right?

b. What would be the co-ordinates of point B if it were reflected across the line $y = -x$?

c. What would be the co-ordinates of point A if it is reflected across the y-axis?

d. What would be the co-ordinates of point C if it is reflected across the line $y = x$?

- $\mathrm{A}(x, y)$ is a point on the Cartesian plane. Without sketching, write down the coordinates of the image of A (A') if A is shifted 1 unit to the left, then 3 units up then reflected across the line $y = x$.

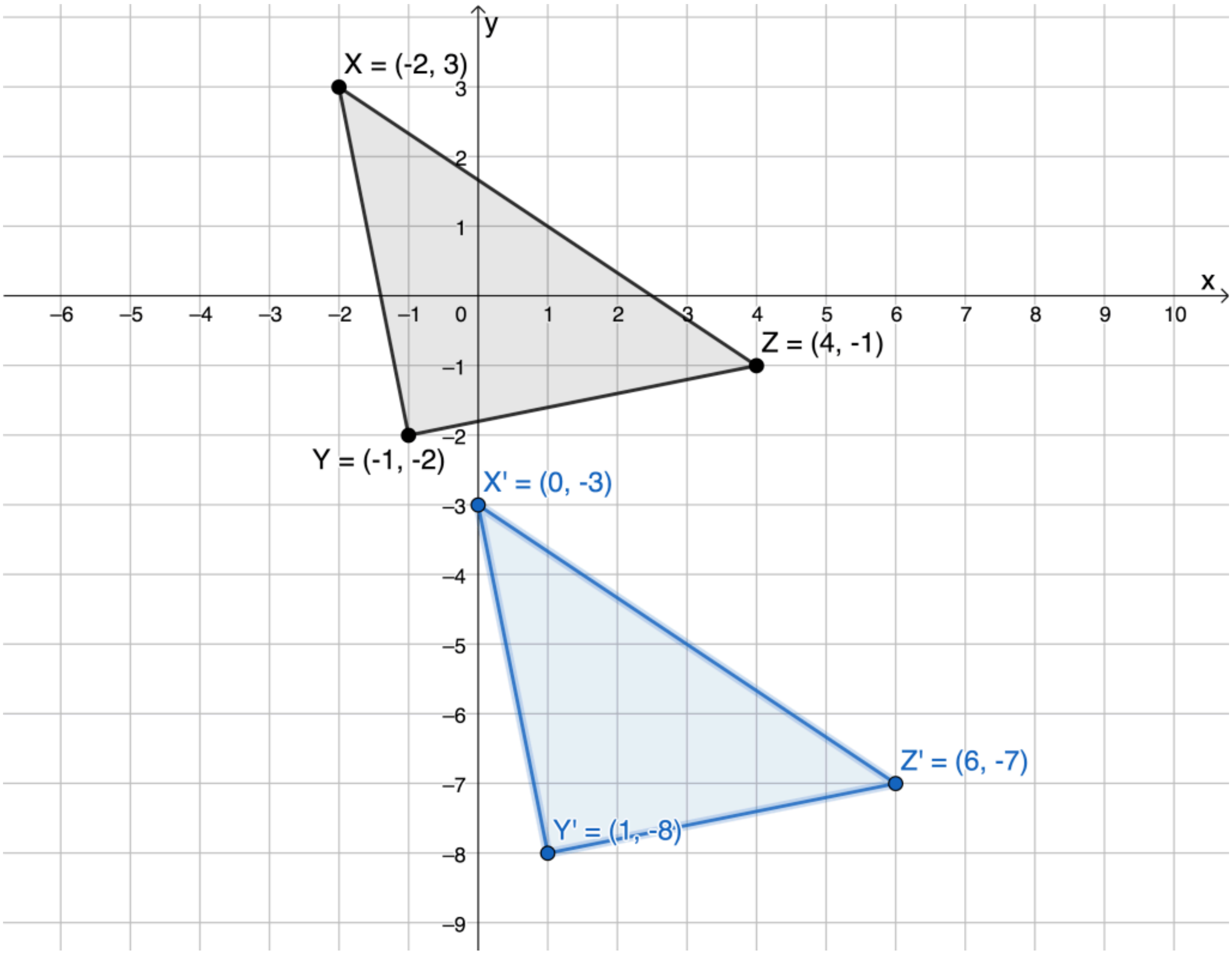
The full solutions are at the end of the unit.

Unit 1: Solutions

Exercise 1.1

1.
    a. If $\mathrm{Q}(-1.5, 5)$ is translated 2 units to the left, then we need to subtract 2 units from the x-coordinate. There is no vertical translation, so the y-coordinate does not change. $\mathrm{Q}(-1.5, 5) \rightarrow \mathrm{Q}'(-3.5, 5)$
    b. If $\mathrm{Q}(-1.5, 5)$ is translated 8 units up and 5 units to the right, then we need to add 8 units to the y-coordinate and add 5 units to the x-coordinate. $\mathrm{Q}(-1.5, 5) \rightarrow \mathrm{Q}'(3.5, 13)$
    c. If $\mathrm{Q}(-1.5, 5)$ is translated 3 units to the right and 3 units down, then we need to add 5 units to the x-coordinate and subtract 3 units from the y-coordinate. $\mathrm{Q}(-1.5, 5) \rightarrow \mathrm{Q}'(1.5, 2)$
    d. If $\mathrm{Q}(-1.5, 5)$ is translated p units to the left and q units up, then we need to subtract p units from the x-coordinate and add q units to the y-coordinate. $\mathrm{Q}(-1.5, 5) \rightarrow \mathrm{Q}'(-1.5 - p, 5 + q)$
2. $\Delta\mathrm{XYZ}$ is translated 2 units to the right and 6 units down. Therefore, each vertex is translated 2 units to the right and 6 units down.

$X(-2, 3) \to X'(0, -3)$
$Y(-1, -2) \to Y'(1, -8)$
$Z(4, -1) \to Z'(6, -7)$

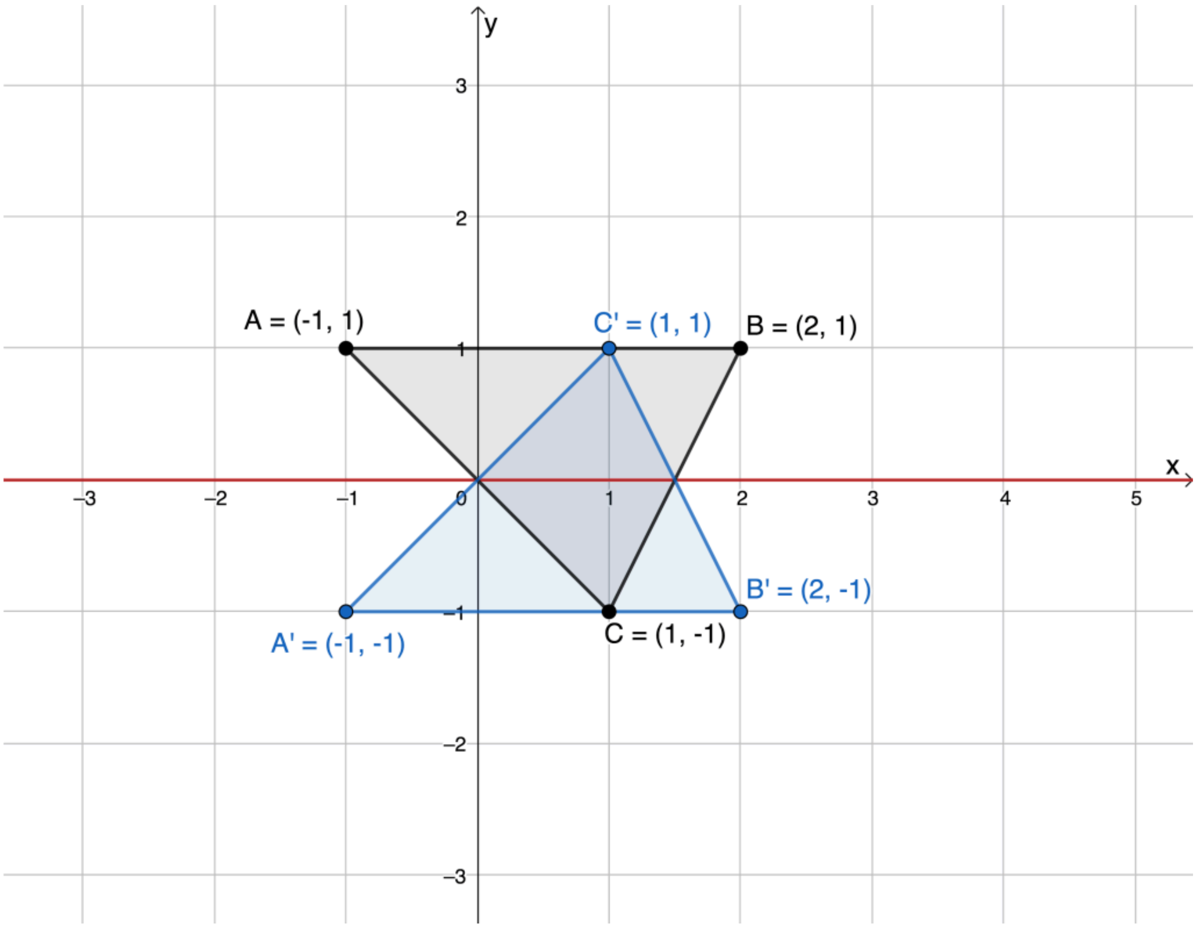


3. Because we are starting with $G'(-6, 3)$ and need to get back to G we have to perform the normal translations in reverse. G was translated 3 units down and 2 units to the right. Therefore, we need to **add** 3 units to the y-coordinate of G' and **subtract** 2 units from the x-coordinate of G'.
$G'(-6, 3) \to G(-8, 6)$

Back to Exercise 1.1

Exercise 1.2

1.
   a. ΔABC is reflected across the x-axis. Therefore, the vertices are transformed as follows:
   $A(-1, 1) \to A'(-1, -1)$
   $B(2, 1) \to B'(2, -1)$
   $C(1, -1) \to C'(1, 1)$

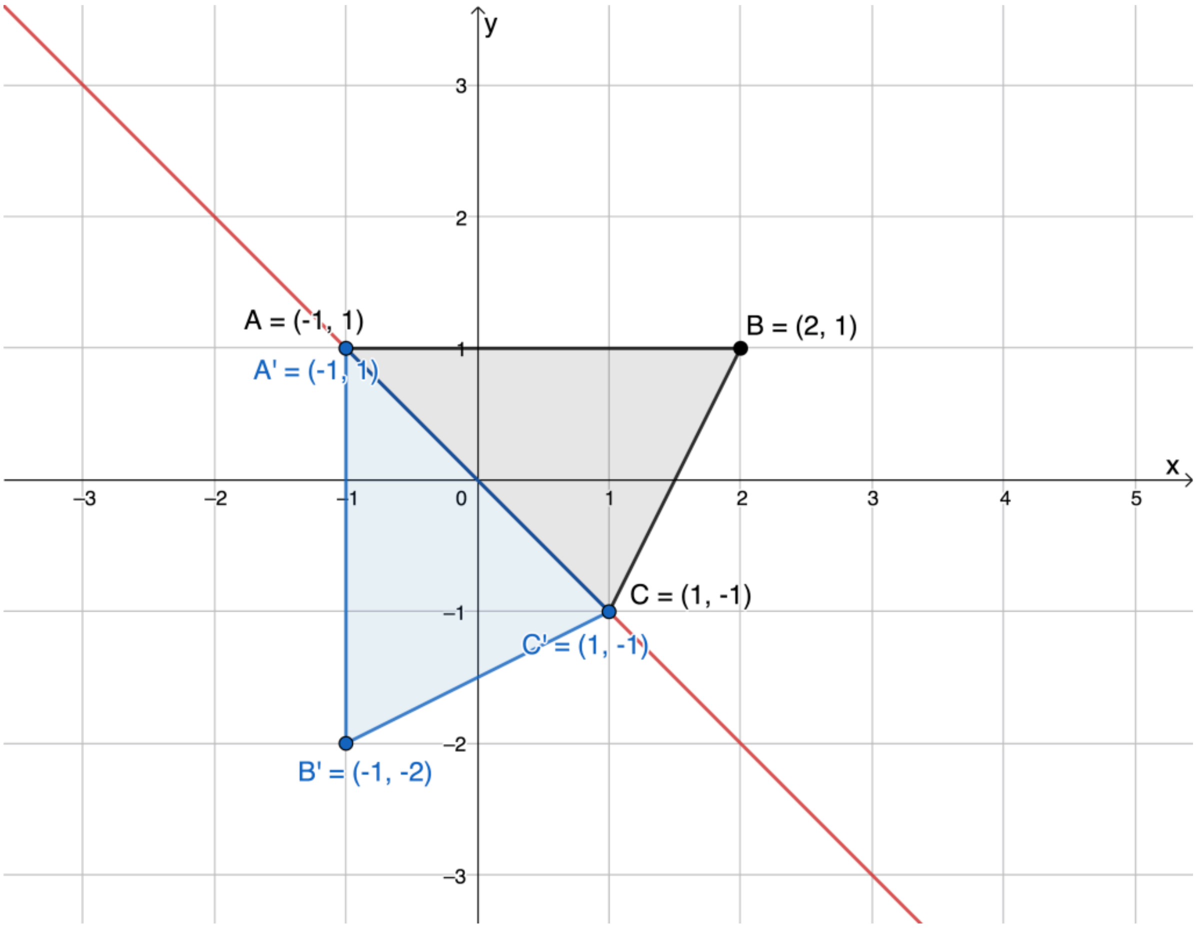


b. $\Delta\mathrm{ABC}$ is reflected across the line $y=-x$. Therefore, the vertices are transformed as follows:

$\mathrm{A}(-1,1) \rightarrow \mathrm{A}'(-1,1)$

$\mathrm{B}(2,1) \rightarrow \mathrm{B}'(-1,-2)$

$\mathrm{C}(1,-1) \rightarrow \mathrm{C}'(1,-1)$



2. $D'(-6,-2)$ is a reflection of D across the line $y=x$. We know that $D(x,y) \rightarrow D'(y,x)$. Therefore $D(-2,-6)$.

Back to Exercise 1.2

Unit 1: Assessment

1.
   a. $A(8,-2)$ is translated 3 units up and 2 units to the right. $A(x,y) \rightarrow A'(x+p, y+q)$. Therefore, $A'(10,1)$.
   b. $B(6,-2)$ reflected across the line $y=-x$. $B(x,y) \rightarrow B'(-y,-x)$. Therefore $B'(2,-6)$.
   c. $A(8,-2)$ is reflected across the y-axis. $A(x,y) \rightarrow A'(-x,y)$. Therefore $A'(-8,-2)$.
   d. $C(3,-7)$ reflected across the line $y=x$. $C(x,y) \rightarrow C'(y,x)$. Therefore $C'(-7,3)$.
2. $A(x,y)$
   Translation 1 unit to the left: $A(x,y) \rightarrow A'(x-1,y)$
   Translation 3 units up: $A'(x-1,y) \rightarrow A'(x-1,y+3)$
   Reflection across the line $y=x$: $A'(x-1,y+3) \rightarrow A'(y+3,x-1)$

Back to Unit 1: Assessment

Media Attributions

SUBJECT OUTCOME X

SPACE, SHAPE AND MEASUREMENT: SOLVE PROBLEMS BY CONSTRUCTING AND INTERPRETING GEOMETRICAL MODELS

Subject outcome 3.5

Solve problems by constructing and interpreting geometrical models.

Learning outcomes

- Investigate the relationship between the sides of a right-angled triangle to develop the Theorem of Pythagoras.
- Use the Theorem of Pythagoras to calculate a missing length in a right-angled triangle leaving answers in the most appropriate form.

Unit outcomes: Unit 1: Theorem of Pythagoras

By the end of this unit you will be able to:

- Identify different types of triangles.
- Investigate the relationship between the sides of a right-angled triangle to develop the Theorem of Pythagoras.
- Use the Theorem of Pythagoras to find missing lengths in right angled triangles.

Unit 1: Theorem of Pythagoras

NATASHIA BEARAM-EDMUNDS

Unit outcomes: Unit 1: Transformation geometry

By the end of this unit you will be able to:

- Identify different types of triangles.
- Investigate the relationship between the sides of a right-angled triangle to develop the Theorem of Pythagoras.
- Use the Theorem of Pythagoras to find missing lengths in right angled triangles.

What you should know

Before you start this unit, make sure you can:

- Identify different types of polygons. To revise the properties of polygons refer back to Subject outcome 3.1 Unit 1.
- Solve algebraic equations. You can revise how to solve algebraic equations and inequalities in Subject outcome 2.3.

Introduction

Pythagoras was a Greek mathematician who developed one of the most famous theorems, which relates the sides of a right-angled triangle, called Pythagoras' Theorem. Before we go into detail about Pythagoras' Theorem, let's recap the basics you need to know about triangles.

A triangle is a three-sided polygon. Triangles can be classified by their sides: equilateral, isosceles and scalene. Triangles can also be classified according to angles: acute-angled, right-angled and obtuse-angled. The different types of triangles are shown in Figure 1.

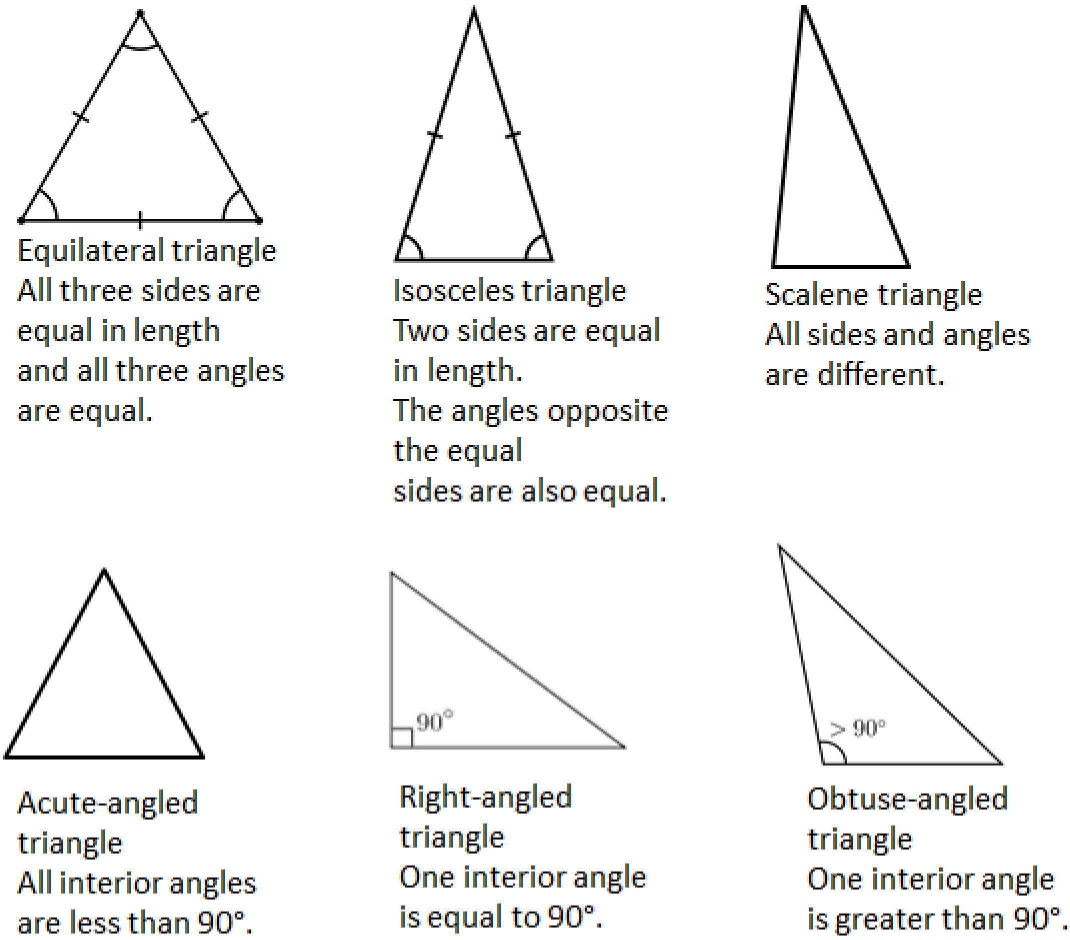


Figure 1: Classification of triangles

Different combinations of these properties are also possible. For example, you can get an obtuse isosceles triangle and a right-angled isosceles triangle. Try to draw these yourself. Remember that the sum of all the angles in any triangle will always be equal to 180°.

Investigating the sides of a right-angled triangle

As we have noted, a right-angled triangle has one 90° angle. The longest side of the right-angled triangle is found opposite the right-angle and is called the **hypotenuse** (pronounced 'high – pot – eh – news').

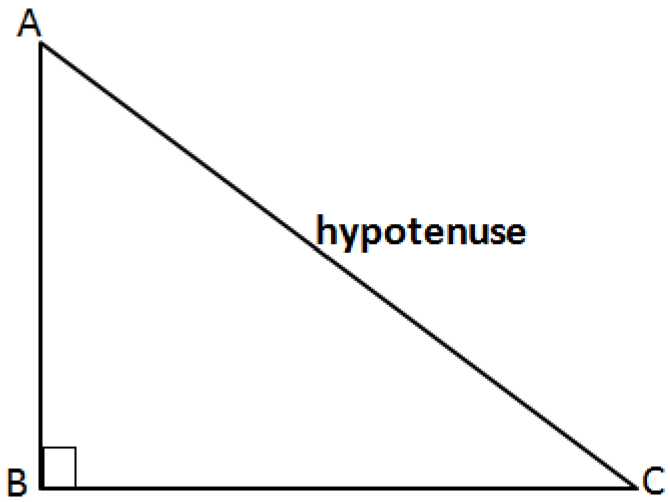


Figure 2: A right-angled triangle △ABC showing the hypotenuse

We use the notation △ABC to refer to a triangle with vertices labelled A, B and C in that order.

Activity 1.1: Investigate squares on the sides of a right-angled triangle

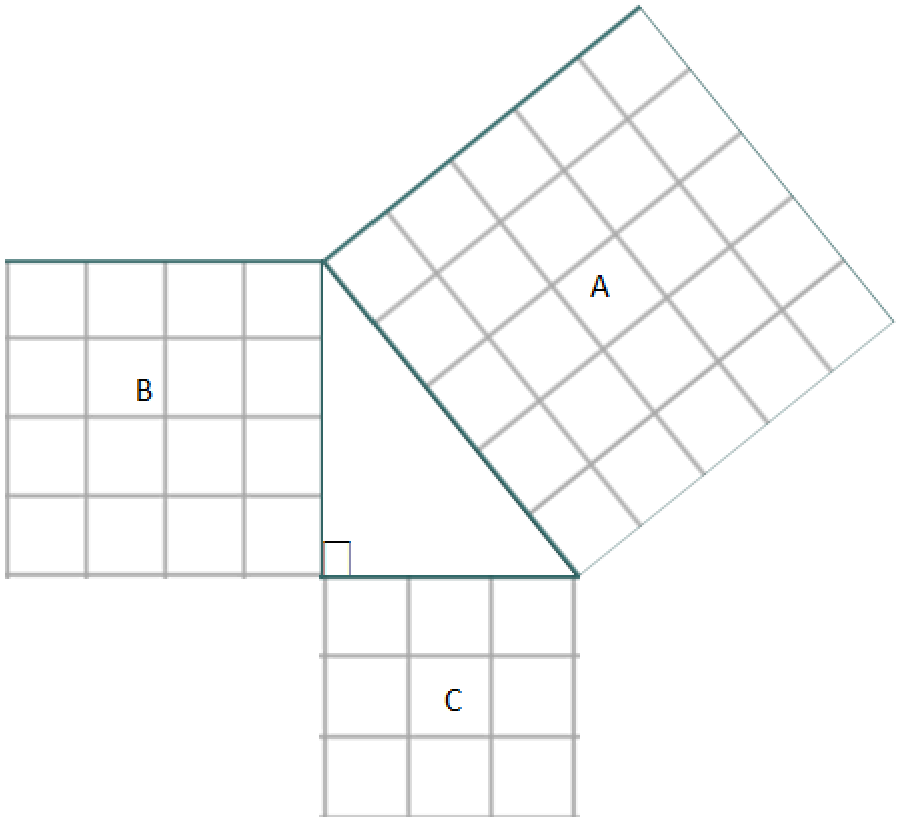
Time required: 15 minutes

What you need:

- a pen and paper

What to do:

Below is a right-angled triangle with squares on each of its sides.



1. Write down the areas of:
   a. Square A
   b. Square B
   c. Square C
2. Add the area of square B and C. What do you get?
3. What do you notice about the areas of B plus C compared to A?
4. What is the length of the hypotenuse?
5. What is the length of each of the shorter sides of the triangle?
6. Wite an equation that relates the sum of the shorter sides of the triangle to the hypotenuse.

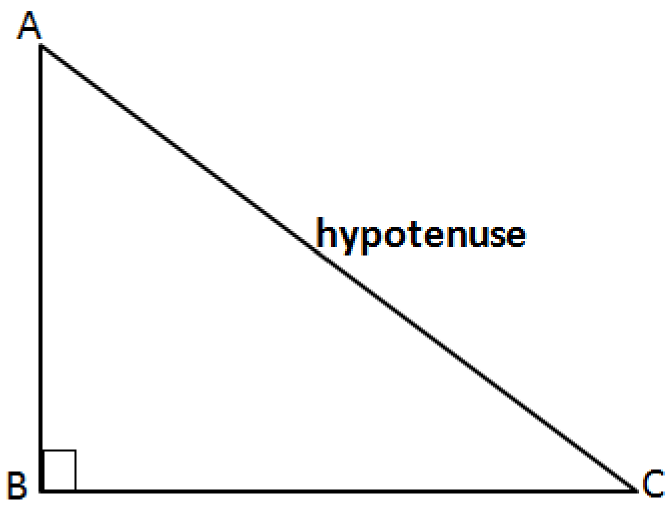
What did you find?

1. By counting you will get the following:
   a. The area of $A = 25$ square units.
   b. The area of $\mathrm{B} = 16$ square units.
   c. The area of $\mathrm{C} = 9$ square units.
2. $\text{Area of square B} + \text{Area of square C} = 16 + 9 = 25.$
3. The areas of the sum of squares B and C is the same as the area of square A.
4. The hypotenuse measures 5 units.
5. The other two sides measure 4 and 3units.
6. $4^2 + 3^2 = 5^2.$

In Activity 1.1, you have discovered Pythagoras' Theorem for right-angled triangles.

The Theorem of Pythagoras

In a right-angled triangle, a square formed on the hypotenuse will have the same area as the sum of the area of the two squares formed on the other sides of the triangle. Another way of saying this is that the square of the hypotenuse is equal to the sum of the squares of the other two sides.



If $\triangle$ABC is right-angled at angle B then $\mathrm{AC}^2 = \mathrm{AB}^2 + \mathrm{BC}^2$.

Conversely, if $\mathrm{AC}^2 = \mathrm{AB}^2 + \mathrm{BC}^2$ then $\triangle$ABC is right-angled with $\hat{\mathrm{B}} = 90^\circ$.

Note

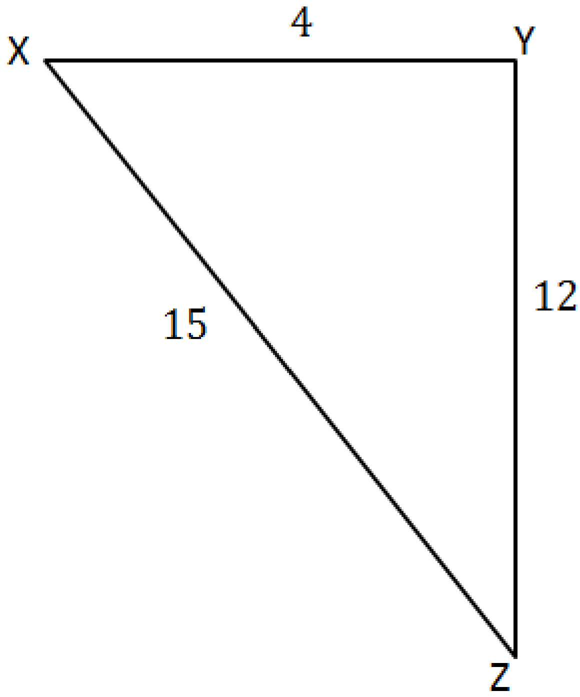
Remember that the Theorem of Pythagoras can only be used in a right-angled triangle.

When you have an internet connection you can view a demonstration of Pythagoras' Theorem by watching the Pythagorean theorem water demo (Duration: 0.44).

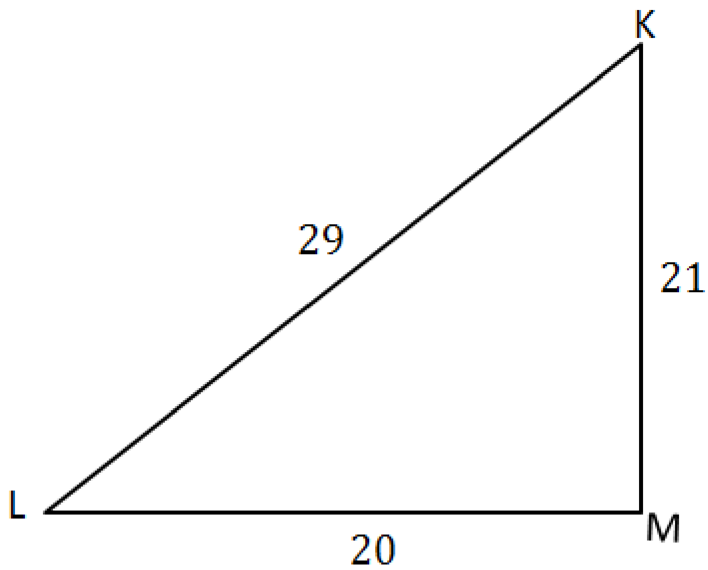
Example 1.1

Use Pythagoras' Theorem to determine whether these triangles are right-angled.

1.



2.



Solutions

1. If the triangle is right-angled the sum of the square of the two shorter sides will be equal to the square of the longest side.

$$\begin{aligned} XY^2 + YZ^2 &= (4)^2 + (12)^2 \\ &= 16 + 144 \\ &= 160 \end{aligned}$$

$$XZ^2 = (15)^2$$
$$= 225$$
$$\therefore XZ^2 \neq XY^2 + YZ^2$$

Therefore, $\triangle$XYZ is not right-angled since the sum of the square of the two shorter sides is not equal to the square of the longest side.

2.

$$LM^2 + KM^2 \quad = (20)^2 + (21)^2$$
$$= 841$$

Now, check KL.

$$KL^2 \quad = (29)^2$$
$$= 841$$
$$\therefore LM^2 + KM^2 = KL^2$$

Therefore, $\triangle$KLM is right-angled at angle M by the converse of Pythagoras.

Exercise 1.1

1. A triangle's side lengths are 8 mm, 15 mm and 17 mm. Prove that it is a right-angled triangle.
2. Is a triangle with sides 12 cm, $16 \text{ cm}$and 20 cmright-angled?

The full solutions are at the end of the unit.

Did you know?

Did you know there are different ways to prove the Theorem of Pythagoras? Watch the video called "How many ways are there to prove the Pythagorean theorem?" to see these proofs.

How many ways are there to prove the Pythagorean theorem? (Duration: 05.16)

Finding missing sides

You can use the Theorem of Pythagoras to find the lengths of missing sides if you know that a triangle is right-angled.

Example 1.2

Calculate the length of the hypotenuse if the lengths of the other two sides are 6 units and 8 units.

Solution

Since we are told that the triangle has a hypotenuse this means the triangle is right-angled.

The sum of the square of the shorter sides is:

$$\begin{aligned} 6^2 + 8^2 &= 36 + 64 \\ &= 100 \end{aligned}$$

To find the length of the hypotenuse we need to take the square root of the sum of the squares of the shorter sides.

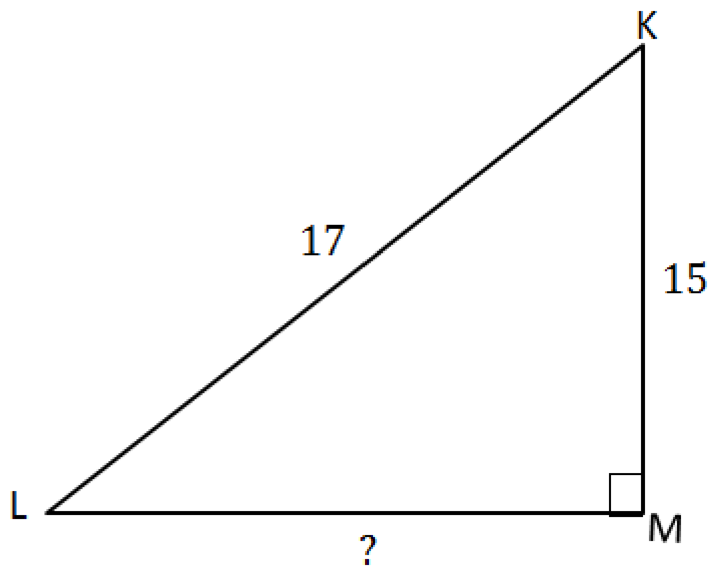
$$\begin{aligned} \text{hypotenuse}^2 &= 100 \\ \therefore \text{hypotenuse} &= 10 \qquad \text{We only take the positive square root since distance is not negative.} \end{aligned}$$

Sometimes the square root of a number is not a whole number or a rational number. In these cases, you can leave the answer under the square root sign. Remember, this is leaving your answer in surd form. Refer back to Unit 5 of Subject outcome 1.2 to revise surds.

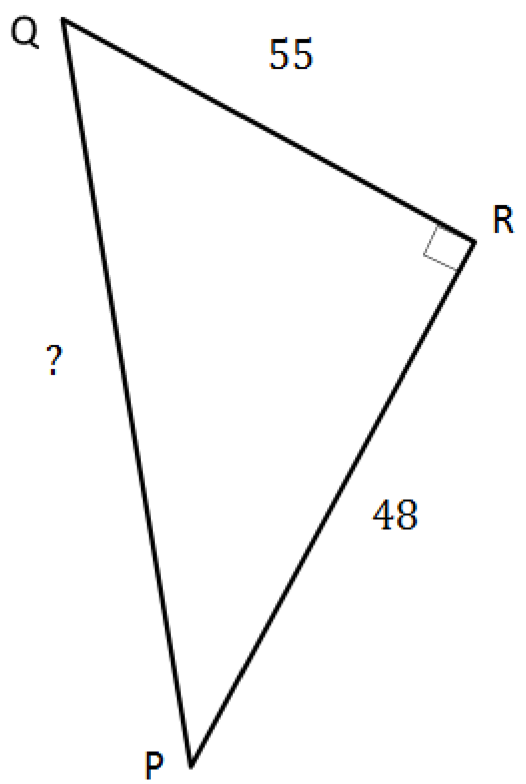
Exercise 1.2

1. Find the length of the missing side in each of the triangles below. Leave the answers in surd form where applicable.

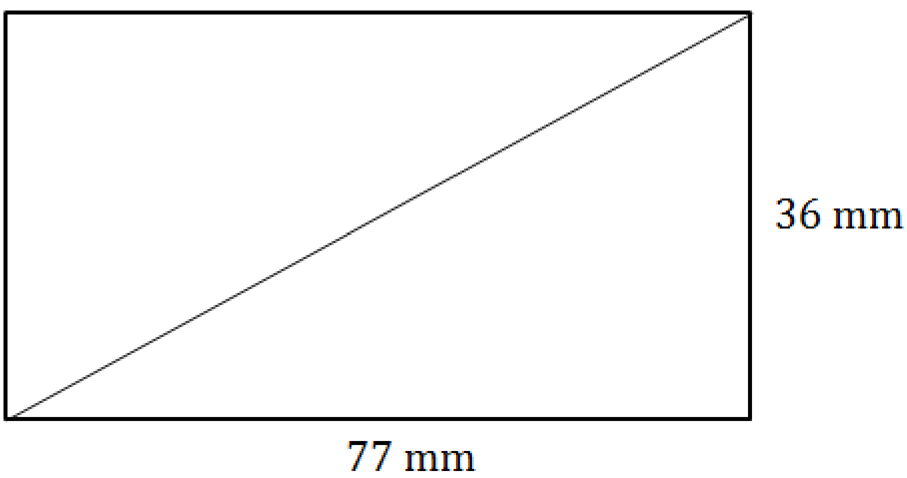
   a.



b.



2. A rectangle has sides with lengths $36\ \text{mm}$ and $77\ \text{mm}$. Find the length of the rectangle's diagonal.



The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

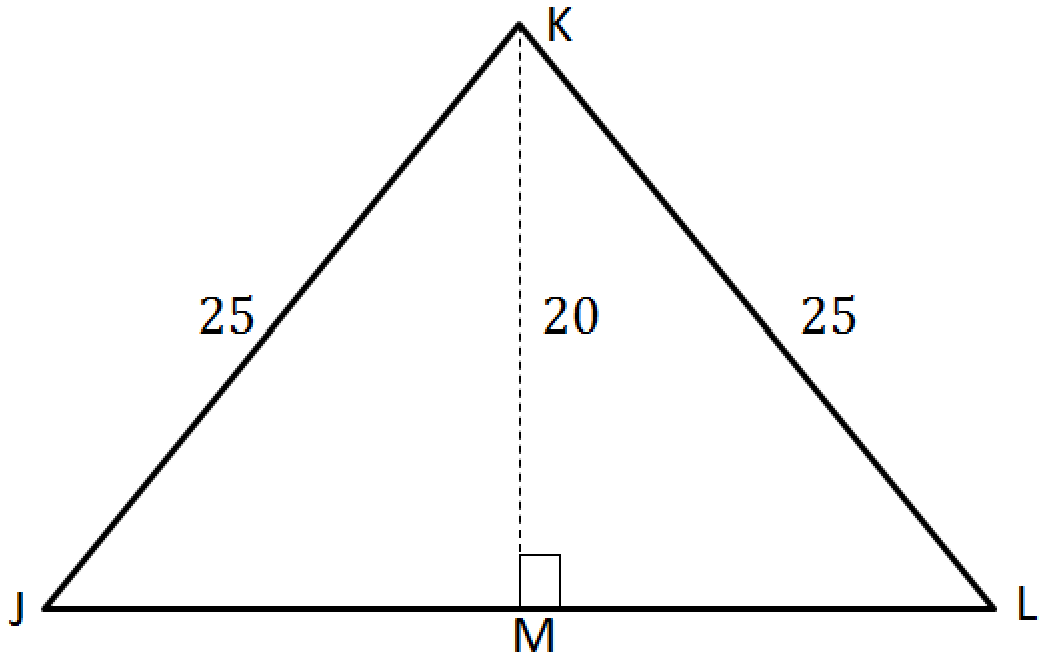
- How to classify triangles according to their sides and angles.
- What the Theorem of Pythagoras states about the relationship between the sides of a triangle.

- How to prove a triangle is right-angled using the Theorem of Pythagoras.
- How to find missing sides using the Theorem of Pythagoras.

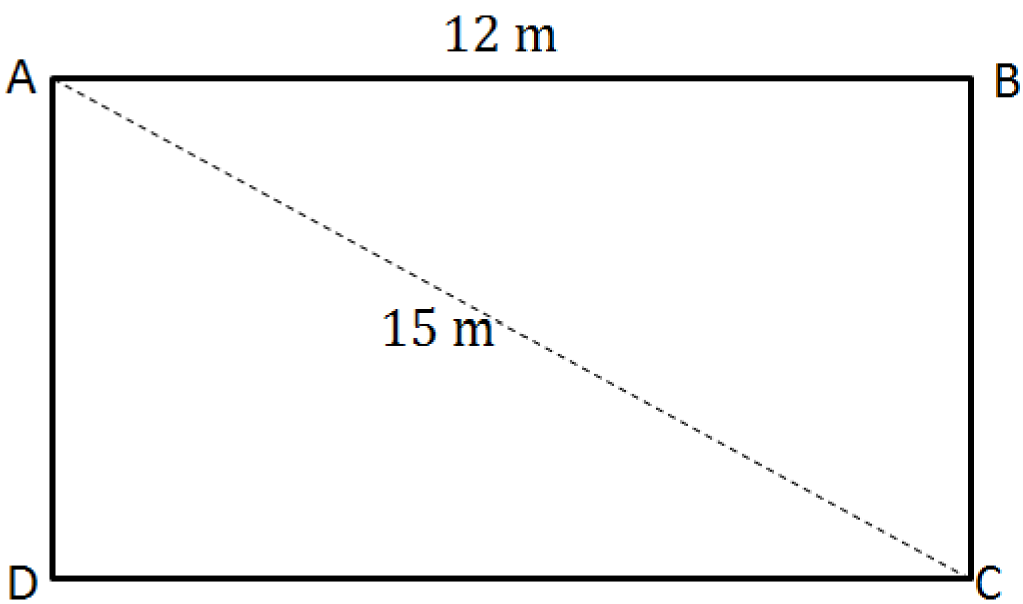
Unit 1: Assessment

Suggested time to complete: 20 minutes

1. In the diagram below, distance JK and distance KL are 25 metres while distance MK is 20 metres. Distance JM is equal to distance ML. Calculate the perimeter of the figure.

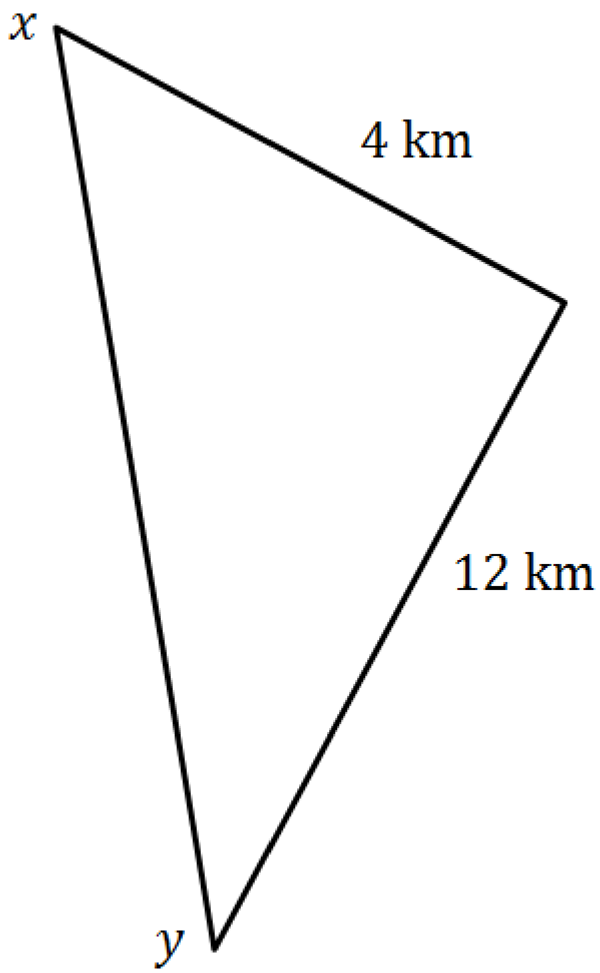


2. An insect is observed walking around a rectangular garden. The distance from A to B is 12 metres and the distance from A to C is 15 metres.



Calculate the total distance that the insect walked.

3. An ambulance leaves the hospital parking (x) to travel to an emergency (y). The path and distance of the journey are shown by the sketch below.



Calculate the straight line distance between points x and y.

The full solutions are at the end of the unit.

Unit 1: Solutions

Exercise 1.1

1.
$$\begin{aligned}(17\text{ mm})^2 &= 289\text{ mm}^2\\ \text{And}\\ (8\text{ mm})^2+(15\text{ mm})^2 &= 64\text{ mm}^2+225\text{ mm}^2\\ &= 289\text{ mm}^2\end{aligned}$$
Therefore, the triangle is right angled by the converse of Pythagoras.

2.

$$
\begin{aligned}
(20\ \text{mm})^2 &= 400\ \text{mm}^2\\
\text{And}\\
(16\ \text{mm})^2 + (12\ \text{mm})^2 &= 256\ \text{mm}^2 + 144\ \text{mm}^2\\
&= 400\ \text{mm}^2
\end{aligned}
$$

Therefore, the triangle is right angled by the converse of Pythagoras.

Back to Exercise 1.1

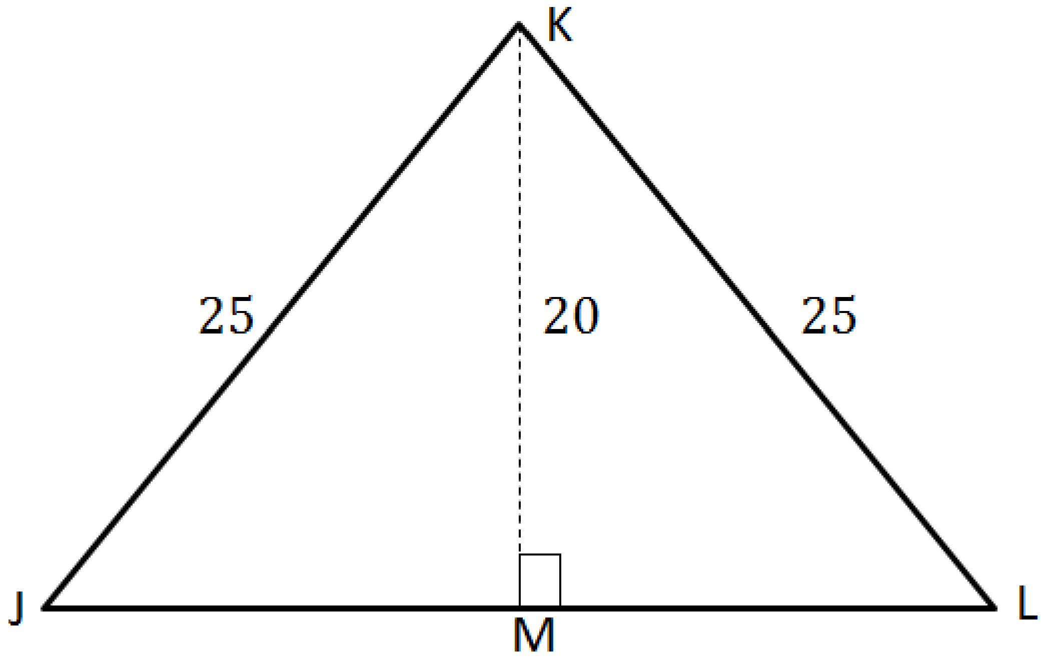
Exercise 1.2

1.
   a.
   $$
   \begin{aligned}
   \text{KL}^2 &= \text{KM}^2 + \text{LM}^2\ \text{(Pythagoras)}\\
   \therefore \text{LM}^2 &= \text{KL}^2 - \text{KM}^2\\
   &= (17)^2 - (15)^2\\
   &= 289 - 225\\
   &= 64\\
   \therefore \text{LM} &= \sqrt{64}\\
   &= 8
   \end{aligned}
   $$
   b.
   $$
   \begin{aligned}
   \text{PQ}^2 &= \text{QR}^2 + \text{PR}^2\ \text{(Pythagoras)}\\
   &= (55)^2 + (48)^2\\
   &= 5329\\
   \therefore \text{PQ} &= \sqrt{5329}\\
   &= 73
   \end{aligned}
   $$
2. A rectangle has interior angles each equal to 90° so we can use the Theorem of Pythagoras to find the length of the diagonal.
   $$
   \begin{aligned}
   (\text{Diagonal})^2 &= (36)^2 + (77)^2\ \text{(Pythagoras)}\\
   &= 7225\\
   \therefore \text{Diagonal} &= \sqrt{7225}\\
   &= 85\ \text{mm}
   \end{aligned}
   $$

Back to Exercise 1.2

Unit 1: Assessment

1. To calculate the perimeter of the figure we need to first find the length of ML.



In ΔKML

$$
\begin{aligned}
KM^2 + ML^2 &= KL^2 \ \text{(Pythagoras)} \\
ML^2 &= KL^2 - KM^2 \\
&= (25)^2 - (20)^2 \\
&= 225 \\
\therefore ML &= \sqrt{225} \\
&= 15 \\
JM &= ML \\
&= 15 \\
\text{Perimeter} &= JK + KL + JL \\
&= 25 + 25 + (15 + 15) \\
&= 80
\end{aligned}
$$

2. To find the distance the insect walked we must find the perimeter of the rectangular garden.

$$
\begin{aligned}
&AB = DC = 12 \text{ m} \quad \text{(opposite sides of a rectangle are equal)} \\
&BC = AD
\end{aligned}
$$

To find BC, use Pythagoras in ΔABC since $\hat{B} = 90°$.

$$
\begin{aligned}
AB^2 + BC^2 &= AC^2 \\
\therefore BC^2 &= AC^2 - AB^2 \\
&= 15^2 - 12^2 \\
&= 81 \\
\therefore BC &= \sqrt{81} \\
&= 9 \text{ m} \\
&= AD
\end{aligned}
$$

The insect walked:

$$
\begin{aligned}
&AB + BC + DC + AD = 12 \text{ m} + 9 \text{ m} + 12 \text{ m} + 9 \text{ m} \\
&= 42 \text{ m}
\end{aligned}
$$

3.

$$\begin{aligned}
(\mathrm{xy})^2 &= 12^2 + 4^2 \\
&= 160 \\
\therefore \mathrm{xy} &= \sqrt{160} && \text{Simplify the surd} \\
&= \sqrt{16 \times 10} \\
&= 4\sqrt{10}
\end{aligned}$$

$$\mathrm{xy} = 4\sqrt{10}\ \mathrm{km}$$

Back to Unit 1: Assessment

Media Attributions

SUBJECT OUTCOME XI

SPACE, SHAPE AND MEASUREMENT: SOLVE PROBLEMS BY CONSTRUCTING AND INTERPRETING TRIGONOMETRIC MODELS

Subject outcome 3.6

Solve problems by constructing and interpreting geometrical models.

Learning outcomes

- Define and use the following trigonometric functions: $\cos\theta$, $\sin\theta$, $\tan\theta$.
- Calculate trigonometric ratios in each of the quadrants where one ratio in that quadrant is given.
  - Example: If $\sin\theta = \frac{3}{5}$ and $90^\circ \le \theta \le 180^\circ$ determine $\cos\theta$.
- Solve problems in two dimensions using the trigonometric ratios $\cos\theta$, $\sin\theta$, $\tan\theta$.
- Express an appreciation of the contribution to the history of the development and the use of geometry and trigonometry by various cultures (NOT EXAMINABLE).

Unit 1: Trigonometric ratios

By the end of this unit you will be able to:

- Define and use the trigonometric ratios of $\cos\theta$, $\sin\theta$ and $\tan\theta$.
- Calculate the trigonometric ratios in each quadrant of the Cartesian plane.
- Calculate the value of expressions containing trigonometric ratios.

Unit outcomes: Unit 2: Problems in two dimensions (2D)

By the end of this unit you will be able to:

- Use a calculator to calculate the value of the three basic trig ratios for different angles.
- Solve problems in two dimensions (2D) using the trigonometric ratios $\cos\theta$, $\sin\theta$, $\tan\theta$.

Unit 1: Trigonometric ratios

DYLAN BUSA

Unit 1: Trigonometric ratios

By the end of this unit you will be able to:

- Define and use the trigonometric ratios of $\cos\theta$, $\sin\theta$ and $\tan\theta$.
- Calculate the trigonometric ratios in each quadrant of the Cartesian plane.
- Calculate the value of expressions containing trigonometric ratios.

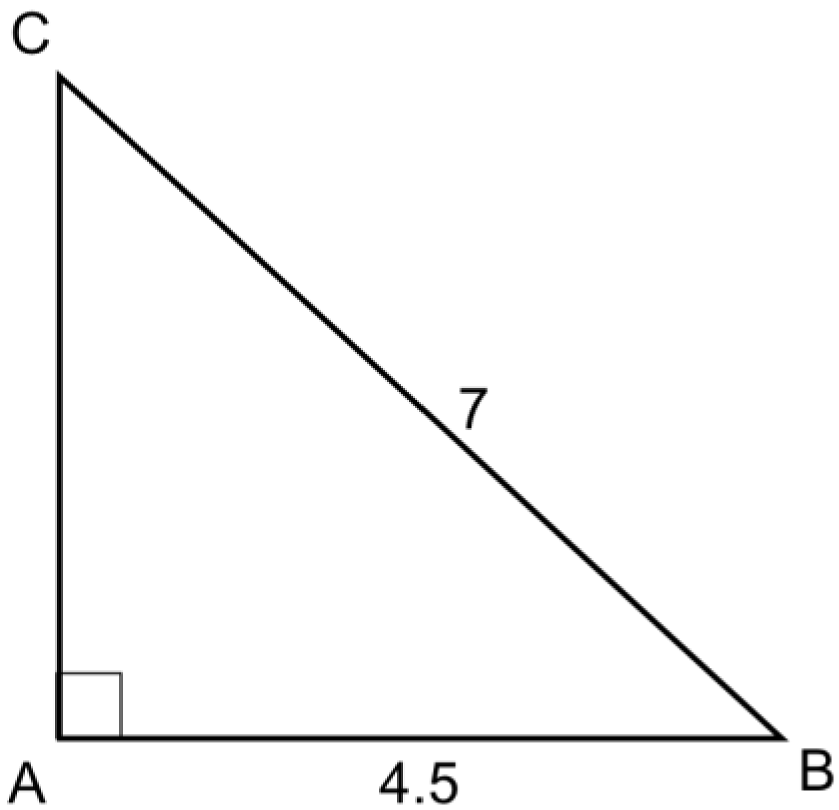
What you should know

Before you start this unit, make sure you can:

- Use the Theorem of Pythagoras to find missing lengths in right-angled triangles. Go over Subject outcome 3.5 Unit 1 if you need help with this.

Here is a short self-assessment to make sure you have the skills you need to proceed with this unit.

In $\Delta\mathrm{ABC}$, $\hat{\mathrm{A}} = 90^\circ$, $\mathrm{AB} = 4.5\ \mathrm{cm}$ and $\mathrm{BC} = 7\ \mathrm{cm}$. Calculate the length of AC.



Solution

$\Delta \mathrm{ABC}$ is a right-angled triangle. Therefore, we can use the Theorem of Pythagoras to calculate the length of the AC.

$$\begin{aligned} \mathrm{BC}^2 &= \mathrm{AB}^2 + \mathrm{AC}^2 \\ \therefore \mathrm{AC}^2 &= \mathrm{BC}^2 - \mathrm{AB}^2 \\ \therefore \mathrm{AC} &= \sqrt{\mathrm{BC}^2 - \mathrm{AB}^2} \\ &= \sqrt{7^2 - 4.5^2} \\ &= \sqrt{49 - 20.25} \\ &= \sqrt{28.75} \\ &= 5.36 \end{aligned}$$

Introduction

What is trigonometry (or 'trig' as it is sometimes called)? A clue lies in the first part of the word 'trigonometry'. 'Tri' means three; as in tricycle (three wheels) and triangle (three angles and three sides). Trigonometry deals with the relationships between the angles and sides of triangles.

Did you know?

The word trigonometry comes from the Greek words for triangle (trigōnon) and measure (metron).

Trigonometry

From about the year 150 A.D., Egyptians and Greeks used trigonometry to measure the distances between objects and places they could not measure directly. They even used it to measure the distances between stars and the circumference of the Earth. Trigonometry uses a technique called **triangulation** to measure distances.

Trigonometry has, therefore, always played a key part in navigation. Many of the modern applications of trigonometry still have to do with navigation. Modern satellite navigation systems still use trigonometry and triangulation to find the distances between landmarks.

Artist's concept of a NAVSTAR Global Positioning System satellite, a space-based radio navigation network.

Other fields that use trigonometry include acoustics, optics, financial market analysis, electronics, probability, statistics, biology, medical imaging, chemistry, cryptology, meteorology, oceanography, land surveying, architecture, phonetics, engineering, computer graphics, and game development, amongst many others.

To lay the foundation for understanding what trigonometry is and how it works, try this next activity.

Activity 1.1: The sides in similar triangles

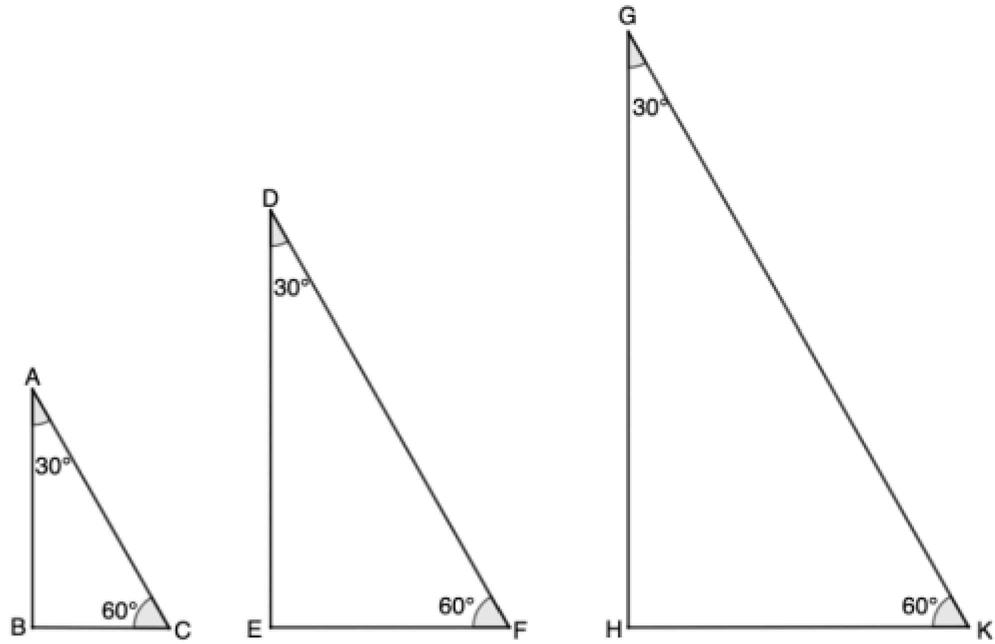
Time required: 20 minutes

What you need:

- a pen or pencil
- paper
- a protractor
- a ruler

What to do:

1. Draw three triangles of different sizes using a protractor and a ruler so that each triangle has interior angles equal to 30°, 60° and 90° as shown below. We call these 'similar triangles' because they are all the same shape, but not the same size. If you do not have a protractor or ruler, you can measure the lengths of the triangles below instead of your own.



2. Now measure the angles and lengths accurately and complete the following table. Leave all your ratios as fractions for now.

$\frac{\text{AB}}{\text{BC}} =$	$\frac{\text{AB}}{\text{AC}} =$	$\frac{\text{BC}}{\text{AC}} =$
$\frac{\text{DE}}{\text{EF}} =$	$\frac{\text{DE}}{\text{DF}} =$	$\frac{\text{EF}}{\text{DF}} =$
$\frac{\text{GH}}{\text{HK}} =$	$\frac{\text{GH}}{\text{GK}} =$	$\frac{\text{HK}}{\text{GK}} =$

3. What do you notice about each of the ratios? You can convert each ratio to a decimal if this helps you to compare. Does it matter what the lengths of the sides of the triangles are if the angles inside the three triangles stay the same size?
4. Draw one more right-angled triangle but this time make the other two angles 40° and 50°. Measure the lengths of the sides as before. Are the sides in the same ratio as the first three triangles that you drew?

What did you find?

You should have found that the value of each of the ratios of corresponding sides in each triangle was always the same. In other words, $\frac{\text{AB}}{\text{BC}} = \frac{\text{DE}}{\text{EF}} = \frac{\text{GH}}{\text{HK}}, \frac{\text{AB}}{\text{AC}} = \frac{\text{DE}}{\text{DF}} = \frac{\text{GH}}{\text{GK}}$ and $\frac{\text{BC}}{\text{AC}} = \frac{\text{EF}}{\text{DF}} = \frac{\text{HK}}{\text{GK}}$.

This means that it is the size of the angles in a triangle that determines the ratios between the lengths of the sides. It does not matter how big or small the triangle is. If the sizes of the angles stay the same, the sides will always be in the same ratio. If we change the sizes of the angles, we change the ratio of the sides.

The basic trigonometric ratios

We use the fact that it is the size of the angles in a triangle that determine the ratio of the sides as the basis of trigonometry and to define the three basic trigonometric ratios.

Have a look at the right-angled triangle in Figure 2. We can name the sides of the triangle with reference to the angle called θ (theta) at A as follows. The **hypotenuse** (AB) is always the side **opposite the right-angle**. The side **next to** θ (AC) is called the **adjacent** side (adjacent means next to) and the side **opposite** θ (BC) is called the **opposite** side.

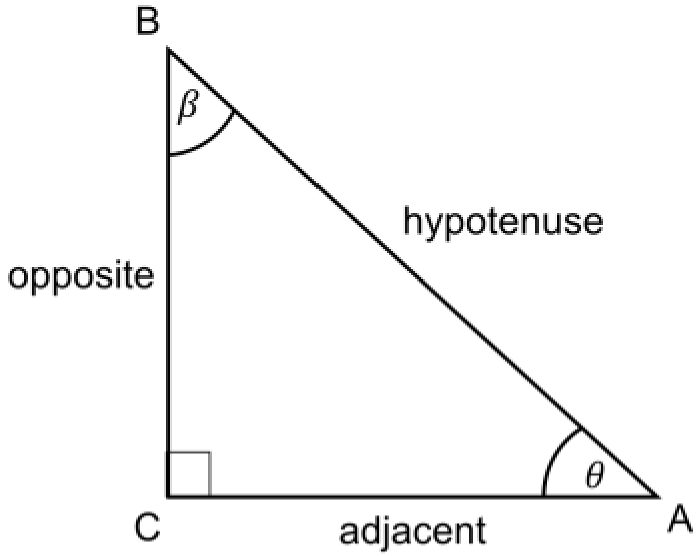


Figure 2: Names of the sides of a right-angled triangle with respect to the angle θ

If we defined the sides with respect to β (beta) at B, then the **adjacent** side would be BC and the **opposite** side would be AC. The hypotenuse would remain unchanged.

Note

The definitions of opposite, adjacent and hypotenuse are only applicable when working with right-angled triangles. Always check to make sure your triangle has a right-angle before you use them.

In Activity 1.1, we saw that we can define the three ratios of the lengths of the sides within any right-angled triangle. We can give each of these three ratios a special name – **sin**e, **cos**ine and **tan**gent – all with respect to the angle θ:

$$\sin\theta = \frac{\text{length of the opposite side}}{\text{length of the hypotenuse}}$$
$$\cos\theta = \frac{\text{length of the adjacent side}}{\text{length of the hypotenuse}}$$
$$\tan\theta = \frac{\text{length of the adjacent side}}{\text{length of the opposite side}}$$

In $\Delta\mathrm{ABC}$ in Figure 2, these three ratios would be defined as:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\mathrm{BC}}{\mathrm{AB}}$$
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\mathrm{AC}}{\mathrm{AB}}$$
$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\mathrm{BC}}{\mathrm{AC}}$$

In $\Delta\mathrm{ABC}$ in Figure 2, we could have also defined the three ratios with respect to β. Remembering that the sides BC and AC would then be 'adjacent' and 'opposite', reflecting their position relative to angle β, the ratios of β are:

$$\sin\beta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\mathrm{AC}}{\mathrm{AB}}$$
$$\cos\beta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\mathrm{BC}}{\mathrm{AB}}$$
$$\tan\beta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\mathrm{AC}}{\mathrm{BC}}$$

These three ratios of **sin**e, **cos**ine and **tan**gent form the basis of all of trigonometry.

Note

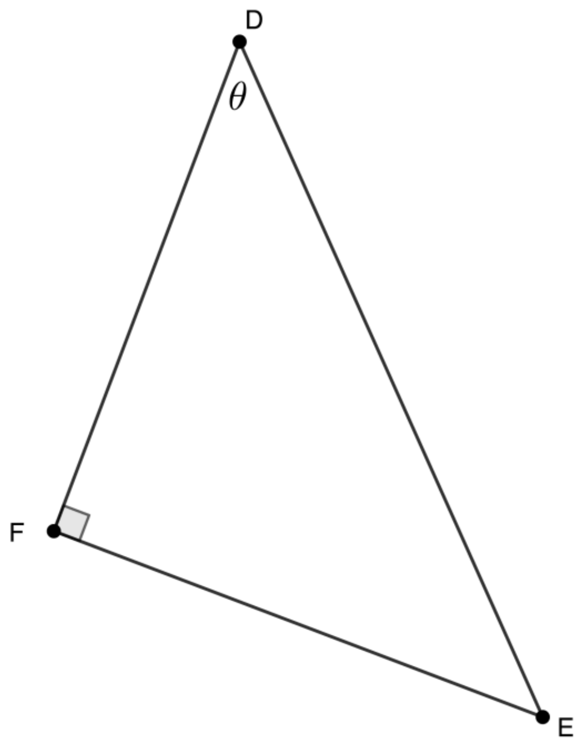
The three basic trigonometric ratios are:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \quad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \quad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

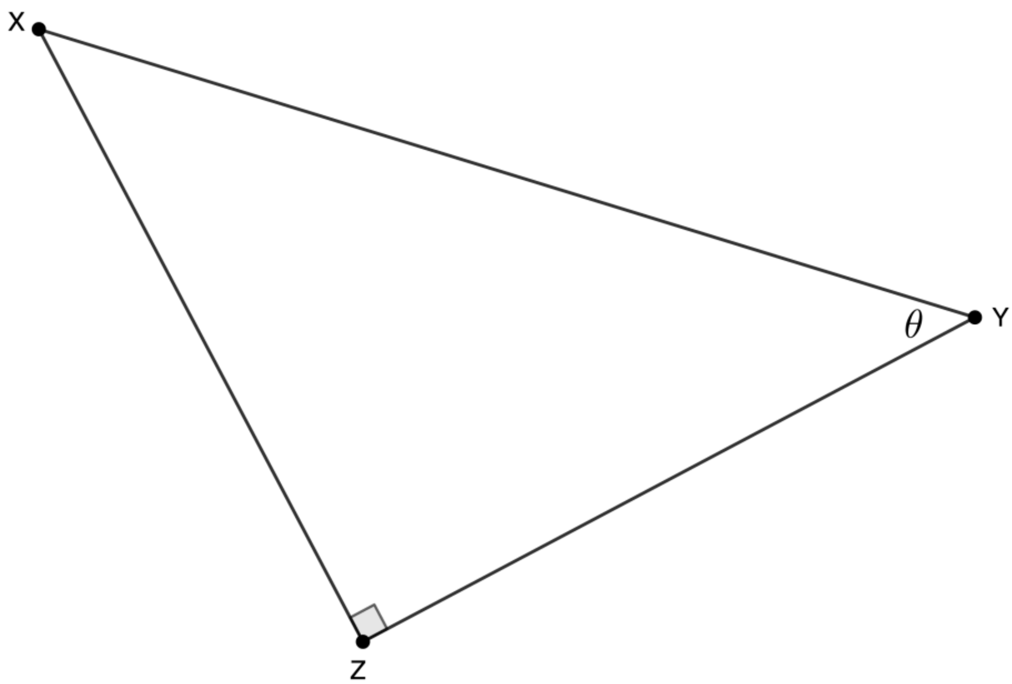
We never define the trigonometric ratios with respect to the right-angle.

Example 1.1

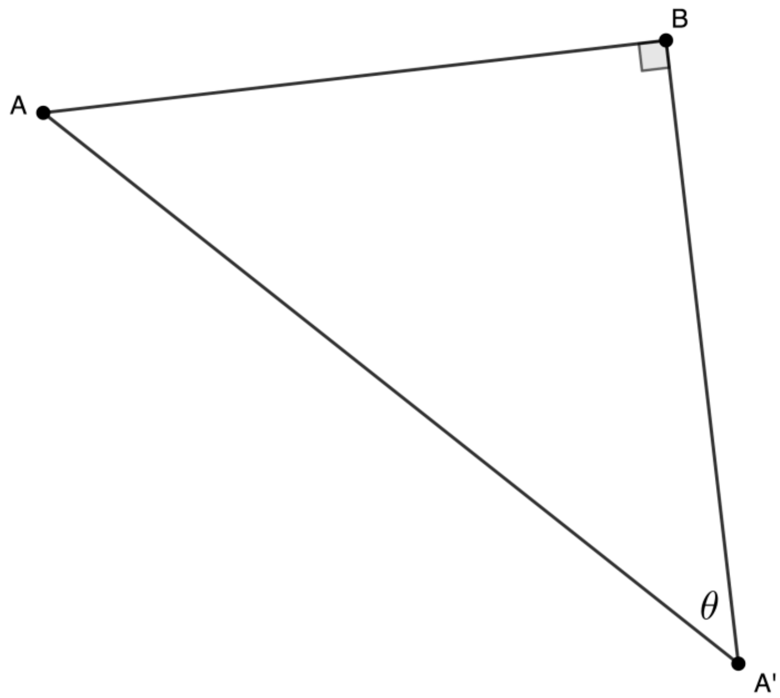
1. State the three basic trigonometric ratios for the following triangles with respect to the angle θ.
   a.



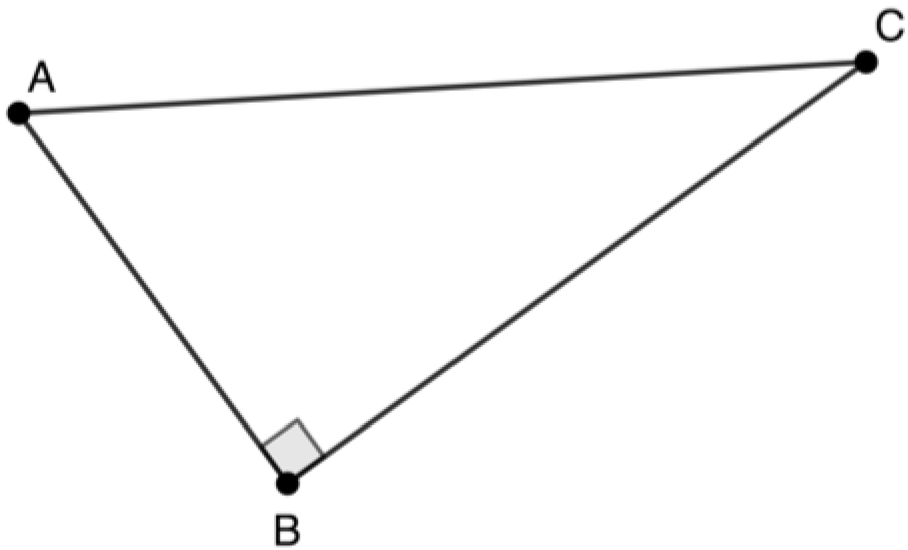
b.



c.



2. Complete each of the following for the given triangle:



a. $\sin \hat{A}$
b. $\cos \hat{A}$
c. $\tan \hat{A}$
d. $\sin \hat{C}$
e. $\cos \hat{C}$
f. $\tan \hat{C}$

Solutions

1.

a. $\sin\theta = \dfrac{\text{opp}}{\text{hyp}} = \dfrac{\text{EF}}{\text{DE}}$

$\cos\theta = \dfrac{\text{adj}}{\text{hyp}} = \dfrac{\text{DF}}{\text{DE}}$

$\tan\theta = \dfrac{\text{opp}}{\text{adj}} = \dfrac{\text{EF}}{\text{DF}}$

b. $\sin\theta = \dfrac{\text{opp}}{\text{hyp}} = \dfrac{\text{XZ}}{\text{XY}}$

$\cos\theta = \dfrac{\text{adj}}{\text{hyp}} = \dfrac{\text{YZ}}{\text{XY}}$

$\tan\theta = \dfrac{\text{opp}}{\text{adj}} = \dfrac{\text{XZ}}{\text{YZ}}$

c. $\sin\theta = \dfrac{\text{opp}}{\text{hyp}} = \dfrac{\text{AB}}{\text{AC}}$

$\cos\theta = \dfrac{\text{adj}}{\text{hyp}} = \dfrac{\text{BC}}{\text{AC}}$

$\tan\theta = \dfrac{\text{opp}}{\text{adj}} = \dfrac{\text{AB}}{\text{BC}}$

2.

a. $\sin \text{A} = \dfrac{\text{opp}}{\text{hyp}} = \dfrac{\text{BC}}{\text{AC}}$

b. $\cos \text{A} = \dfrac{\text{adj}}{\text{hyp}} = \dfrac{\text{AB}}{\text{AC}}$

c. $\tan \text{A} = \dfrac{\text{opp}}{\text{adj}} = \dfrac{\text{BC}}{\text{AB}}$

d. $\sin \text{C} = \dfrac{\text{opp}}{\text{hyp}} = \dfrac{\text{AB}}{\text{AC}}$

e. $\cos \text{C} = \dfrac{\text{adj}}{\text{hyp}} = \dfrac{\text{BC}}{\text{AC}}$

f. $\tan \text{C} = \dfrac{\text{opp}}{\text{adj}} = \dfrac{\text{AB}}{\text{BC}}$

Note

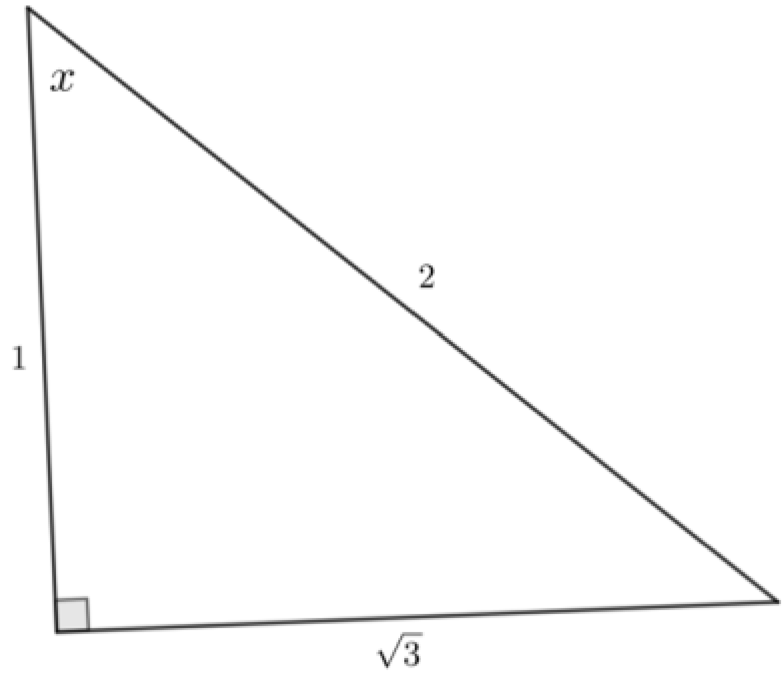
You can use **Soh Cah Toa** to help you remember how each of the trig ratios are defined.

Soh **Cah** **Toa**

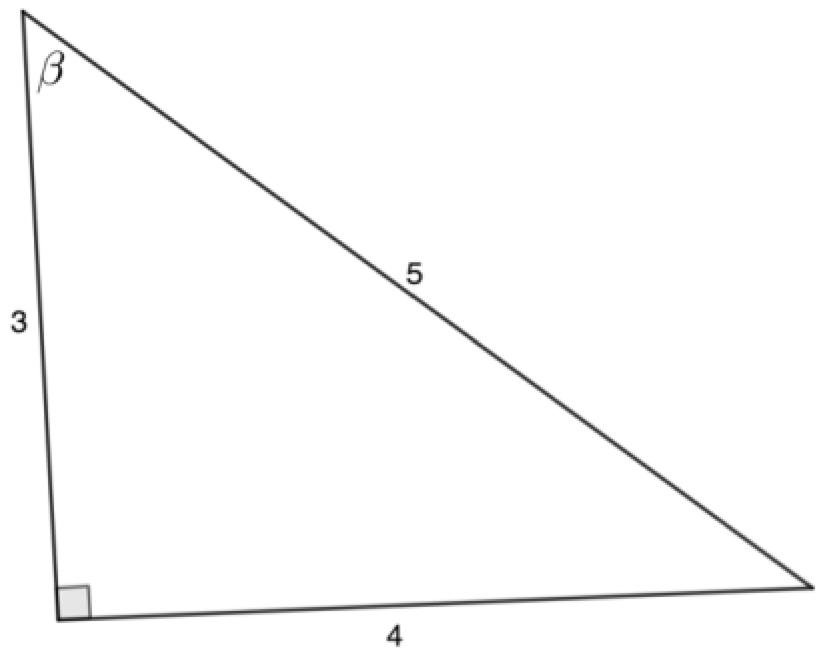
$\sin\theta = \dfrac{\text{opposite}}{\text{hypotenue}}$ $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenue}}$ $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$

Exercise 1.1

1. Define x in this triangle in three different ways.



2. Which of the following statements about β in the triangle is correct?



a. $\cos\beta = \dfrac{4}{5}$

b. $\sin\beta = \dfrac{4}{5}$

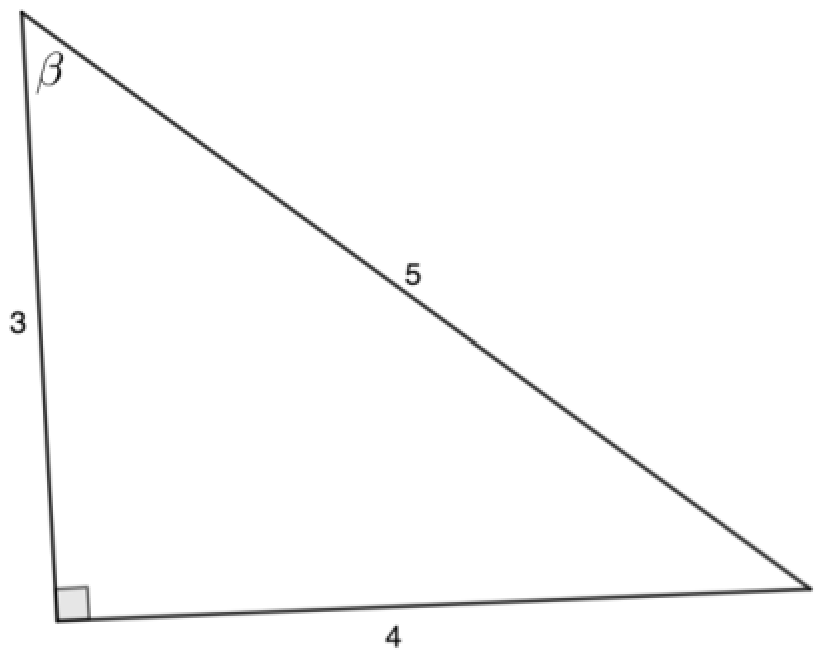
c. $\cos\left(\dfrac{3}{5}\right) = \beta$

d. $\tan\beta = 0.75$

The full solutions are at the end of the unit.

Evaluate basic trigonometric ratios

Have a look at this triangle from exercise 1.1 question 2 again.



We can see that $\sin\beta = \dfrac{4}{5}$. This means that the expression $\sin\beta$ has a definite value and it depends on the size of β. In the triangle above, what do you think $\sin\beta + \sin\beta$ is equal to?

Hopefully you said that $\sin\beta + \sin\beta = \dfrac{4}{5} + \dfrac{4}{5} = \dfrac{8}{5}$.
We could also say that $\sin\beta + \sin\beta = 2\sin\beta = 2 \times \dfrac{4}{5} = \dfrac{8}{5}$.

When we do calculations involving the three basic trig ratios, we can treat them the same way we treat numbers and variables. So long as we do not split the trig ratio from its angle, they obey all the normal arithmetic rules.

Therefore, $\sin\theta + \sin\theta = 2\sin\theta$ the same way that $x + x = 2x$. We treat $\sin\theta$ as a single entity. **We do not split it up**. This means that $\sin\theta + \sin\theta \neq \sin 2\theta$. $\sin 2\theta$ means that we are finding the sine of a bigger angle the size of two times the angle θ.

$\cos\alpha \times \cos\alpha \times \cos\alpha = \cos^3\alpha$ the same way that $x \times x \times x = x^3$. We position the exponent above the trig ratio.

This means that $\cos\alpha \times \cos\alpha \times \cos\alpha \neq \cos\alpha^3$. $\cos\alpha^3$ means that we are finding the cosine of the cube of angle α. We could say that $\cos\alpha \times \cos\alpha \times \cos\alpha = (\cos\alpha)^3$ but we tend not to write it this way.

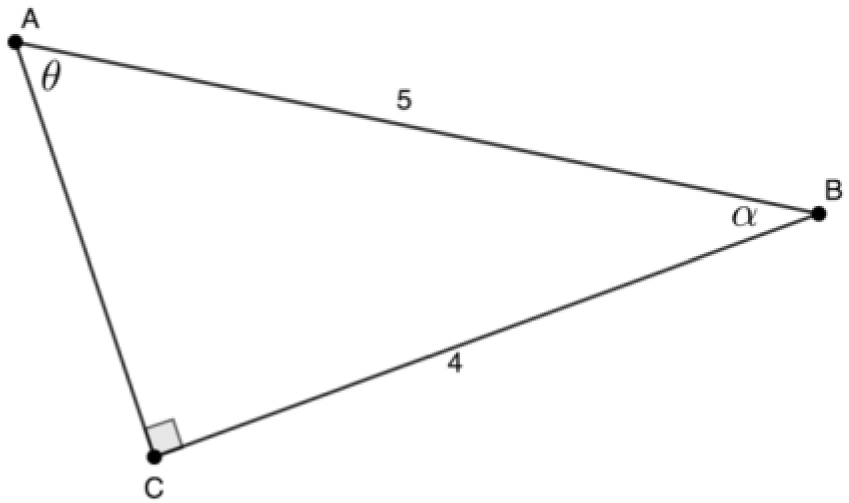
Note

- $\sin\theta + \sin\theta = 2\sin\theta$
- $3\sin\theta - \sin\theta = 2\sin\theta$
- $\sin\theta \times \sin\theta = \sin^2\theta$
- $\dfrac{\sin^3\theta}{\sin\theta} = \sin^2\theta$
- $\dfrac{2\sin\theta}{\sin\theta} = 2$

If you ever get confused, you can always replace the trig ratio and its operative angle with x.

Example 1.2

Given ΔABC with $\hat{C} = 90°$, $\hat{A} = \theta$ and $\hat{B} = \alpha$.



Determine the value of the following, showing all calculations.

1. $\cos\alpha$
2. $\tan\theta$
3. $1 + \tan^2\theta$
4. $\dfrac{\sin\alpha}{\cos\alpha}$

Solutions

1. $\cos\alpha = \frac{\text{adj}}{\text{hyp}} = \frac{\text{BC}}{\text{AB}} = \frac{4}{5}$

2. $\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{\text{BC}}{\text{AC}}$. But we do not have the length of AC, so we can use Pythagoras' Theorem to find AC.

$$\begin{aligned} \text{AB}^2 &= \text{AC}^2 + \text{BC}^2 \\ \therefore \text{AC}^2 &= \text{AB}^2 - \text{BC}^2 \\ \therefore \text{AC} &= \sqrt{\text{AB}^2 - \text{BC}^2} \\ &= \sqrt{5^2 - 4^2} \\ &= \sqrt{25 - 16} \\ &= \sqrt{9} \\ &= 3 \end{aligned}$$

Therefore, $\tan\theta = \frac{opp}{adj} = \frac{\text{BC}}{\text{AC}} = \frac{4}{3}$

3.

$$1 + \tan^2\theta = 1 + \left(\frac{4}{3}\right)^2$$

$$= 1 + \frac{16}{9}$$

$$= \frac{9 + 16}{9}$$

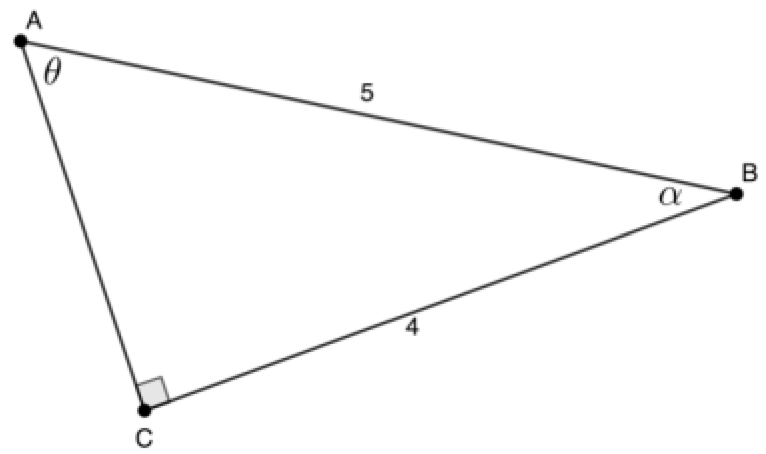
$$= \frac{25}{9}$$

4.

$$\frac{\sin\alpha}{\cos\alpha} = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{5} \times \frac{5}{4} = \frac{3}{4}$$

Exercise 1.2

Given ΔABC with $\hat{\text{C}} = 90^\circ$, $\hat{\text{A}} = \theta$ and $\hat{\text{B}} = \alpha$.



Determine the value of the following, showing all calculations.

1. $\sin\alpha$
2. $\cos\theta + \tan\theta$
3. $1 + \sin^2\theta$
4. $\sin^2\alpha + \cos^2\alpha$

The full solutions are at the end of the unit.

The trigonometric ratios on the Cartesian plane

One of the many powerful features of trigonometry is that it helps us explore things that repeat or cycle. To see why this is the case, we need to extend the definitions of the basic trig ratios to the Cartesian plane.

Let's start by plotting a point on the Cartesian plane. Let's plot the point $(3, 4)$ and then join this point to the origin with a straight line (see Figure 3). Let's also call the angle made between this line and the x-axis α.

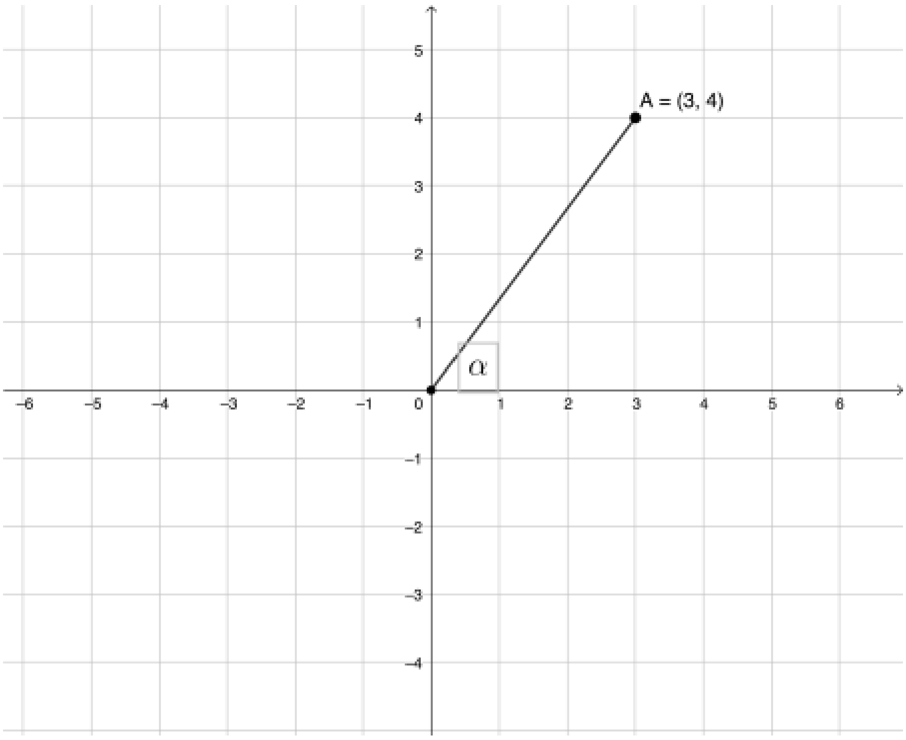


Figure 3

Now, if we drop a perpendicular line from A to the x-axis, we will form a right-angled triangle (see Figure 4). With respect to α, we can see that the length of the opposite side is given by the y-coordinate of A which is 4 units and the length of the adjacent side is given by the x-coordinate of A which is 3 units. So, in this triangle we can say that $\tan\alpha = \dfrac{\text{opp}}{\text{adj}} = \dfrac{y}{x} = \dfrac{4}{3}$.

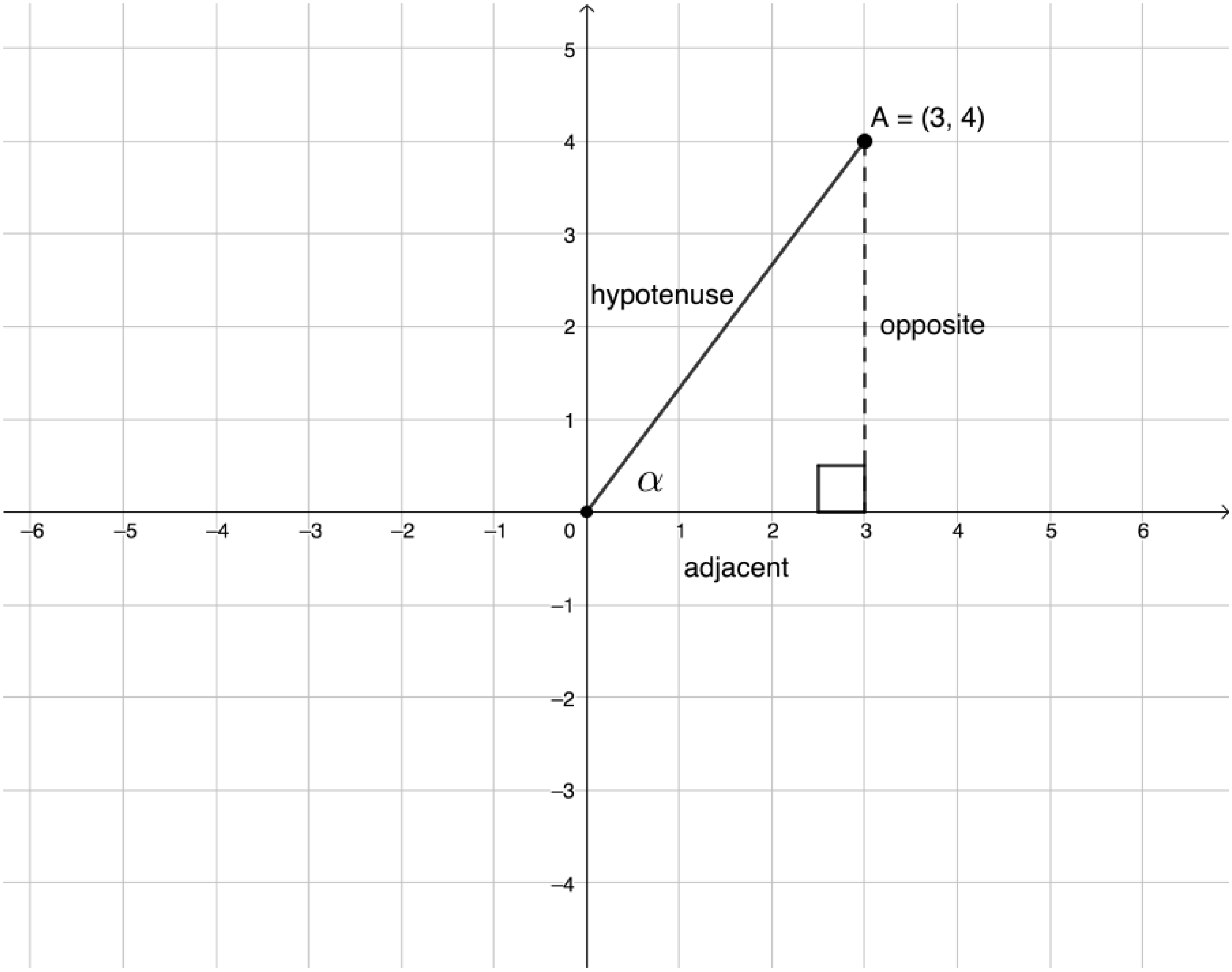


Figure 4

We can now use Pythagoras' Theorem to find that the length of the hypotenuse is 5 units, which we will call r. Hence, we see that $\sin\alpha = \frac{\text{opp}}{\text{hyp}} = \frac{y}{r} = \frac{4}{5}$ and $\cos\alpha = \frac{\text{adj}}{\text{hyp}} = \frac{x}{r} = \frac{3}{5}$.

Now, what if we draw a point A' symmetrical to A but in the second quadrant? The coordinates of A' would be $(4, -3)$. We could drop a perpendicular from A' to the x-axis. Obviously, the angle created inside this right-angled triangle would be the same size as α. Let's call it a (see Figure 5). But what would the values of the trig ratios be for this second quadrant triangle?

The length of r (the hypotenuse) would still be 5 because it is a length. Therefore, $\sin a = \frac{\text{opp}}{\text{hyp}} = \frac{y}{r} = \frac{4}{5}$.

However, $\cos a = \frac{\text{adj}}{\text{hyp}} = \frac{x}{r} = \frac{-3}{5} = -\frac{3}{5}$ and $\tan a = \frac{\text{opp}}{\text{adj}} = \frac{y}{x} = \frac{4}{-3} = -\frac{4}{3}$.

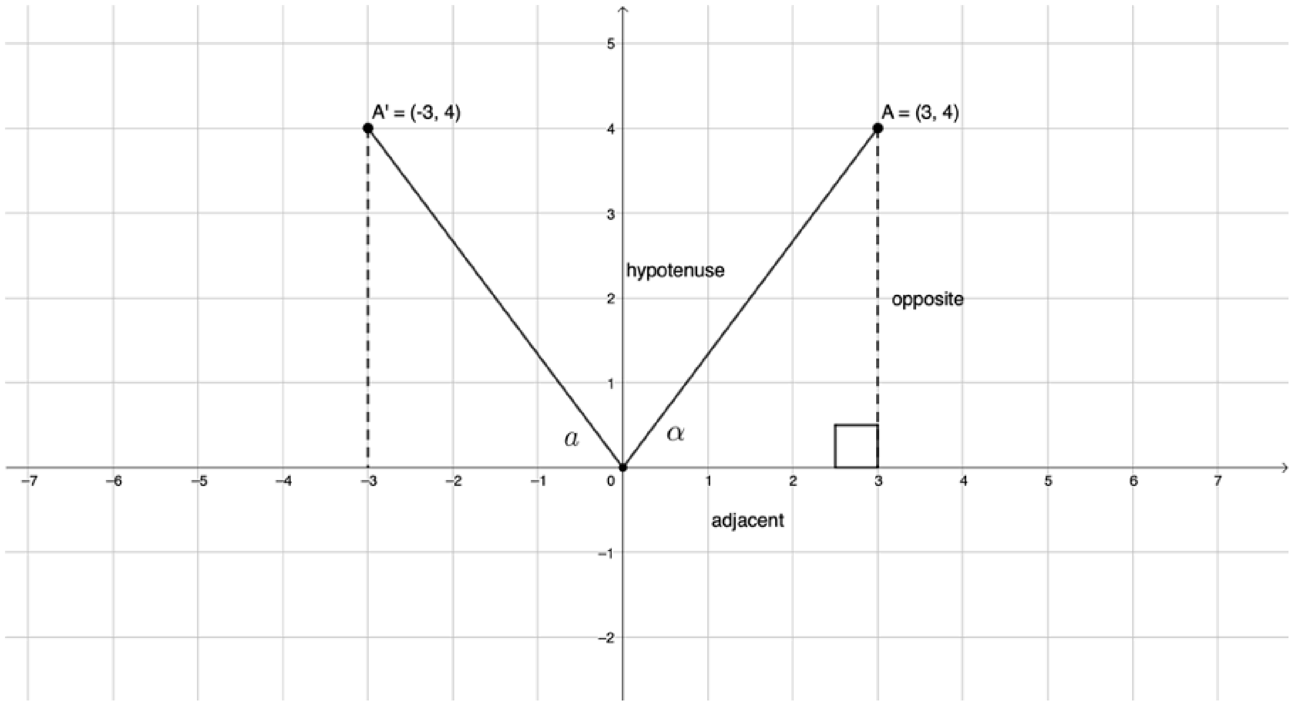


Figure 5

The values of the ratios for cosine and tangent are **numerically** the same but are **negative** because the value of x in the second quadrant is negative. Only **sine is positive**.

What do you think will happen to the signs of the three trig ratios if we reflect point A into the third and fourth quadrants? Make your own sketches to see if you can figure it out before you read on.

In the third quadrant, the values of x and y are both negative. Remember r is a length so is always positive.

$\sin a = \dfrac{\text{opp}}{\text{hyp}} = \dfrac{y}{r} = \dfrac{-4}{5} = -\dfrac{4}{5}$, $\cos a = \dfrac{\text{adj}}{\text{hyp}} = \dfrac{x}{r} = \dfrac{-3}{5} = -\dfrac{3}{5}$ and $\tan a = \dfrac{\text{opp}}{\text{adj}} = \dfrac{y}{x} = \dfrac{-4}{-3} = \dfrac{4}{3}$.

This time only **tangent is positive** (see Figure 6).

In the fourth quadrant, the y value is negative. Therefore,
$\sin a = \dfrac{\text{opp}}{\text{hyp}} = \dfrac{y}{r} = \dfrac{-4}{5} = -\dfrac{4}{5}$, $\cos a = \dfrac{\text{adj}}{\text{hyp}} = \dfrac{x}{r} = \dfrac{3}{5} = \dfrac{3}{5}$ and $\tan a = \dfrac{\text{opp}}{\text{adj}} = \dfrac{y}{x} = \dfrac{-4}{3} = -\dfrac{4}{3}$.

This time only **cosine is positive** (see Figure 6).

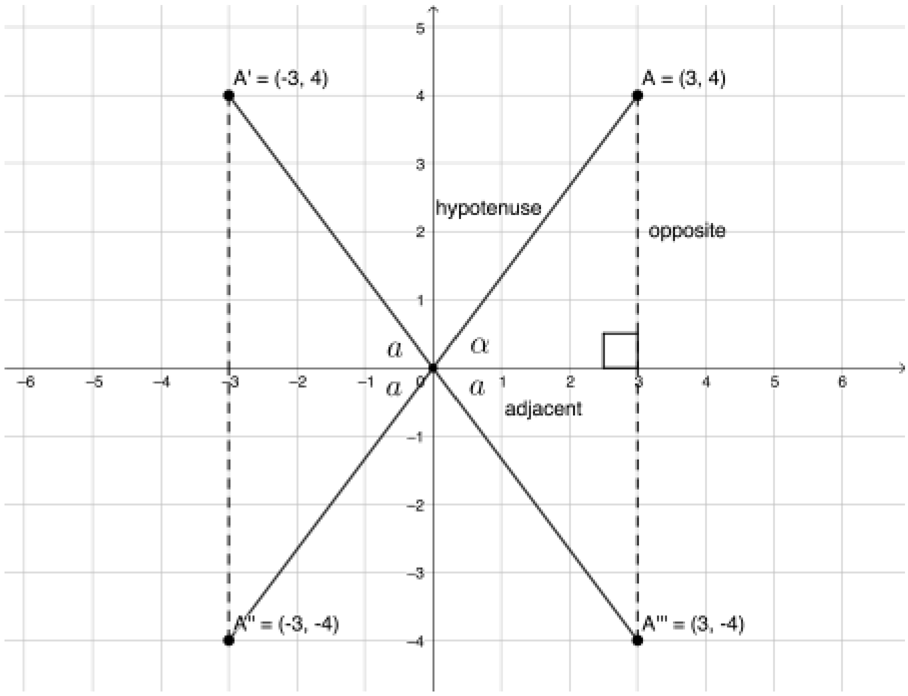


Figure 6

What we have actually done is to draw a circle, radius $r = 5$, with its centre at the origin (see Figure 7). As we move point A around the circumference of the circle, the radius makes an angle α with the positive x-axis. At any time, we can drop a perpendicular from A to the x-axis to create a right-angled triangle and calculate the trig ratios of this angle α.

As we have discovered, if A is in the first quadrant ($0^\circ < \alpha < 90^\circ$), then all the trig ratios are positive. If A is in the second quadrant ($90^\circ < \alpha < 180^\circ$) then only sine is positive. If A is in the third quadrant ($180^\circ < \alpha < 270^\circ$) then only tangent is positive and if A is in the fourth quadrant ($270^\circ < \alpha < 360^\circ$), then only cosine is positive.

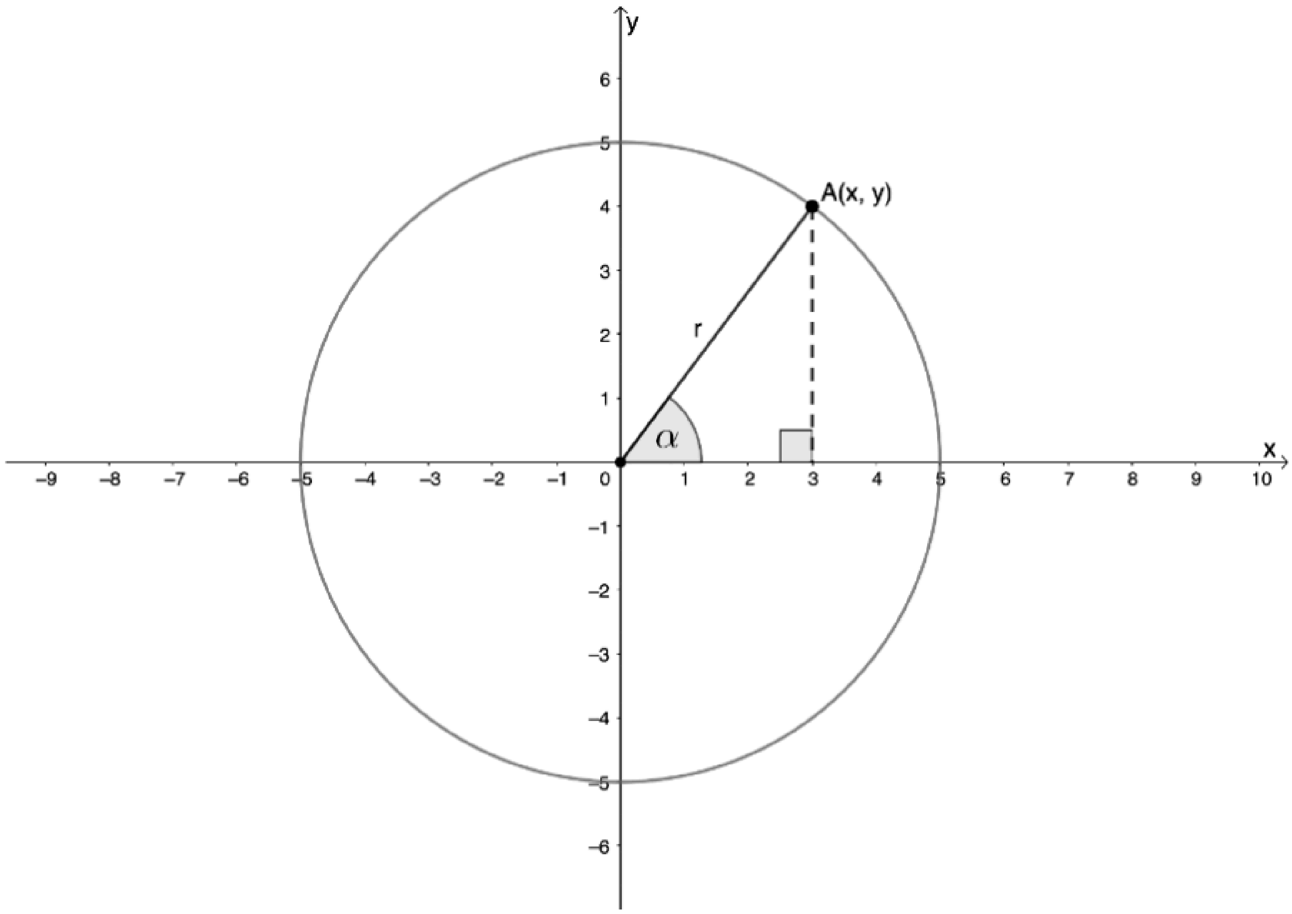


Figure 7: Circle radius 5 with centre the origin

There is a useful tool called the CAST diagram (see Figure 8) that helps us remember which trig ratios are positive in which quadrants.

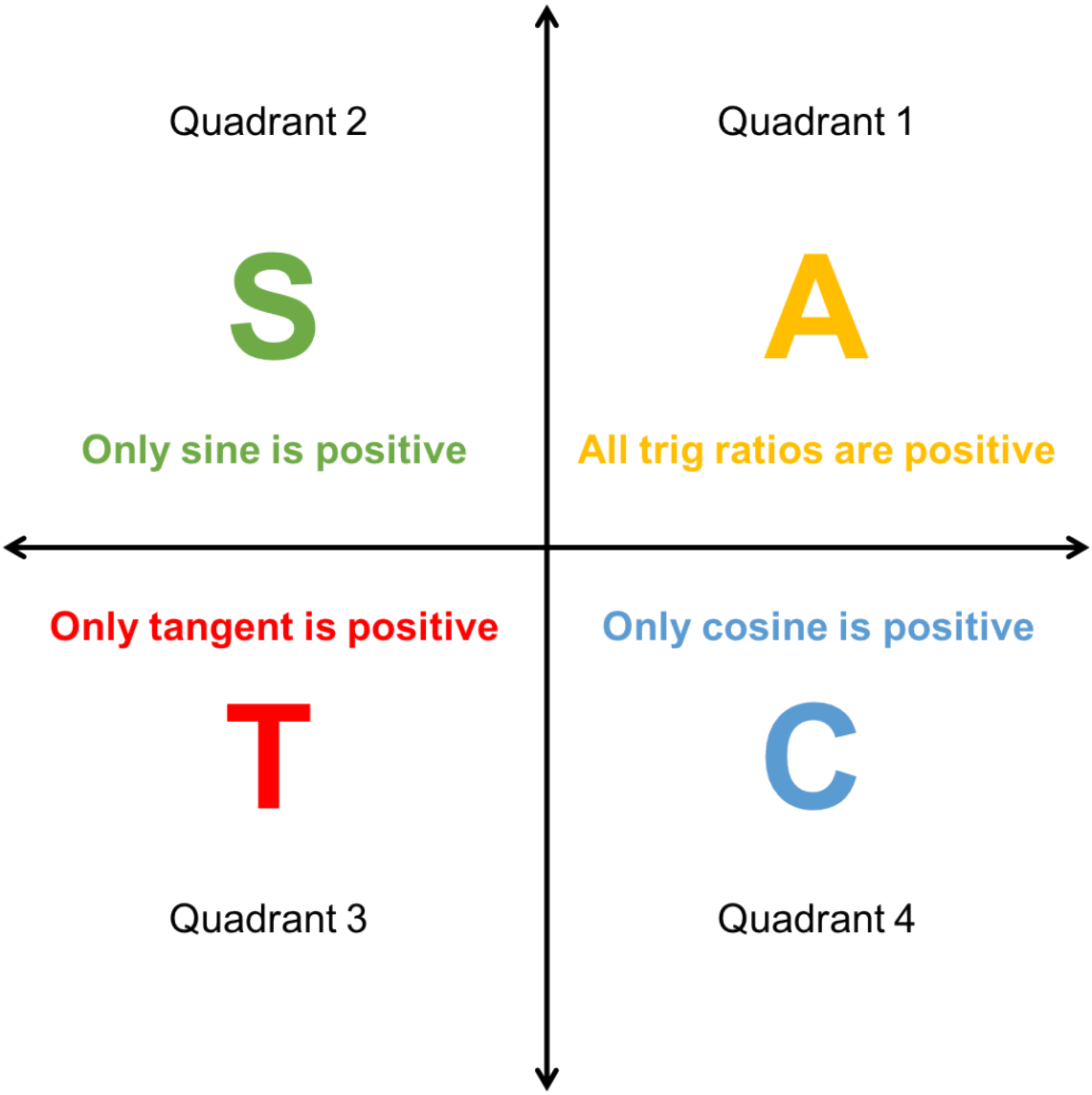
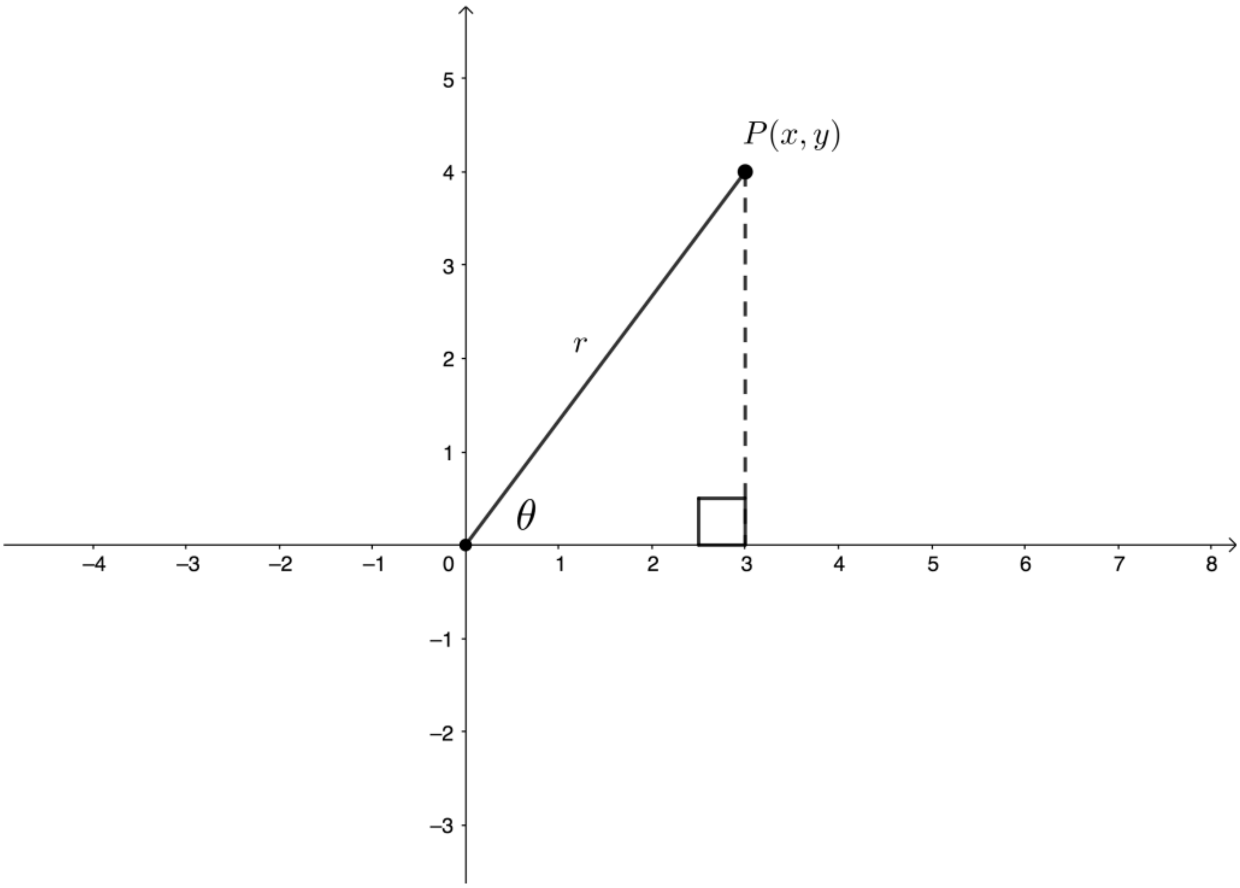


Figure 8: The CAST diagram

Take note!

On the Cartesian plane, we define the trig ratios based on the coordinates of a point $\mathrm{P}(x, y)$ and the angle (θ) made between the positive x-axis and the line (r) drawn from the origin to P.



$\sin\theta = \frac{y}{r}$ $\cos\theta = \frac{x}{r}$ $\tan\theta = \frac{y}{x}$

If you have an internet connection, spend some time playing with the interactive simulation called CAST diagram.

Click and drag point P to move it around the circle and notice how the values of the trig ratios change. Pay special attention to how their signs change as P moves from one quadrant to another.

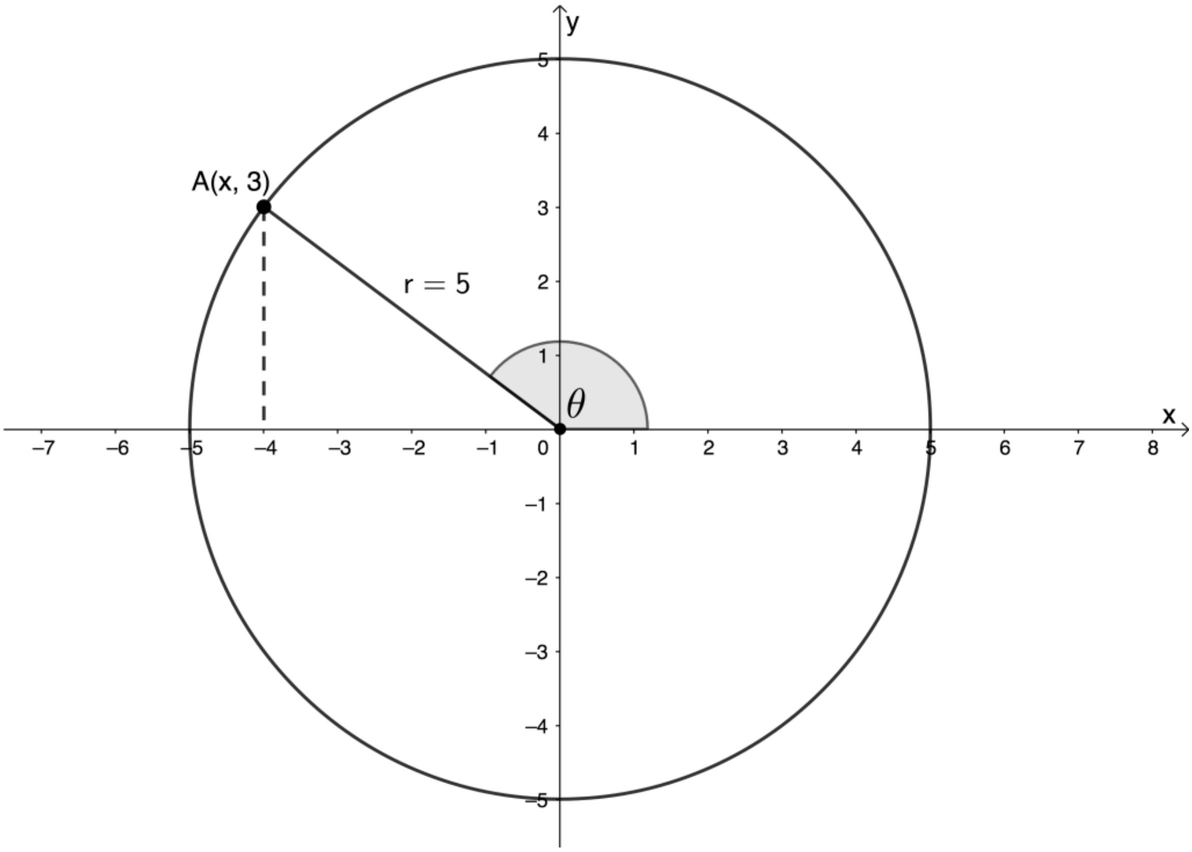
Example 1.3

1. If $\sin\theta = \frac{3}{5}$ and $90^\circ \leq \theta \leq 180^\circ$ determine:
   a. $\cos\theta$
   b. $\tan^2\theta + 1$

2. If $\tan \beta = -\sqrt{3}$ and $270° \leq \beta \leq 360°$ determine:
   a. $\sin \beta + \cos \beta$
   b. $\sin^2 \beta + \cos^2 \beta$
   c. $\dfrac{1}{2\cos \beta}$

Solutions

1. It is always best to make a quick sketch of the situation. We are told that $90° \leq \theta \leq 180°$. This means that we are working in the second quadrant. We are also told that $\sin \theta = \dfrac{3}{5}$. This means that $r = 5$ and the y-coordinate of the point on the circumference of the circle is 3.



We can use Pythagoras' Theorem to calculate the x-coordinate of A.

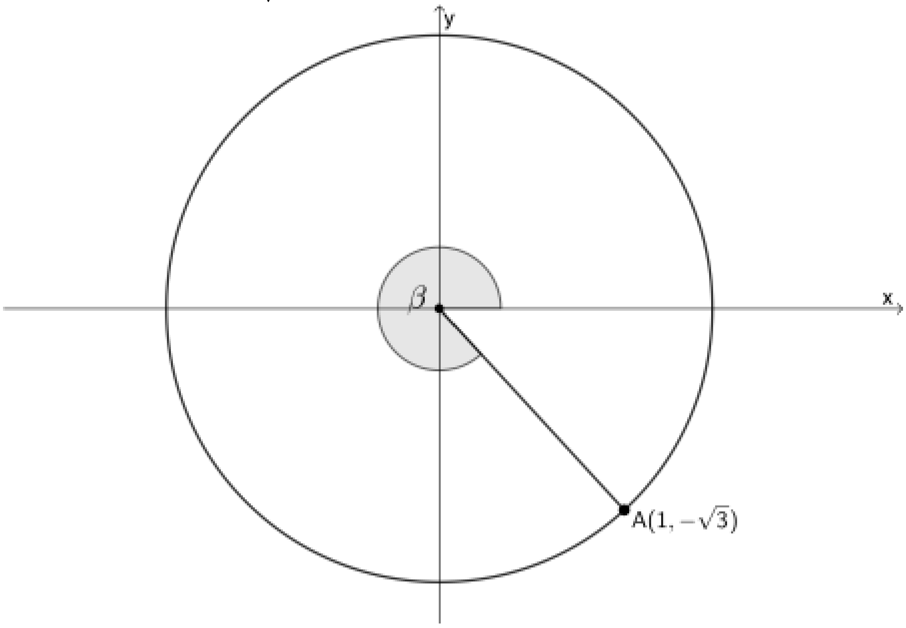
$$r^2 = x^2 + y^2$$
$$\therefore x^2 = r^2 - y^2$$
$$\therefore x = \sqrt{r^2 - y^2}$$
$$= \sqrt{25 - 9}$$
$$= \sqrt{16}$$
$$= \pm 4$$

But this is a point in the second quadrant so $x = -4$.

   a. $\cos \theta = \dfrac{x}{r} = \dfrac{-4}{5} = -\dfrac{4}{5}$
   b. $\tan \theta = \dfrac{y}{x} = \dfrac{3}{-4} = -\dfrac{3}{4}$

$$\therefore \tan^2\theta + 1 = \left(-\frac{3}{4}\right)^2 + 1$$
$$= \frac{9}{16} + 1$$
$$= \frac{9+16}{16}$$
$$= \frac{25}{16}$$

2. We are told that $270^\circ \leq \beta \leq 360^\circ$. This means that we are working in the fourth quadrant. We are also told that $\tan\beta = -\sqrt{3} = -\frac{\sqrt{3}}{1}$. This means that the y-coordinate of the point on the circumference of the circle is $-\sqrt{3}$ and the x-coordinate is 1.



We can use Pythagoras' Theorem to calculate the length of the radius of the circle.

$$r^2 = 1^2 + \left(-\sqrt{3}\right)^2$$
$$\therefore r = \sqrt{1+3} = 2$$

a.

$$\sin\beta + \cos\beta = \frac{-\sqrt{3}}{2} + \frac{1}{2}$$
$$= \frac{-\sqrt{3}+1}{2}$$

b.

$$\sin^2\beta + \cos^2\beta = \left(\frac{-\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2$$
$$= \frac{3}{4} + \frac{1}{4}$$
$$= 1$$

c.

$$\begin{aligned}\frac{1}{2\cos\beta} &= \frac{1}{2\left(\frac{1}{2}\right)} \\ &= \frac{1}{1} \\ &= 1\end{aligned}$$

Exercise 1.3

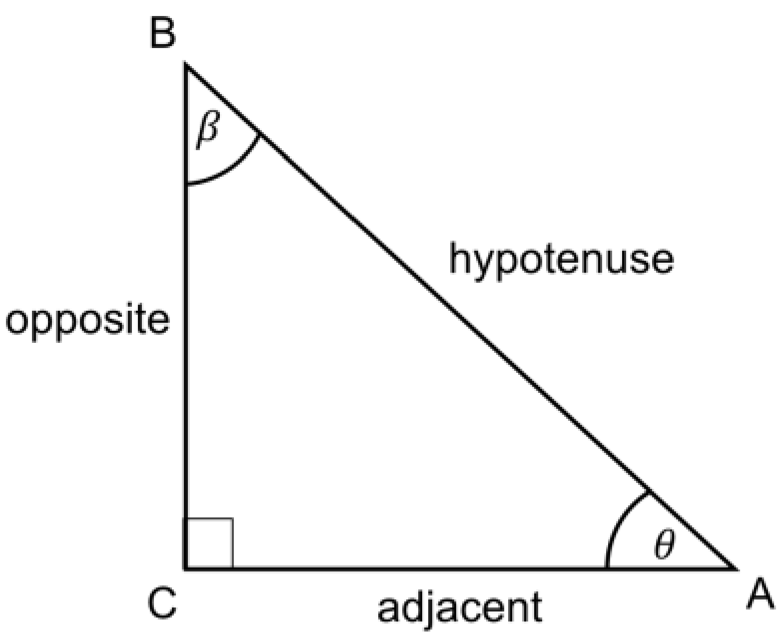
1. $\cos\alpha = -\frac{\sqrt{3}}{2}$ and $0^\circ \leq \alpha \leq 180^\circ$. Determine:
   a. $\tan^2\alpha - 1$
   b. $3\sin\alpha + 2(1+\cos\alpha)$
2. If $\tan\theta = \frac{6}{8}$ and $90^\circ \leq \theta \leq 360^\circ$, show that $\sin\theta(1+\cos\theta) = \frac{-3}{25}$.

The full solutions are at the end of the unit.

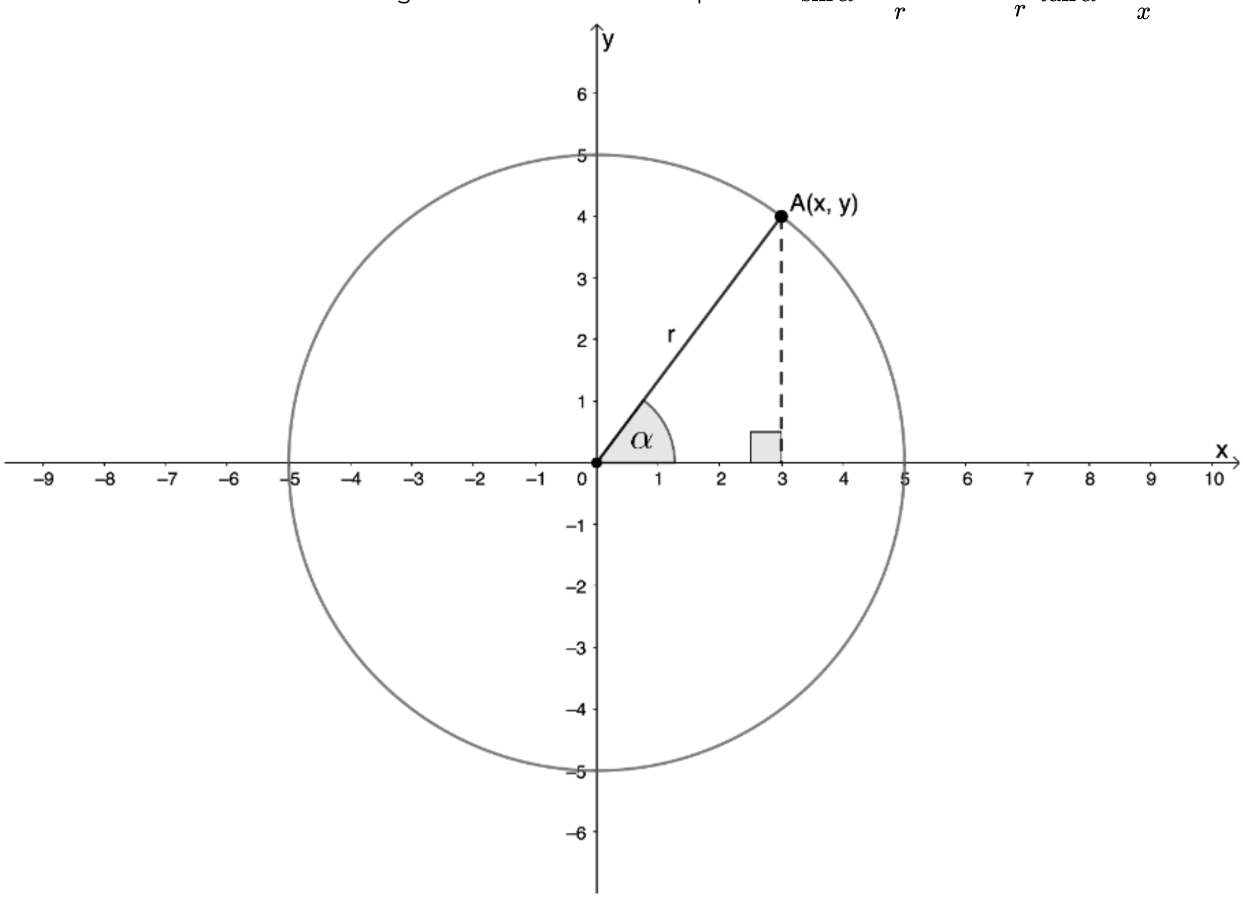
Summary

In this unit you have learnt the following:

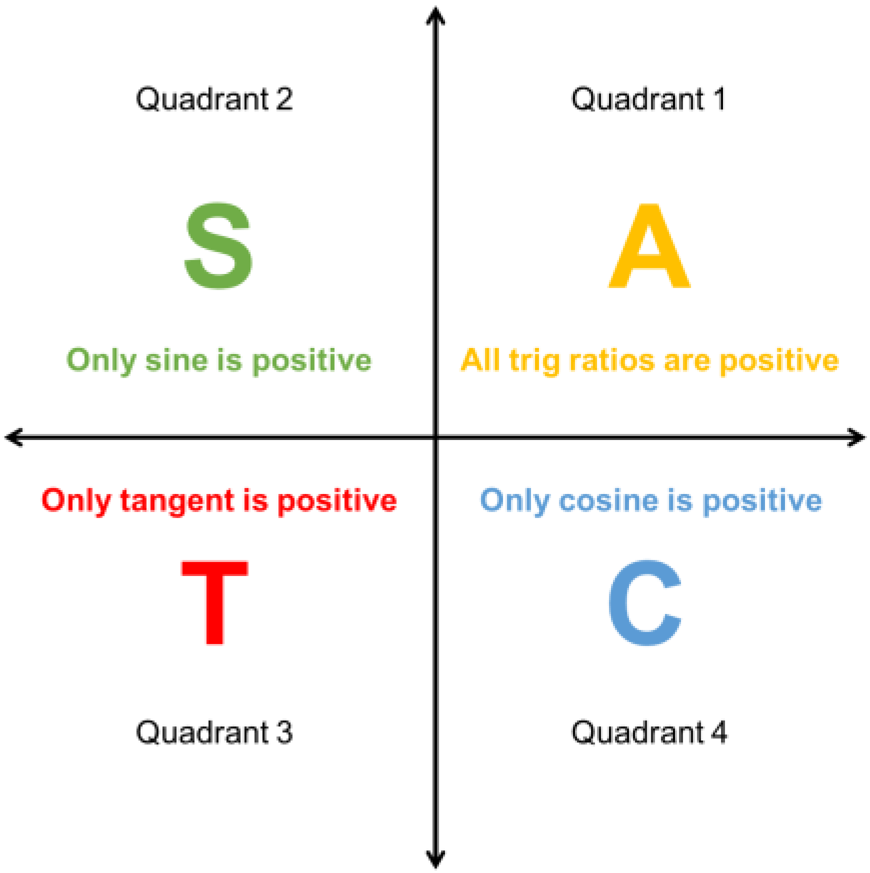
- The length of the sides of similar triangles are always in the same ratios.
- We define the three basic trig ratios in a right-angled triangle as:$\sin\theta = \frac{\text{opp}}{\text{hyp}}$ $\cos\theta = \frac{\text{adj}}{\text{hyp}}$ $\tan\theta = \frac{\text{opp}}{\text{adj}}$.



- We can define the three basic trig ratios on the Cartesian plane as: $\sin\alpha = \dfrac{y}{r}$ $\cos\alpha = \dfrac{x}{r}$ $\tan\alpha = \dfrac{y}{x}$



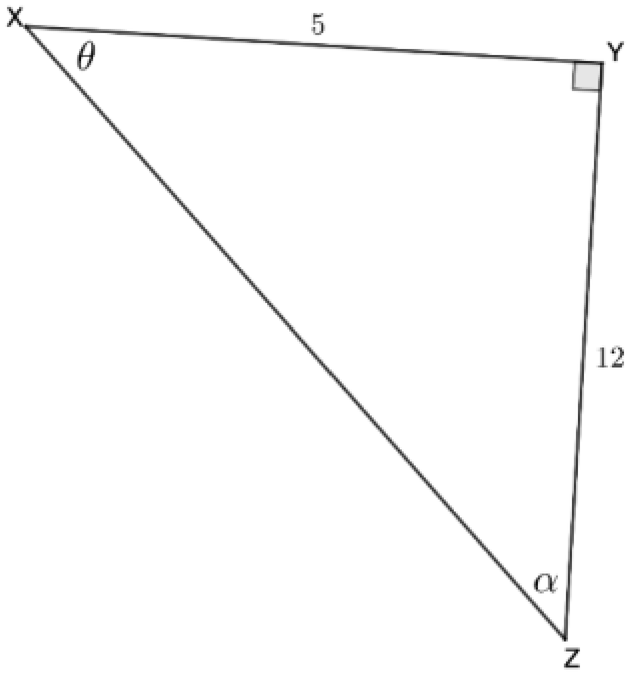
- All three basic trig ratios are positive in the first quadrant. Only sine is positive in the second quadrant. Only tangent is positive in the third quadrant. Only cosine is positive in the fourth quadrant.



Unit 1: Assessment

Suggested time to complete: 35 minutes

1. Given $\Delta\mathrm{XYZ}$ as shown.
   a. Determine the value of:



      i. $\sin\theta$
      ii. $\cos^2\alpha$

   b. Show that $(1+\tan\theta)(1-\tan\theta)=1-\tan^2\theta$

2. $\cos\theta=\dfrac{7}{25}$ and $90^\circ\leq\theta\leq360^\circ$. Determine the value of:
   a. $\sin\theta$
   b. $\tan\theta-\cos\theta$
   c. $\dfrac{\tan^2\theta}{\sin^2\theta}$

The full solutions are at the end of the unit.

Unit 1: Solutions

Exercise 1.1

1. $\sin x=\dfrac{\sqrt{3}}{2}$ $\cos x=\dfrac{1}{2}$ $\tan x=\dfrac{\sqrt{3}}{1}=\sqrt{3}$
2.

a. $\cos\beta = \frac{\text{adj}}{\text{hyp}} = \frac{3}{5} \neq \frac{4}{5}$ Therefore, this is not correct.

b. $\sin\beta = \frac{\text{opp}}{\text{hyp}} = \frac{4}{5}$ Therefore, this is correct.

c. $\cos\left(\frac{3}{5}\right) = \beta$ is not correctly stated. The trig ratios are defined in terms of the angle. In this case the angle is β and the ratio is $\frac{3}{5}$.

d. $\tan\beta = \frac{\text{opp}}{\text{adj}} = \frac{4}{3} \neq \frac{3}{4}$ or 0.75. Therefore, this is not correct.

Therefore only option b. is correct.

Back to Exercise 1.1

Exercise 1.2

1.

a.

$$\begin{aligned} \text{AB}^2 &= \text{AC}^2 + \text{BC}^2 \\ \therefore \text{AC}^2 &= \text{AB}^2 - \text{BC}^2 \\ \therefore \text{AC} &= \sqrt{\text{AB}^2 - \text{BC}^2} \\ &= \sqrt{25 - 16} \\ &= 3 \end{aligned}$$

$$\sin\alpha = \frac{3}{5}$$

b.

$$\sin\alpha = \frac{3}{5}$$

$$\begin{aligned} \cos\theta + \tan\theta &= \frac{3}{5} + \frac{3}{4} \\ &= \frac{15 + 12}{20} \\ &= \frac{37}{20} \end{aligned}$$

c.

$$\begin{aligned} 1 + \sin^2\theta &= 1 + \left(\frac{4}{5}\right)^2 \\ &= 1 + \frac{16}{25} \\ &= \frac{25 + 16}{25} \\ &= \frac{41}{25} \end{aligned}$$

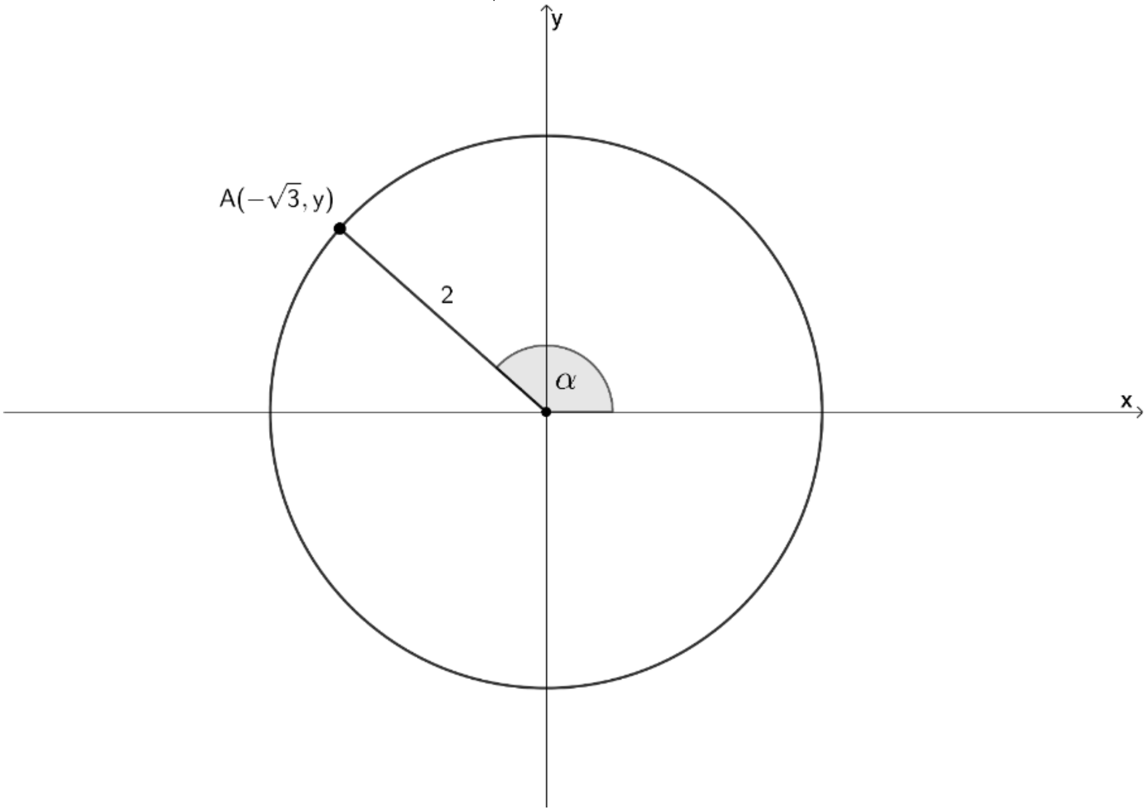
d.

$$\begin{aligned} \sin^2\alpha + \cos^2\alpha &= \left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 \\ &= \frac{9}{25} + \frac{6}{25} \\ &= \frac{25}{25} \\ &= 1 \end{aligned}$$

Back to Exercise 1.2

Exercise 1.3

1. $\cos\alpha = -\dfrac{\sqrt{3}}{2}$ and $0^\circ \le \alpha \le 180^\circ$. Therefore, the angle is in the second quadrant, $r = 2$ and the x-coordinate of the point on the circle is $-\sqrt{3}$.



$$\begin{aligned} r^2 &= x^2 + y^2 \\ \therefore y^2 &= r^2 - x^2 \\ \therefore y &= \sqrt{r^2 - x^2} \\ &= \sqrt{4-3} \\ &= 1 \end{aligned}$$

a.

$$\begin{aligned} \tan^2\alpha - 1 &= \left(\frac{1}{-\sqrt{3}}\right)^2 - 1 \\ &= \frac{1}{3} - 1 \\ &= -\frac{2}{3} \end{aligned}$$

b.

$$\begin{aligned} 3\sin\alpha + 2(1+\cos\alpha) &= 3\left(\frac{1}{2}\right) + 2\left(1 + \frac{(-\sqrt{3})}{2}\right) \\ &= \frac{3}{2} + 2\left(\frac{2-\sqrt{3}}{2}\right) \\ &= \frac{3}{2} + 2 - \sqrt{3} \\ &= \frac{7}{2} - \sqrt{3} \end{aligned}$$

2. If $\tan\theta = \dfrac{6}{8}$ and $90^\circ \le \theta \le 360^\circ$,then as tangent is positive, the angle must be in the third quadrant. The coordinates of the point on the circumference of the circle are $(-8, -6)$. Therefore, the

radius of the circle is 10 (using Pythagoras).

$$\begin{aligned}\sin\theta(1+\cos\theta) &= \frac{-6}{10}\left(1-\frac{8}{10}\right)\\ &= \frac{-6}{10}\left(\frac{2}{10}\right)\\ &= \frac{-12}{100}\\ &= \frac{4\times -3}{4\times 25}\\ &= \frac{-3}{25}\end{aligned}$$

Back to Exercise 1.3

Unit 1: Assessment

1.

a.

$$\begin{aligned}\mathrm{XZ}^2 &= \mathrm{XY}^2+\mathrm{YZ}^2\\ \therefore \mathrm{XZ} &= \sqrt{\mathrm{XY}^2+\mathrm{YZ}^2}\\ &= \sqrt{25+144}=\sqrt{169}\\ &= 13\end{aligned}$$

i. $\sin\theta = \dfrac{\text{opp}}{\text{hyp}} = \dfrac{12}{13}$

ii. $\cos^2\alpha = \left(\dfrac{12}{13}\right)^2 = \dfrac{144}{169}$

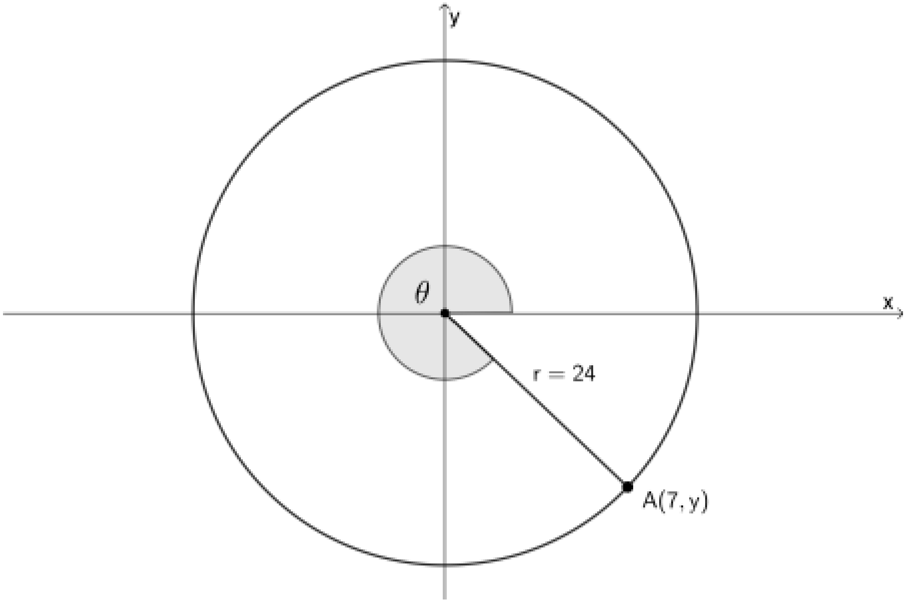
b.

$$\begin{aligned}(1+\tan\theta)(1-\tan\theta) &= \left(1+\frac{12}{5}\right)\left(1-\frac{12}{5}\right)\\ &= \left(\frac{5+12}{5}\right)\left(\frac{5-12}{5}\right)\\ &= \left(\frac{17}{5}\right)\left(\frac{-7}{5}\right)\\ &= -\frac{119}{25}\end{aligned}$$

$$\begin{aligned}1-\tan^2\theta &= 1-\left(\frac{12}{5}\right)^2\\ &= 1-\frac{144}{25}\\ &= \frac{25-144}{25}\\ &= -\frac{119}{25}\end{aligned}$$

Therefore, $(1+\tan\theta)(1-\tan\theta) = 1-\tan^2\theta$.

2. $\cos\theta = \dfrac{7}{25}$ and $90^\circ \leq \theta \leq 360^\circ$.



$$\begin{aligned} r^2 &= x^2 + y^2 \\ \therefore y^2 &= r^2 - x^2 \\ \therefore y &= \sqrt{r^2 - x^2} \\ &= \sqrt{25^2 - 7^2} \\ &= \sqrt{625 - 49} \\ &= \sqrt{576} \\ &= \pm 24 \end{aligned}$$

But the point is in the fourth quadrant. Therefore, $y = -24$.

a. $\sin\theta = -\dfrac{24}{25}$

b.

$$\begin{aligned} \tan\theta - \cos\theta &= -\frac{24}{7} - \frac{7}{25} \\ &= \frac{-600 - 49}{175} \\ &= -\frac{649}{175} \end{aligned}$$

c.

$$\begin{aligned} \frac{\tan^2\theta}{\sin^2\theta} &= \frac{\left(-\frac{24}{7}\right)^2}{\left(-\frac{24}{25}\right)^2} \\ &= \frac{\frac{576}{49}}{\frac{576}{625}} \\ &= \frac{576}{49} \times \frac{625}{576} \\ &= \frac{625}{49} \end{aligned}$$

You could take this result further as follows:

$$\frac{625}{49} = \frac{25^2}{7^2} = \left(\frac{25}{7}\right)^2 = \left(\frac{1}{7/25}\right)^2 = \left(\frac{1}{\cos\theta}\right)^2 = \frac{1}{\cos^2\theta}$$

Back to Unit 1: Assessment

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Unit 2: Problems in two dimensions (2D)

DYLAN BUSA

Unit outcomes: Unit 2: Problems in two dimensions (2D)

By the end of this unit you will be able to:

- Use a calculator to calculate the value of the three basic trig ratios for different angles.
- Solve problems in two dimensions (2D) using the trigonometric ratios $\cos\theta$, $\sin\theta$, $\tan\theta$.

What you should know

Before you start this unit, make sure you can:

- Define and use the trigonometric ratios of $\cos\theta$, $\sin\theta$ and $\tan\theta$.
- Calculate the value of expressions containing trigonometric ratios.

To revise these skills, work through Unit 1 of this Subject outcome.

Introduction

In Unit 1 we defined the three basic trig ratios of sine, cosine and tangent, and we evaluated some expressions that contained these ratios. But we never actually solved any real problems. In this unit, we are going to solve some practical problems with trigonometry. Many of the approaches that we will take are the same as those used every day in the real world.

Finding the lengths of unknown sides

It's time to start using trigonometry to solve real problems. One of the most commons types of problems you will encounter is to find the length of an unknown side in a right-angled triangle.

Have a look at the triangle in Figure 1. Because it is a right-angled triangle and because we know the length of two sides, we can easily find the length of the third side BC by using Pythagoras' Theorem.

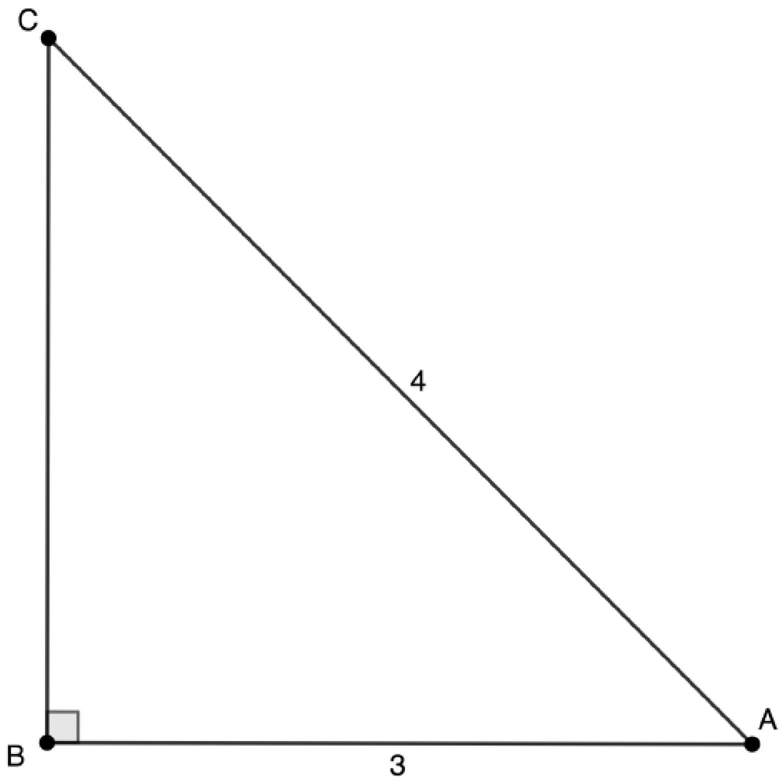


Figure 1

Now look at the triangle in Figure 2. In this case, we only know the length of one of the sides so we cannot use Pythagoras. But we do have another piece of information. We know the size of one of the non-right angles.

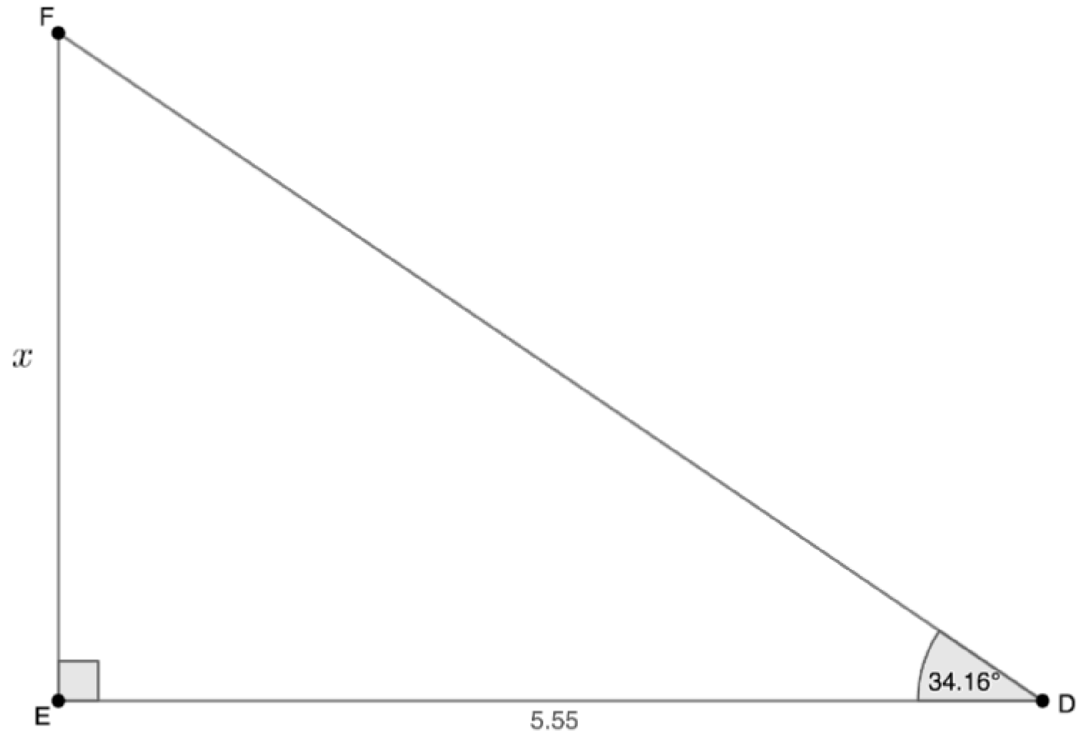


Figure 2

With respect to the known angle, which trig ratio combines the length of the side we know (DE) and the length of x, the side we are trying to find (EF)?

With respect to $\hat{\text{D}}$, we know the length of the **adjacent** side, and we want to find the length of the **opposite** side. Therefore, we can use tangent and say that $\tan 34.16^\circ = \frac{\text{opp}}{\text{adj}} = \frac{\text{EF}}{\text{DE}} = \frac{x}{5.55}$. In other words, $\tan 34.16^\circ = \frac{x}{5.55}$.

Now we have an equation that we can solve.

$$\begin{aligned} \tan 34.16^\circ &= \frac{x}{5.55} \\ \therefore x &= \tan 34.16^\circ \times 5.55 \end{aligned}$$

But what is the value of $\tan 34.16^\circ$? We can use any scientific calculator to find the value of any of the trig ratios for any angle. See the sequence of key presses required for different calculators. You will need to check which sequence your calculator uses.

Casio calculators (direct input): $\boxed{\tan} \rightarrow \boxed{34.16} \rightarrow \boxed{)} \rightarrow \boxed{=}$

Other calculators (indirect input): $\boxed{34.16} \rightarrow \boxed{\tan} \rightarrow \boxed{=}$

In either case, you should get that $\tan 34.16^\circ = 0.678579... \approx 0.679$. Note that in this unit we will mostly show the Casio calculator key sequence.

Note

In most situations the answers you get from your calculator for the value of a trig ratio are non-terminating, non-repeating decimals. For the greatest accuracy, round off only your **final answer** and round off to **three decimal places** unless told otherwise.

Now we can solve for x in our original equation.

$$\begin{aligned}\tan 34.16^\circ &= \frac{x}{5.55}\\ \therefore x &= \tan 34.16^\circ \times 5.55\\ &= 3.766\end{aligned}$$

$\boxed{\tan} \rightarrow \boxed{34.16} \rightarrow \boxed{)} \rightarrow \boxed{\times} \rightarrow \boxed{5.55} \rightarrow \boxed{=}$

Example 2.1

Use your calculator to calculate the following correct to three decimal places:

1. $\cos 58^\circ$
2. $2 \times \sin 13.56^\circ$
3. $2\sin^2 76.32^\circ$
4. $\dfrac{\cos 67.3^\circ}{\sin 41.97^\circ}$

Solutions

1. $\boxed{\cos} \rightarrow \boxed{58} \rightarrow \boxed{)} \rightarrow \boxed{=} \rightarrow 0.530$
2. $\boxed{2} \rightarrow \boxed{\times} \rightarrow \boxed{\sin} \rightarrow \boxed{13.56} \rightarrow \boxed{)} \rightarrow \boxed{=} \rightarrow 0.469$
3. $\boxed{2} \rightarrow \boxed{\times} \rightarrow \boxed{\sin} \rightarrow \boxed{76.32} \rightarrow \boxed{)} \rightarrow \boxed{x^2} \rightarrow \boxed{=} \rightarrow 1.888$
4. $\boxed{\cos} \rightarrow \boxed{67.3} \rightarrow \boxed{)} \rightarrow \boxed{\div} \rightarrow \boxed{\sin} \rightarrow \boxed{41.97} \rightarrow \boxed{)} \rightarrow \boxed{=} \rightarrow 0.577$

Note

When calculating the value of trig ratios, make sure that your calculator is in DEGREE mode. You should see a small D or DEG symbol on screen. If you do not, consult your calculator's manual for how to change back to DEGREE mode.

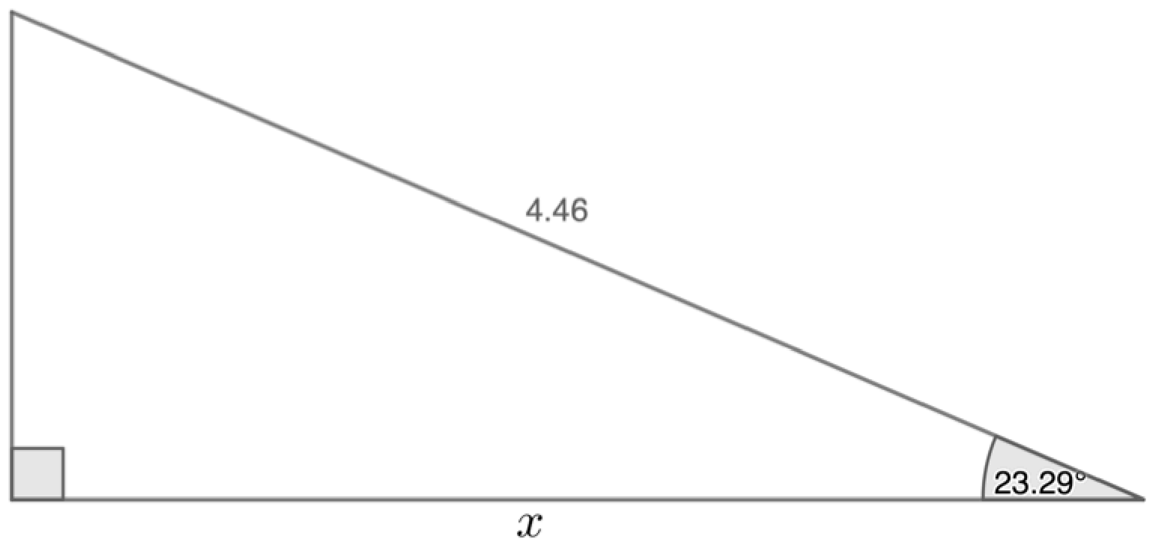
Exercise 2.1

1. $\tan 192°$
2. $\tan^2 12.1°$
3. $\sqrt{\sin 40.2°}$
4. $\sin^2 6.67° + \cos^2 267.7°$

The full solutions are at the end of the unit.

Example 2.2

Find the length of x (correct to three decimal places).



Solution

With respect to the angle that we have, we know the hypotenuse and we want the adjacent side. Cosine is the ratio that combines adjacent and hypotenuse.

$$\begin{aligned}\cos 23.29° &= \frac{x}{4.46} \\ \therefore x &= 4.46 \times \cos 23.29° \\ &= 4.097\end{aligned}$$

$\boxed{4.46} \rightarrow \boxed{\times} \rightarrow \boxed{\cos} \rightarrow \boxed{23.29} \rightarrow \boxed{)} \rightarrow \boxed{=} \rightarrow 4.097$

Note

If you would like to see some additional simple worked examples watch these two videos:

SOHCAHTOA – Finding Missing Sides PART 1 (Duration: 04.37)

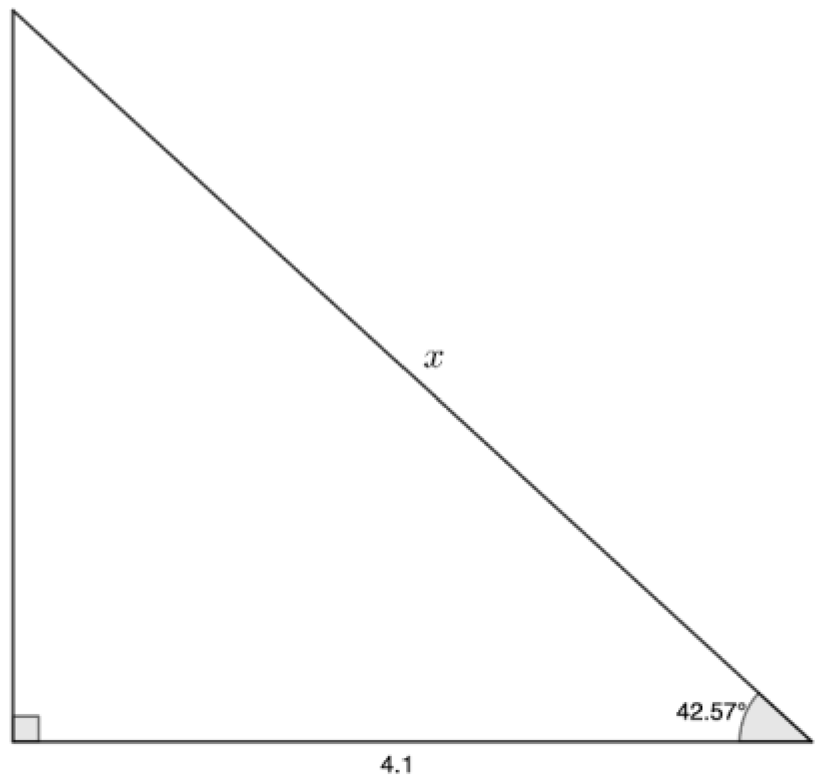
SOHCAHTOA – Finding Missing Sides PART 2 (Duration: 02.34)

Exercise 2.2

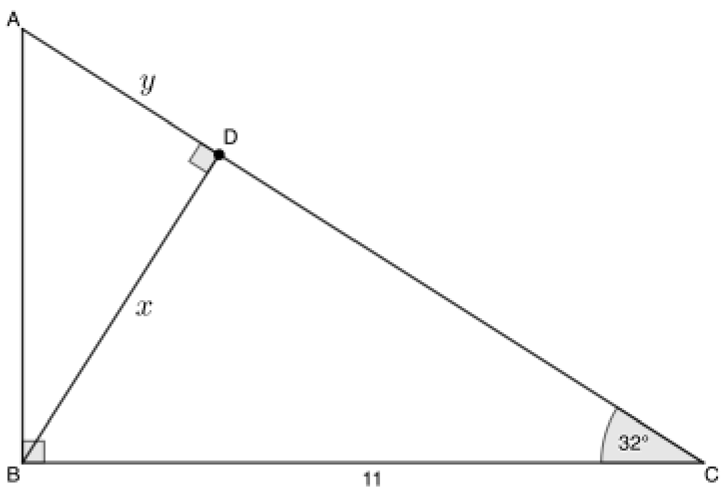
1. Find the length of x in the following triangles:

   a.

   b.



2. Calculate x and y.



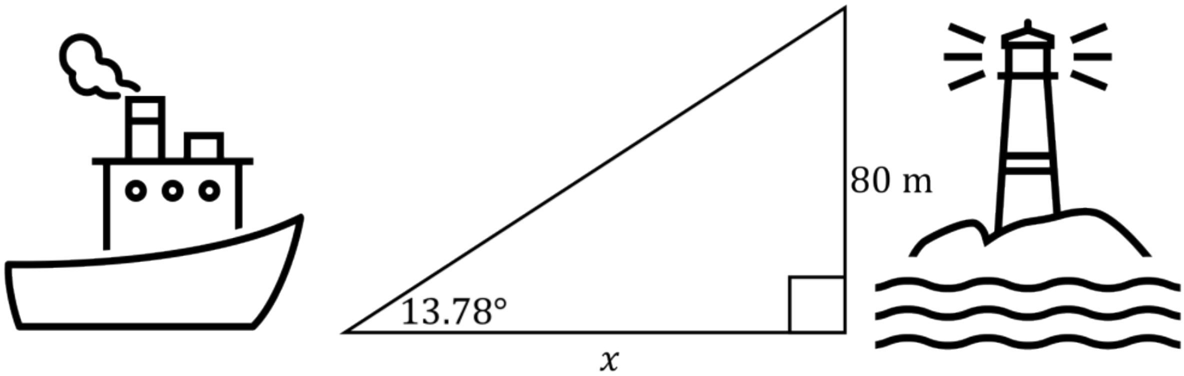
The full solutions are at the end of the unit.

Example 2.3

A lighthouse is 80 m tall. A ship out at sea measures the angle to the top of the lighthouse as $13.78°$. How far away is the ship from the lighthouse?

Solution

It is often best to draw a sketch of the situation and write down the information given.



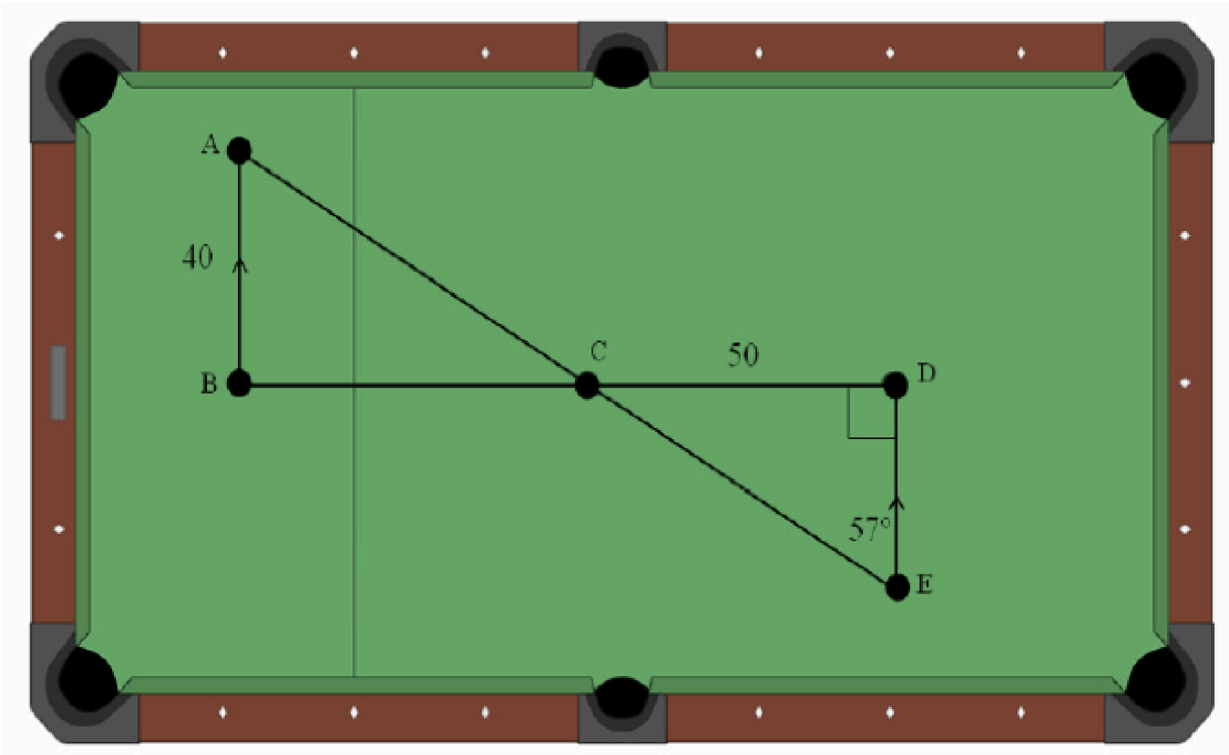
Relative to the angle that we have, we know the opposite side and we want the adjacent side. Tangent is the ratio that combines opposite and adjacent.

$$\begin{aligned} \tan 13.78° &= \frac{80}{x} \\ \therefore x &= \frac{80}{\tan 13.78°} \\ &= 326.193 \text{ m} \end{aligned}$$

Don't forget to include the units (metres) in your final answer.

Exercise 2.3

Five balls are placed on a pool table as shown in the figure below. The distance between Ball A and Ball B is 40 cm while the distance between Ball C and Ball D is 50 cm. Using the sketch below calculate the distance of Ball A from Ball E (AE) if AB is parallel to DE and $\text{C}\hat{\text{E}}\text{D} = 57°$.



The full solutions are at the end of the unit.

Finding the size of unknown angles

If $\sin 30^\circ = \frac{1}{2}$ then it means that if the length of the side opposite an angle and the hypotenuse in a right-angled triangle are in the ratio of 1 : 2, then the angle must be 30°.

Just like we can use our calculators to find the ratio of the sides when we are given an angle, we can also use them to find the angle if we know the ratio of the sides.

If we want to calculate the angle (θ) that makes the side opposite it and the hypotenuse to be in the ratio 0.652, we write that as $\sin\theta = 0.652$. Remember that $\sin\theta$ is one entity. We cannot split it. We use a calculator to find the size of θ as follows:

Casio calculators (direct input): $\boxed{\text{shift}/\text{2nd function}} \rightarrow \boxed{\sin} \rightarrow \boxed{0.652} \rightarrow \boxed{)} \rightarrow \boxed{=} \rightarrow 40.693^\circ$

Other calculators (indirect input): $\boxed{0.652} \rightarrow \boxed{\text{shift}/\text{2nd function}} \rightarrow \boxed{\sin} \rightarrow \boxed{=} \rightarrow 40.692^\circ$

By pressing the SHIFT or 2nd FUNC key, we gain access to the inverse sine or sin-1 function which returns an angle when given a ratio (see Figure 3).

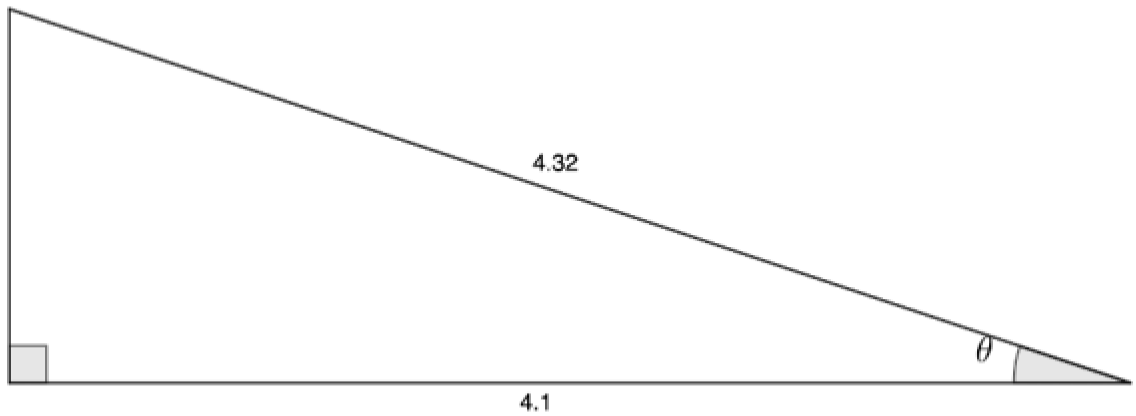
Figure 3: Casio fx-991ZA calculator

Note

When calculating the value of an angle, make sure that your calculator is in DEGREE mode. You should see a small D or DEG symbol on screen. If you do not, consult your calculator's manual for how to change back to DEGREE mode.

Example 2.4

Calculate the value of θ.



Solution

With respect to the angle we want to find, we know the adjacent side and the hypotenuse. Cosine is the ratio that combines adjacent and hypotenuse.

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{4.1}{4.32} \approx 0.949$$
$$\therefore \theta = 18.364^\circ$$

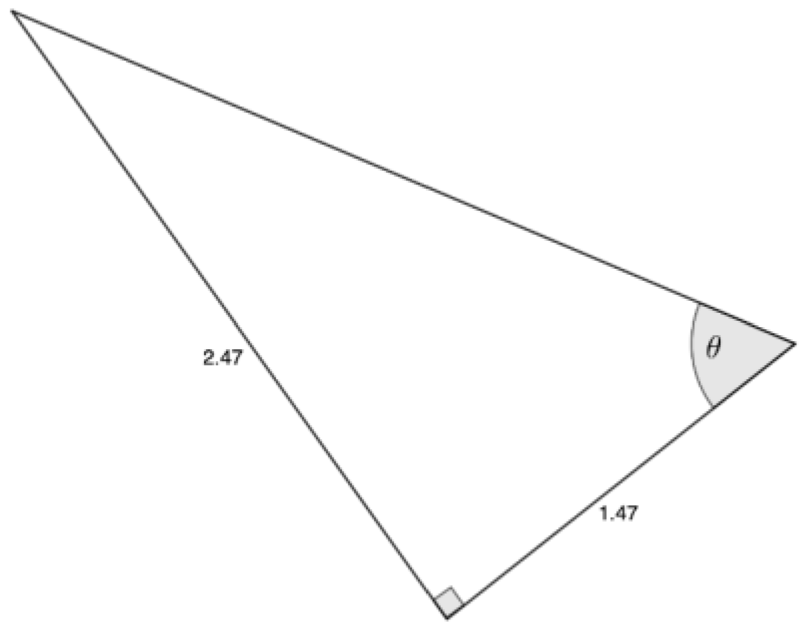
$\boxed{\text{shift/2nd function}} \rightarrow \boxed{\cos} \rightarrow \boxed{4.1} \rightarrow \boxed{\div} \rightarrow \boxed{4.32} \rightarrow \boxed{)} \rightarrow \boxed{=} \rightarrow 18.364^\circ$

Note

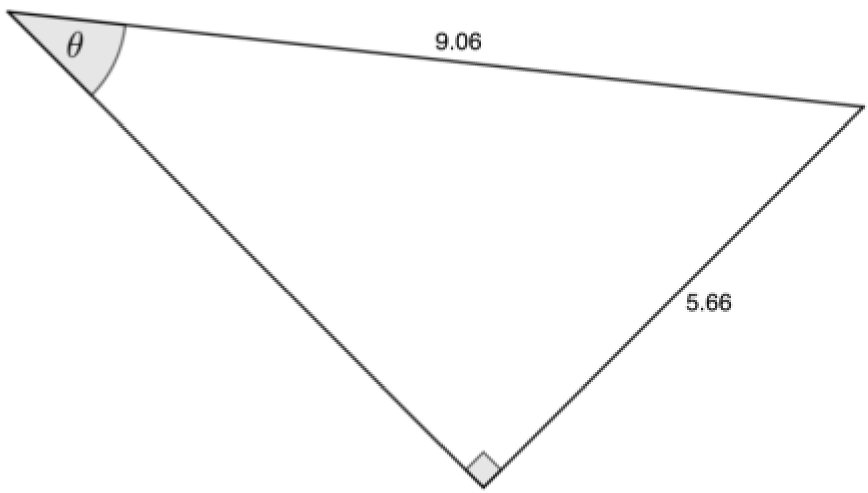
In most situations the answer you get from your calculator for the value of an angle based on a ratio is a non-terminating, non-repeating decimal. For the greatest accuracy, round off only your **final answer** and round off to **three decimal places** unless told otherwise.

Exercise 2.4

1. Find θ in each case:
   a.



b.



2. A river is $25\ \mathrm{m}$ wide with parallel banks. A swimmer wants to reach a point $123\ \mathrm{m}$ downstream on the opposite bank. At what angle to the bank of the river does the swimmer need to swim to reach this point?

The full solutions are at the end of the unit.

Summary

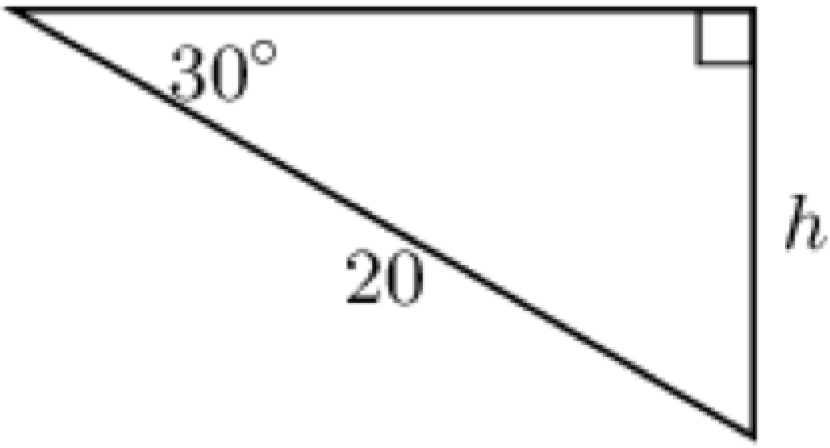
In this unit you have learnt the following:

- How to use a calculator and the trigonometric ratios to calculate the length of a side in a right-angle triangle if you know one angle and the length of one other side.
- How to use a calculator and the trigonometric ratios to calculate the size of an angle in a right-angled triangle if you know the length of two of the sides.
- Solve problems in two dimensions (2D) using the trigonometric ratios $\cos\theta, \sin\theta, \tan\theta$.

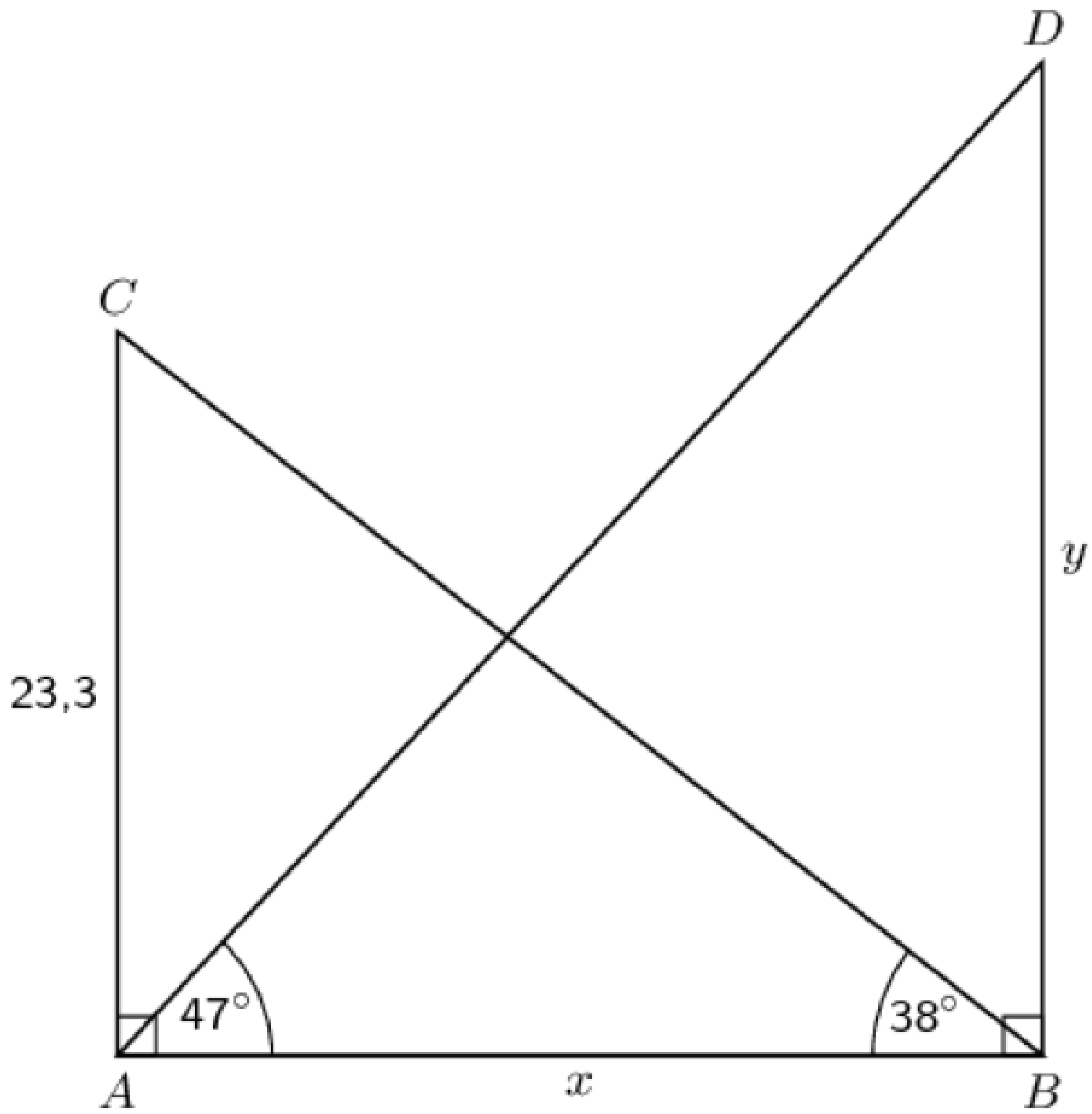
Unit 2: Assessment

Suggested time to complete: 25 minutes

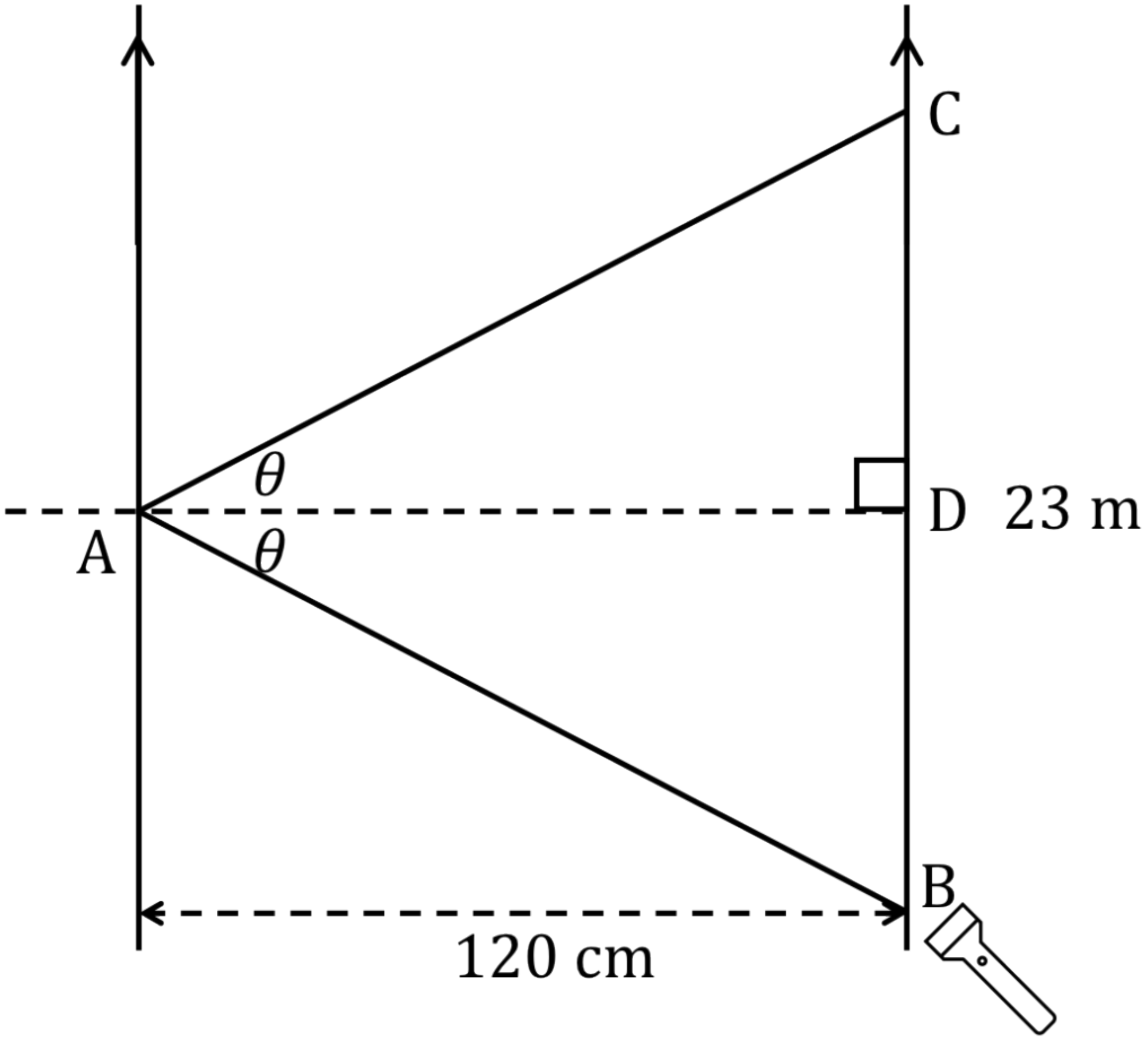
1. Determine h in the following triangle:



2. In ΔMNP, $\text{N} = 90^{\circ}$, $\text{MP} = 20$ and $\text{P} = 40^{\circ}$. Calculate NP and MN correct to two decimal places.
3. Determine x and y.



4. Light from a laser strikes a flat mirror at an angle of incidence of θ. It reflects off the mirror and strikes a point on a wall $23\ \text{m}$ away from the laser. If light is reflected at the same angle at which it strikes a mirror and if the mirror and the wall are parallel and a distance of $120\ \text{cm}$ apart, at what angle does the laser strike the mirror?



The full solutions are at the end of the unit.

Unit 2: Solutions

Exercise 2.1

1. $\boxed{\tan} \rightarrow \boxed{192} \rightarrow \boxed{)} \rightarrow \boxed{=} \rightarrow 0.213$
2. $\boxed{\tan} \rightarrow \boxed{12.1} \rightarrow \boxed{)} \rightarrow \boxed{x^2} \rightarrow \boxed{=} \rightarrow 0.046$
3. $\boxed{\sqrt{}} \rightarrow \boxed{\sin} \rightarrow \boxed{40.2} \rightarrow \boxed{)} \rightarrow \boxed{=} \rightarrow 0.803$
4. $\boxed{\sin} \rightarrow \boxed{6.67} \rightarrow \boxed{)} \rightarrow \boxed{x^2} \rightarrow \boxed{+} \rightarrow \boxed{\cos} \rightarrow \boxed{)} \rightarrow \boxed{x^2} \rightarrow \boxed{=} \rightarrow 0.015$

Back to Exercise 2.1

Exercise 2.2

1.

a.

$$\begin{aligned} \sin 42.28^\circ &= \frac{x}{13.19} \\ \therefore x &= 13.19 \sin 42.28^\circ \\ &= 8.874 \end{aligned}$$

b.

$$\begin{aligned} \cos 42.57^\circ &= \frac{4.1}{x} \\ \therefore x &= \frac{4.1}{\cos 42.57^\circ} \\ &= 5.567 \end{aligned}$$

2.

In $\Delta \mathrm{BCD}$:

$$\begin{aligned} \sin 32^\circ &= \frac{x}{11} \\ \therefore x &= 11 \sin 32^\circ \end{aligned}$$

In $\Delta \mathrm{ABC}$:

$$\begin{aligned} &= 5.829 \\ \mathrm{B\hat{A}C} &= 180^\circ - 90^\circ - 32^\circ \text{ (angles in triangle supplementary)} \end{aligned}$$

In $\Delta \mathrm{ABD}$:

$$\begin{aligned} &= 58^\circ \\ \tan 58^\circ &= \frac{x}{y} = \frac{5.829}{y} \\ \therefore y &= \frac{5.829}{\tan 58^\circ} \\ &= 3.642 \end{aligned}$$

Back to Exercise 2.2

Exercise 2.3

In $\Delta \mathrm{CDE}$:

$$\begin{aligned} \sin 57^\circ &= \frac{50}{\mathrm{CE}} \\ \therefore \mathrm{CE} &= \frac{50}{\sin 57^\circ} \\ &= 59.618 \text{ cm} \end{aligned}$$

$\mathrm{B\hat{A}C} = \mathrm{C\hat{E}D} = 57^\circ$ (alternate angle equal, $\mathrm{AB} \| \mathrm{DE}$)

In $\Delta \mathrm{ABC}$:

$$\begin{aligned} \cos 57^\circ &= \frac{40}{\mathrm{AC}} \\ \therefore \mathrm{AC} &= \frac{40}{\cos 57^\circ} \\ &= 73.443\mathrm{cm} \end{aligned}$$

$$\begin{aligned} \mathrm{AE} &= \mathrm{AC} + \mathrm{CE} \\ &= 59.618 + 73.443 \\ &= 133.061 \text{ cm} \end{aligned}$$

Back to Exercise 2.3

Exercise 2.4

1.

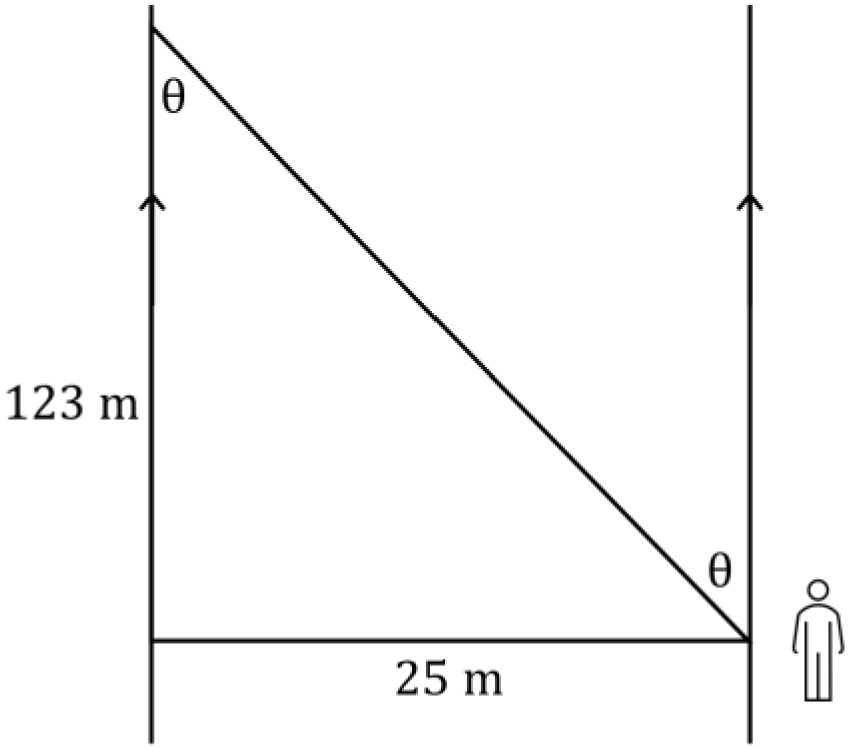
a.

$$\tan\theta = \frac{2.47}{1.47}$$
$$\therefore \theta = 59.241^\circ$$

b.

$$\sin\theta = \frac{5.66}{9.06}$$
$$\therefore \theta = 38.662^\circ$$

2.



The angle inside the triangle is also θ. Because the banks are parallel, the alternate angles are equal.

$$\tan\theta = \frac{25}{123}$$
$$\therefore \theta = 11.489^\circ$$

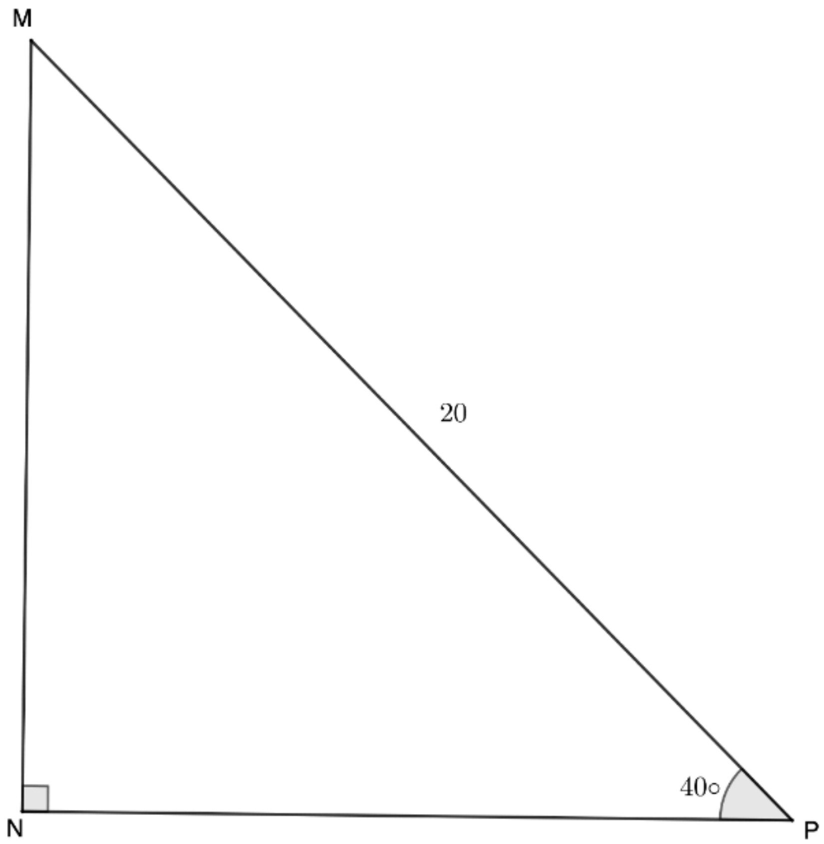
Back to Exercise 2.4

Unit 2: Assessment

1.

$$\sin 30^\circ = \frac{h}{20}$$
$$\therefore h = 20\sin 30^\circ$$
$$= 10$$

2.



$$\cos 20^\circ = \frac{\text{NP}}{20}$$
$$\therefore \text{NP} = 20\cos 20^\circ$$
$$= 18.794$$
$$\sin 20^\circ = \frac{\text{MN}}{20}$$
$$\therefore \text{MN} = 20\sin 20^\circ$$
$$= 6.840$$

3. In ΔABC:

$$\tan 38^\circ = \frac{23.3}{x}$$
$$\therefore x = \frac{23.3}{\tan 38^\circ}$$ In ΔABD:
$$= 29.823$$
$$\tan 47^\circ = \frac{y}{x} = \frac{y}{29.823}$$
$$\therefore y = 29.823\tan 47^\circ$$
$$= 31.981$$

4. ΔABD and ΔACD are congruent (same shape and size). Therefore, $\text{BD} = \text{CD}$. But $\text{BC} = 23$. Therefore, $\text{BD} = \text{CD} = 11.5\ \text{m}$.
   In ΔABD:
   $$\tan\theta = \frac{11.5}{1.2}$$
   $$\therefore \theta = 84.043^\circ$$
   Note: $120\ \text{cm} = 1.2\ \text{m}$

Back to Unit 2: Assessment

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SUBJECT OUTCOME XII

STATISTICAL AND PROBABILITY MODELS: CALCULATE CENTRAL TENDENCIES AND DISPERSION OF DATA

Subject outcome 4.1

Calculate central tendencies and dispersion of data.

Learning outcomes

- Calculate central tendency of ungrouped data namely the mean, median and mode.
- Calculate measures of dispersion including range, percentiles, quartiles, inter-quartile range and semi-inter-quartile range.

Unit 1: Data collection

By the end of this unit you will be able to:

- Define the types of data.
- Differentiate between grouped and ungrouped data.
- Create an ungrouped data frequency distribution and understand when to use it.
- Collect, organise and interpret univariate numerical data.

Unit 2: Measures of central tendency of ungrouped data

By the end of this unit you will be able to:

- Differentiate between the mean, median and mode.
- Calculate the mean, median and mode for ungrouped data.

Unit 3: Measures of dispersion of ungrouped data

By the end of this unit you will be able to:

- Calculate the range, inter-quartile range and semi-inter-quartile range.
- Calculate percentiles and quartiles.

Unit 1: Data collection

DYLAN BUSA

Unit 1: Data collecting

By the end of this unit you will be able to:

- Define the types of data.
- Differentiate between grouped and ungrouped data.
- Create an ungrouped data frequency distribution and understand when to use it.
- Collect, organise and interpret univariate numerical data.

What you should know:

Before you start this unit, make sure you can:

- Use a calculator to do simple arithmetic calculations

Introduction

Most great discoveries start with a question, a desire to know more about something. Very often, to answer this question, we need to collect data. But collecting data alone is not enough. We need to organise it so that we can analyse it, interpret it, draw conclusions from it and then present our findings to others. This is what statistics is all about.

Statistics

So, statistics is about answering questions with data and can be applied to fields of study as varied as science, psychology, sport, art, economics, politics, and, of course, social media and technology. We all know that Google and Facebook are constantly collecting huge amounts of data about us by tracking our online habits. They then organise, analyse, and interpret this data to determine what adverts or stories and links they think we will like. In fact, social media is collecting so much data about us that we call it BIG DATA!

Statistics has been around for a long time, dating back to 8th century Arab mathematicians. The foundations of modern statistics were laid in the 17th century with the development of probability theory and it really came into its own in the late 19th and early 20th centuries.

Watch this video called “What is Statistics” for an excellent introduction to the topic.

What Is Statistics (Duration: 04.56)

Did you know?

Data is the plural form of datum. Therefore, it is not correct to say 'The data is correct.' Instead, you should say 'The data **are** correct.' Or 'The datum **is** correct.' However, very few people follow this strict convention. What is definitely wrong, though, is saying that 'The datas are correct.' because **data** is the plural.

Before we start learning more about statistics, it is important to realise that sometimes statistics can be used to deceive, manipulate and even outright lie. Benjamin Disraeli once said that "there are three types of lies – lies, damn lies and statistics." By studying statistics, you will be better able to spot when someone is trying to manipulate you or persuade you of something that is not actually true. To get an idea of how statistics can be used in these ways, watch the following two videos. The second one is quite a bit longer but well worth the time if you have it.

How statistics can be misleading (Duration: 04.18)

This is How Easy It Is to Lie With Statistics (Duration: 18.54)

Types of data

All statistics relies on data. Without data there is nothing to organise, analyse, interpret, or present. But what are data? Data are a collection of unorganised observations or records about people, places, things, events or anything else and can contain thousands to millions of entries.

Data can be of two main types. **Quantitative** data is **numerical** – it is represented as numbers – like height, time, and cost. **Qualitative** data is **not numerical** and deals with descriptions and observations that cannot be measured like colour, appearance, and type.

Quantitative data can be split into **discrete** data where the values can only be **whole numbers** (like the number of ants in a nest) and **continuous** data where each value can be any **real number** (like the heights of students in a class).

To help you remember the difference remember that 'quantitative' is like 'quantity'.

Types of data:

- **Quantitative data:** numerical data that can be measured
  Examples: length, height, weight, time, cost, and number of people
  - **Discrete data** can only take on whole number values
    Example: number of females in a room.
  - **Continuous data** can have any real number value
    Example: height of each male in the room.
- **Qualitative data:** things that can be observed but not measured.
  Examples: colours, sizes, tastes and appearance.

Statistics is divided into two sub-topics: **descriptive statistics** and **inferential statistics**.

- Descriptive statistics deals with actual data collected from or about a group of the people, places, events or things we are studying, called a **sample** (see Figure 1).
- Inferential statistics deals with the predictions and inferences we make about an entire population based on the data we collected from only a subgroup or sample of the population.

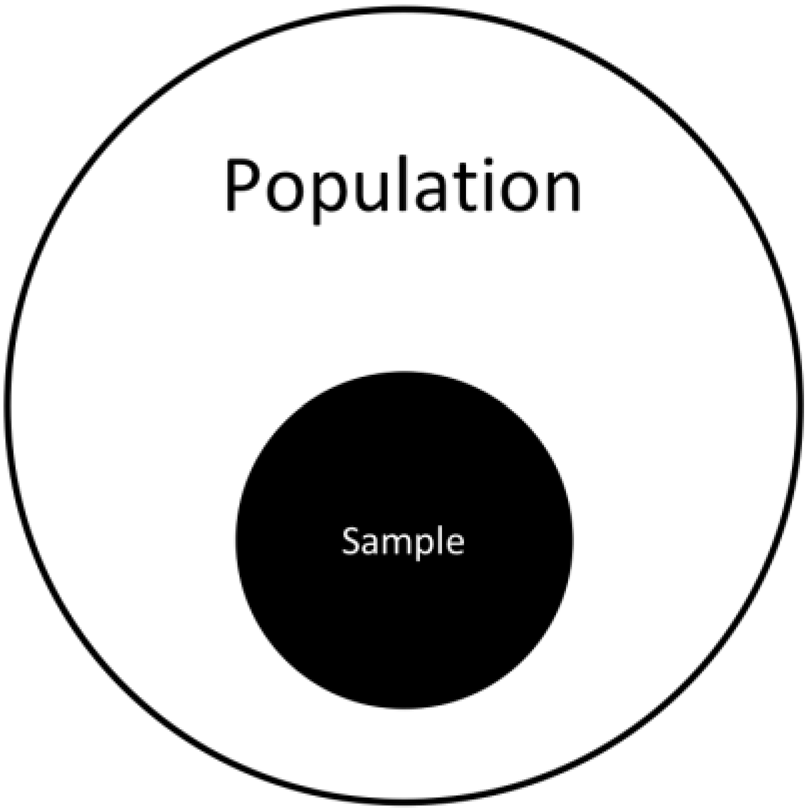


Figure 1: Descriptive statistics deals with data about a sample of an entire population

In this unit, we will only consider quantitative descriptive statistics. You will be working with numerical data from a sample of a whole population.

To learn more about the differences between descriptive and inferential statistics, watch the video called "Descriptive vs Inferential Statistics".

Descriptive vs Inferential Statistics (Duration: 4:04)

Example 1.1

Jeff wants to sell airtime vouchers to other students at his college. He surveys some students to find out how much data (in Mb) they used in the past week. Next, he asked each of these students which mobile network they used.

1. Is the data collected on mobile data used quantitative or qualitative?
2. Is the data collected on mobile networks used quantitative or qualitative?

Solutions

1. The data collected on mobile data used is quantitative because the data values can be written as numbers.
2. The data collected on mobile networks is qualitative because each response is not a number but the name of a company.

Exercise 1.1

1. The following data set is of the careers seven college students wish to pursue: 'Electrician', 'Plumber', 'Fitter and turner', 'Hairdresser', 'Accounts clerk', 'Chef', 'Machine operator'. Is this data quantitative or qualitative?
2. The following data set is of the number of nails in 8 packets bought from a supplier.
   $23;\ 25;\ 22;\ 26;\ 27;\ 25;\ 21;\ 28$
   Categorise the data as fully as possible.
3. Categorise the following data set of sizes of online videos (in Mb) as fully as possible.
   $134.76,\ 674.52,\ 897.25,\ 789.82,\ 438.52,\ 863.86$

The full solutions are at the end of the unit.

Organising data

Suppose a survey was conducted with 30 ladies to find out which is their favourite flower. Figure 2 shows the raw data that was collected from this survey.

Roses	Tulips	Roses
Daffodils	Asters	Daffodils
Tulips	Roses	Roses
Asters	Roses	Tulips
Tulips	Roses	Asters
Asters	Tulips	Roses
Asters	Daffodils	Tulips
Roses	Asters	Daffodils
Asters	Roses	Asters
Roses	Daffodils	Tulips

Figure 2: Raw data from a survey of 30 ladies about their favourite flowers DHET March 2012 Paper 2

How many ladies liked roses the most? How many ladies like tulips the most? It is quite difficult to see, at a glance, what the data is telling us. To make sense of the data we need to condense it or organise it in ways that make analysis and interpretation easier. To do this, we can summarise the data in the form of a table as shown in Table 1.

Flower type	Tally	Frequency
Roses	𝍸 𝍸	10
Daffodils	𝍸	5
Tulips	𝍸 ||	7
Asters	𝍸 |||	8
Total	30	

Table 1

A table like this is called a **frequency distribution**. Which representation (the raw data or the frequency distribution) do you think is better to get an overall idea of the data collected and to answer some basic questions?

Take note!

Frequency distribution

Frequency is how often (or frequently) something occurs. A **frequency distribution**, also called a **frequency table**, is used to organise qualitative and quantitative raw data.

Hopefully you agree that the table is a better way to represent the data if we want to get a basic sense of what it is telling us. As we will see later on in this topic in Subject outcome 4.2, data can also be represented in graphs and charts to get a quick overall picture. Before you summarise data, it is important to understand how to organise data using **frequency distributions** like that shown in Table 1.

In this subject outcome, however, we will mostly look at frequency distributions of **ungrouped data** where there are no **intervals**. We will look at **grouped data frequency distributions** in Subject outcome 4.2.

Watch the video called "Intro to data handling" to see a simple example of how to organise raw ungrouped data into a frequency table.

Intro to data handling (Duration: 05.50)

As the size of the data set grows it becomes even more difficult to handle it in its raw form, so we almost always need to organise data before analysing it.

Activity 1.1: Organise data using a frequency table

Time required: 10 minutes

What you need:

- a pen or pencil
- paper

What to do:

The following is a sample of raw data collected on the number of digital devices owned per household in a suburb.

Number of digital devices per household:

5	2	4	1	1	3	2	4	2	2
3	6	3	2	4	1	3	4	2	4
2	2	3	1	2	2	3	6	4	2
7	1	2	8	2	1	2	5	7	2
1	4	1	6	2	3	2	4	1	3

1. How many households in total were surveyed?
2. Rewrite the data values from smallest to biggest. Can you group similar values together?
3. How many households had two devices? How many had three devices? How many households had more than five devices?
4. Create a table, which shows the number of households with one device, two devices, three devices, etc.
5. How many devices did most households have?

6. The fibre provider will install fibre if 50% of households have three or more devices. Based on your frequency distribution will the fibre provider install fibre in this suburb?

What did you find?

1. By counting the number of entries, we see that 50 households were surveyed.
2. Writing the entries from smallest to greatest gives us the following list.
$1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,$
$3,\ 3,\ 3,\ 3,\ 3,\ 3,\ 3,\ 3,\ 4,\ 4,\ 4,\ 4,\ 4,\ 4,\ 4,\ 4,\ 5,\ 5,\ 6,\ 6,\ 6,\ 7,\ 7,\ 8$
3. To count how many households had two devices, we need to count the number of '2' entries in our list. When we do, we see that 17 households had two devices. Eight households had three devices. To work out how many households had more than five devices we need to add the households with six, seven and eight devices together. There were 3 households with six devices, 2 with seven and 1 with eight devices. Altogether there were 6 households with more than five devices.
4. You can set up a basic table with two columns to record the number of households with different numbers of devices.

Number of devices in household	**Frequency**
1	9
2	17
3	8
4	8
5	2
6	3
7	2
8	1
Total	50

5. From the frequency table, it is easy to see that 2 is the value that is recorded the greatest number of times (it has the highest frequency) so we can say most households have two devices.
6. Since there are 50 households, 25 or more must have three or more devices. There are 8 households with three devices, 8 with four devices, 2 with five, 3 with six, 2 with seven and 1 with eight devices. Therefore, there is a total of 24 households with three or more devices. This is not quite enough for the fibre provider. There are less than 50% of households with three or more devices so they will not install in this suburb.

Summary

In this unit you have learnt the following:

- Statistics is about collecting, organising, analysing, interpreting, and presenting data.
- Quantitative data is data that is numerical and can be measured like length, height, weight, time, cost, and number of items.
- Qualitative data is data that can be observed but not numerically measured like colour, appearance, and type.
- Quantitative data can be either discrete (whole numbers) or continuous (any real number).

- Descriptive statistics deals with data based on a subgroup of all the people, places, events or things we are studying, called a **sample**.
- Inferential statistics deals with observations about an entire population or group being studied.
- Ungrouped data is data in its raw form.
- Grouped data is data organised into intervals and usually presented in the form of a frequency table.
- Frequency is how often something occurs and a **frequency distribution**, also called a **frequency table**, is used to organise qualitative and quantitative raw data.

Unit 1: Assessment

Suggested time to complete: 20 minutes

1. The mathematics marks, out of 50, for a class of learners are given below:
   $46,\ 40,\ 12,\ 10,\ 47,\ 23,\ 26,\ 8,\ 29,\ 34,\ 37,\ 17,\ 40,\ 50,\ 18,\ 23,\ 33,\ 23,$
   $24,\ 15,\ 35,\ 23,\ 19,\ 22,\ 28,\ 35,\ 27,\ 42,\ 29,\ 26,\ 46,\ 33,\ 27,\ 19,\ 28$
   a. Complete the following frequency table using the above marks.

Interval of scores	**Frequency**
$0-10$	
$11-20$	
$21-30$	
$31-40$	
$41-50$	
Total	

   b. How many learners wrote the test?
   c. In which interval did most learners score?
   d. If the pass mark was 21 out of 50, what percentage of learners passed the test?
   e. What percentage of learners scored more than 80% in the test?
   f. How many learners scored between 21 and 40 marks for the test?
2. The employees of a small company were surveyed about their retirement savings. The following frequency distribution shows the numbers of years to retirement for the 101 employees in the company.

Years to retirement	Frequency
10	2
11	1
12	2
13	2
14	3
15	3
16	3
17	4
18	6
19	11
20	2
21	2
22	10
23	7
24	7
25	9
26	5
27	3
28	6
29	4
30	9

a. How many employees will retire in less than 12 years?
b. If retirement age is 65, how many employees are 50 years old or older?
c. How many employees are younger than 40?
d. What percentage of the total employees will retire in 20 years or more?
e. What percentage of the total employees will retire in 20 to 25 years' time?
f. If you were to create a grouped frequency table of the data with seven groups, what size would each interval need to be?
g. Create a grouped frequency table for the data with intervals of three years.
h. From the grouped frequency table is it possible to tell how many people will retire in 13 years or less? Why or why not.
i. From the grouped frequency table, how many employees are 40 years old or younger?

The full solutions are at the end of the unit.

Unit 1: Solutions

Exercise 1.1

1. The data is qualitative data.
2. The data is discrete quantitative data.
3. The data is continuous quantitative data.

Back to Exercise 1.1

Unit 1: Assessment

1.
    a.

Interval of scores	**Frequency**
$0-10$	2
$11-20$	6
$21-30$	14
$31-40$	8
$41-50$	5
Total	35

    b. 35 learners wrote the test.
    c. Most learners scored between 21 and 30 marks out of 50.
    d. Total learners scoring 21 marks or more: $14+8+5=27$.
    Percentage of total learners scoring 21 marks or more: $\dfrac{27}{35}\times 100=77.1\%$
    e. $80\%=\dfrac{40}{50}$. Five learners scored more than 40 marks.
    Percentage of total learners scoring more than 40 marks: $\dfrac{5}{35}\times 100=14.3\%$
    f. Total learners scoring between 21 and 40 marks: $14+8=22$
2.

Years to retirement	Frequency
10	2
11	1
12	2
13	2
14	3
15	3
16	3
17	4
18	6
19	11
20	2
21	2
22	10
23	7
24	7
25	9
26	5
27	3
28	6
29	4
30	9

a. Employees retiring in less than 12 years are those retiring in 11 or 10 years: $1 + 2 = 3$

b. Employees aged 50 would retire in 15 years. So total employees aged 50 or older are those that will retire in 15 years or less: $3 + 3 + 2 + 2 + 1 + 2 = 13$ employees.

c. Employees younger than 40 would retire in more than 25 years i.e. 26 years or more: $5 + 3 + 6 + 4 + 9 = 27$ employees.

d. Employees retiring in 20 years' or more: $2 + 2 + 10 + 7 + 7 + 9 + 5 + 3 + 6 + 4 + 9 = 64$
Percentage of total employees: $\frac{64}{101} \times 100 = 63.4\%$

e. Employees retiring in 20 to 25 years' time: $2 + 2 + 10 + 7 + 7 + 9 = 37$
Percentage of total employees: $\frac{37}{101} \times 100 = 36.6\%$

f. There are a total of 21 intervals. Therefore, each of the seven groups would include three intervals.

g. A grouped frequency table for the data with intervals of three years.

Years to retirement	Frequency
$10-12$	5
$13-15$	8
$16-18$	13
$19-21$	15
$22-24$	24
$25-27$	17
$28-30$	19

h. No. We can tell how many people will retire in 12 years or less. People retiring in 13 years are part of a group and so we do not know from the grouped data how many people this is exactly.

i. Number of employees 40 years or younger that will retire in 25 years or more:
$17+19=36$

Back to Unit 1: Assessment

Media Attributions

Unit 2: Measures of central tendency of ungrouped data

DYLAN BUSA

Unit 2: Measures of central tendency of ungrouped data

By the end of this unit you will be able to:

- Differentiate between the mean, median and mode.
- Calculate the mean, median and mode for ungrouped data.

What you should know

Before you start this unit, make sure you can:

- Define the types of data.
- Differentiate between grouped and ungrouped data.

If you need help with this, review Unit 1 in this subject outcome.

Introduction

Have a look at the test marks of learners in two different groups.

Group A:
$\{45\%,\ 60\%,\ 65\%,\ 72\%,\ 85\%,\ 67\%,\ 74\%,\ 87\%,\ 24\%,\ 36\%,\ 55\%\}$

Group B:
$\{48\%,\ 52\%,\ 68\%,\ 76\%,\ 83\%,\ 73\%,\ 69\%,\ 38\%,\ 75\%,\ 75\%,\ 79\%,\ 81\%\}$

On 'average' did Group A do better than Group B? It's hard to tell by just looking at the raw data. We need a way to statistically compare them. Luckily, we have a number of statistical tools we can use to find where the middle or the average of a set of data lies. These are called **measures of central tendency**.

Calculating measures of central tendency

Measures of central tendency are single numbers that provide summary information about an entire set of data, without listing every single data value. These single values represent the middle or centre values of the data and are helpful in comparing different sets of data. There are three main measures of central tendency: the **mean**, **median** and **mode**. We use different methods to calculate measures of central tendency for ungrouped and grouped data.

The mean

You are probably familiar with the concept of an average. You may have calculated your average test or exam result to see how well you have done. You add all your results together and then divide by the total number of results.

What you are actually calculating is the mean of the marks. We can calculate the mean result for each group above as a way to compare them. Work out what the mean result of each of these groups is.

Group A:

$$\frac{45\% + 60\% + 65\% + 72\% + 85\% + 67\% + 74\% + 87\% + 24\% + 36\% + 55\%}{11}$$
$$= \frac{670}{11}$$
$$= 60.9\%$$

Group B:

$$\frac{48\% + 52\% + 68\% + 76\% + 83\% + 73\% + 69\% + 38\% + 75\% + 75\% + 79\% + 81\%}{12}$$
$$= \frac{817}{12}$$
$$= 68.1\%$$

Group A had a mean mark of 60.9% and Group B had a mean mark of 68.1%. Therefore, we can say that, on average, Group B did better.

Mean

The mean is the sum of a set of values, divided by the number of values in the set.

It can be expressed in mathematical notation as $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{x_1 + x_2 + \ldots + x_n}{n}$

where $\bar{x}$ is the symbol used for mean and $\sum_{i=1}^{n} x_i$ means add up all x values in the set from the first ($i = 1$) to the last ($i = n$).

The median

Another measure of central tendency we can use to compare data sets is the median. The median is that value in the data set that splits the whole data set into a lower half and an upper half. To work out the median, we first need to sort the elements in the data set in ascending order.

So, to calculate the median of the Group A results above, we do the following:

1. List the results in ascending order.
   $24\%, 36\%, 45\%, 55\%, 60\%, 65\%, 67\%, 72\%, 74\%, 85\%, 87\%$
2. Find the middle value that splits the whole data set into a lower half and an upper half.
   $24\%, 36\%, 45\%, 55\%, 60\%, \boxed{65\%}, 67\%, 72\%, 74\%, 85\%, 87\%$

The median value of Group A is 65%.

In this case, finding the median was simple because there were an odd number (11) of values. But what happens when a data set has an even number of values, like Group B?

1. List the results in ascending order.

$38\%, 48\%, 52\%, 68\%, 69\%, 73\%, 75\%, 75\%, 76\%, 79\%, 81\%, 83\%$

2. Find the middle two values that split the whole data set into a lower half and an upper half.
   $38\%, 48\%, 52\%, 68\%, 69\%, \boxed{73\%, 75\%}, 75\%, 76\%, 79\%, 81\%, 83\%$
3. Find the mean of these two values.
   $\frac{73+75}{2} = \frac{148}{2} = 74$

The median value of Group B is 74%.

Take note!

Median

The median is the middle value, when the data set has been arranged from the lowest to the highest value.

The mode

The **mode** is that value that is **repeated most often** in a data set. If there is no **one** value that is repeated most often, then a data set has no mode or is multi-modal (as more than one mode).

Do Group A and Group B have modes. If so, what are these?

In Group A, there is no one value that is repeated most often, In fact, all values appear only once. In Group B, however, 75% is repeated – it appears twice. Every other value appears only once. Therefore, Group B has a mode and it is 75%.

The mode is not used very often when describing data, but it can be useful in certain circumstances. It is an appropriate measure to use with qualitative data, for example, a survey on car colour preferences.

Take note!

Mode

The mode is that value that occurs most often in the set. The mode is the most frequent or most common value in the data set.

Note

Most often a continuous data set will have no mode. Since continuous values can lie anywhere on the real line, any particular value will almost never repeat. This means that the frequency of each value in the data set will be 1 and that there will be no mode.

Example 2.1

A high school has two cricket teams: a junior and a senior team. The junior team consists of 17 players (including reserves) and the senior team consists of 16 players (including reserves). The mass of each team member is given below. Use the data to answer the questions that follow.

Junior team masses (kg)
$\{56,\ 60,\ 67,\ 45,\ 51,\ 53,\ 64,\ 49,\ 56,\ 48,\ 42,\ 51,\ 64,\ 52,\ 64,\ 49,\ 50\}$

Senior team masses (kg)
$\{88,\ 81,\ 53,\ 62,\ 83,\ 68,\ 70,\ 62,\ 91,\ 78,\ 64,\ 74,\ 73,\ 54,\ 62,\ 62\}$

1. What is the mean mass of the senior team?
2. Arrange the masses of the senior team in ascending order.
3. Determine the mode of the senior team.
4. Determine the median of the senior team.
5. Calculate the mean of the masses of the junior team correct to one decimal digit.
6. Calculate the median of the masses of the junior team.
7. Calculate the mode of the masses of the junior team.
8. Look at the answers you found for the junior and senior teams. Which measure do you think gives the best measure of the real 'average' of each data set?

Solutions

1.
$$\begin{aligned}&\text{Mean}\\ &= \frac{88+81+53+62+83+68+70+62+91+78+64+74+73+54+62+62}{16}\\ &= \frac{1\ 125}{16}\\ &= 70.31\ \text{kg}\end{aligned}$$

2. $53,\ 54,\ 62,\ 62,\ 62,\ 62,\ 64,\ 68,\ 70,\ 73,\ 74,\ 78,\ 81,\ 83,\ 88,\ 91$

3. $53, 54, \boxed{62, 62, 62, 62}, 64, 68, 70, 73, 74, 78, 81, 83, 88, 91$
   62 is the value that appears most often (four times) in the data set. Therefore, the mode is 62 kg.

4. $53, 54, 62, 62, 62, 62, 64, \boxed{68, 70}, 73, 74, 78, 81, 83, 88, 91$
   $\frac{68+70}{2} = \frac{138}{2} = 69$
   Therefore, the median is 69 kg.

5.
$$\begin{aligned}&\text{Mean}\\ &= \frac{56+60+67+45+51+53+64+49+56+48+42+51+64+52+64+49+50}{17}\\ &= \frac{921}{17}\\ &= 54.2\ \text{kg}\end{aligned}$$

6. $42, 45, 48, 49, 49, 50, 51, 51, \boxed{52}, 53, 56, 56, 60, 64, 64, 64, 67$
   The median is 52 kg.

7. $42, 45, 48, 49, 49, 50, 51, 51, 52, 53, 56, 56, 60, \boxed{64, 64, 64}, 67$
   The mode is 64 kg, because this value appears most often.
8. The following table contains a summary of the measures of central tendency for each team.

Measure	Junior team	Senior team
Mean	54.2 kg	70.31 kg
Median	52 kg	69 kg
Mode	64 kg	62 kg

We can see that, for both teams, the median value is close to the mean, but the modes are not. Of all these measures, therefore, either the mean or the median seems to be the best representations of the 'average' player mass.

The mode of the junior team is actually higher than the mode of the senior team and is much greater than either the mean or the median. The mode of the senior team is much less than either the mean or the median.

In Example 2.1, we looked at the masses of players in a sports team. Generally, due to natural variations we would expect all the players to have different weights. However, we would also expect these measures to be more or less equally distributed about the mean, with half of them being less than the mean and half of them being greater than the mean. We call this a **normal distribution**. This is illustrated in Figure 1.

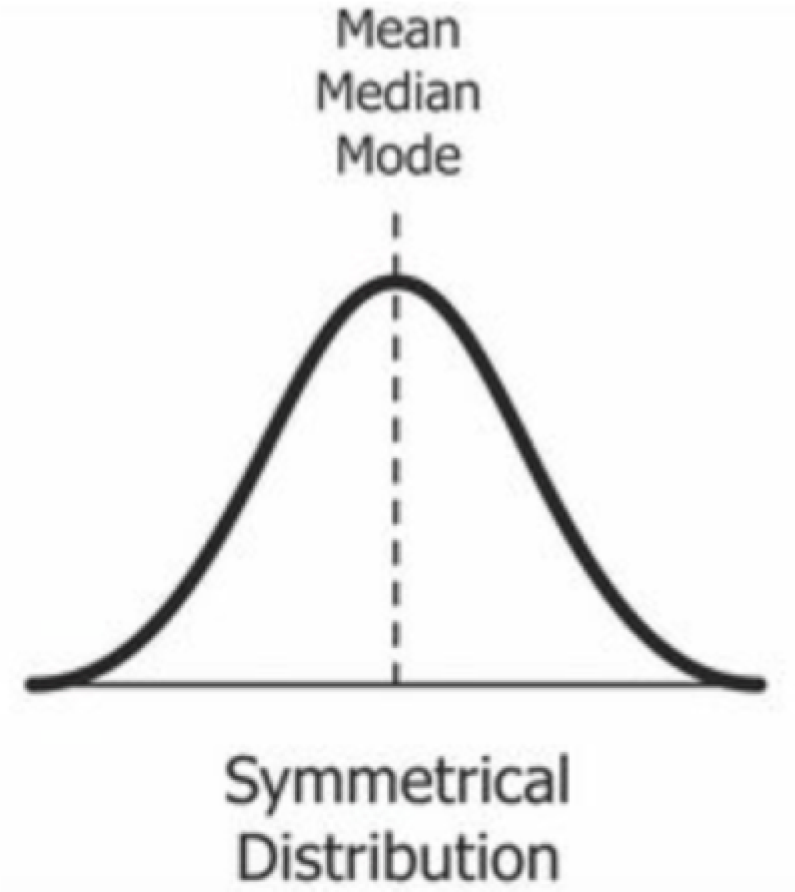


Figure 1: A normal or symmetrical distribution

In a normal distribution, because all the values of the data set are equally distributed above and below the mean, the median tends to be the same (or similar to) the mean. The data is said to have **no skew**. If there is a mode, it is also most often similar to the mean and the median as well (but not necessarily so).

However, in some cases, many more of the data values actually lie above the mean with a few extreme values below the mean. In this case, the median value is **greater than** the mean. Because the 'long tail' of the few, but extreme, values are on the 'negative' side of the mean, we call this a **negative skew**. Any value that is very different from the majority of the other values in a data set is called an **outlier**.

Take note!

Outlier

An outlier is a value in the data set that is not typical of the rest of the set. It is usually a value that is much greater or much smaller than most of the other values in the data set.

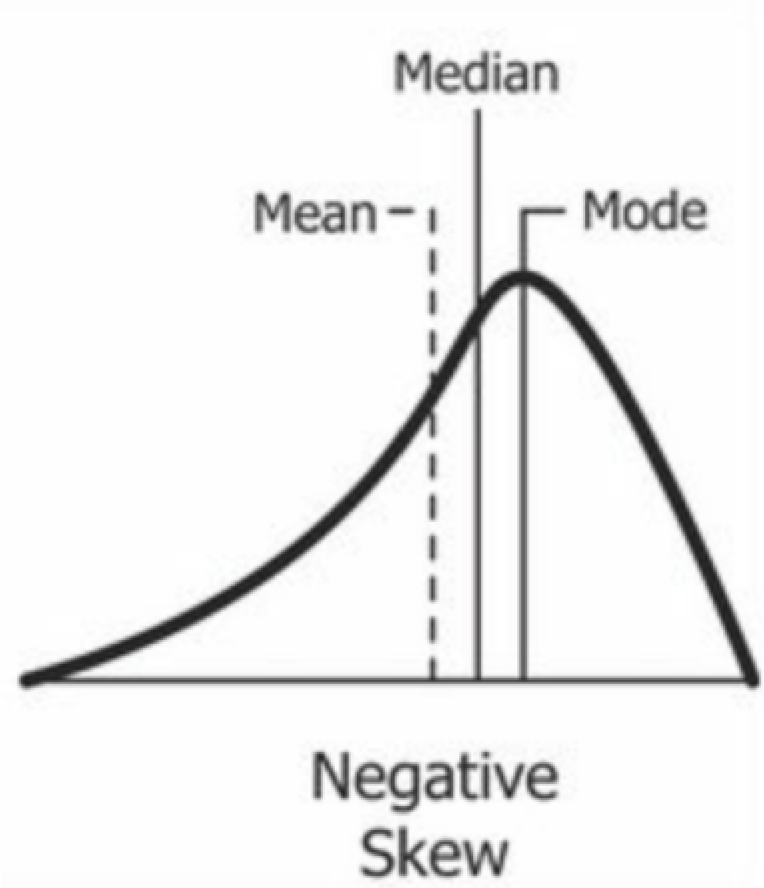


Figure 2: A negative skew

The opposite can also happen. A **positively skewed** data set has a long tail of a few relatively extreme values on the positive side of the mean. In this case the median is **less than** the mean.

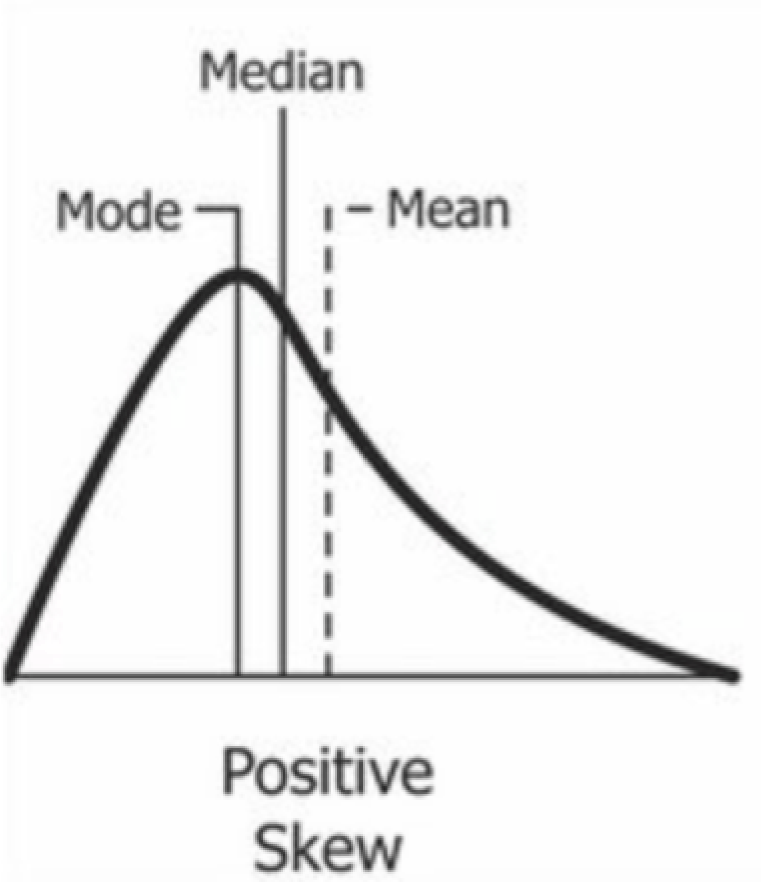


Figure 3: A positive skew

If the mean and median of a data set are very different, this could indicate that the data set is skewed in some way.

If you have an internet connection, watch this excellent video on the subject of skewness called “Statistics: Skewness and Measures of Center”.

Statistics: Skewness and Measures of Center (Duration: 04.14)

Take note!

Skewness

If the value of the calculation of the difference between mean and median (mean – median) is very close to 0, the data set is symmetrical.

If the value of the calculation of mean – median is greater than 0 (or positive), the data is skewed right or is positively skewed.

If the value of the calculation of mean – median is less than 0 (or negative), the data is skewed left or is negatively skewed.

Exercise 2.1

Question adapted from Everything Maths Grade 10

1. South African regulations stipulate that, if the mass of a loaf of bread is not given, it must weigh between $760\ \text{g}$ and $880\ \text{g}$. The mass of ten newly baked loaves of bread were recorded each day for one week. The results, in grams, are given in the following table.

Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
802.4	787.8	815.7	807.4	801.5	786.6	799.0
796.8	798.9	809.7	798.7	818.3	789.1	806.0
819.6	812.6	809.1	791.1	805.3	817.8	801.0
789.0	796.3	787.9	799.8	789.5	802.1	802.2
808.8	780.4	812.6	801.8	784.7	792.2	809.8
796.2	817.6	799.1	826.0	807.9	806.7	780.2
801.2	795.9	795.2	820.4	806.6	819.5	796.7
802.4	790.8	792.4	789.2	815.6	799.4	791.2
802.5	809.3	785.4	793.6	787.7	801.5	799.4
796.2	780.2	806.7	826.0	807.9	799.1	817.6

   a. Is this data set qualitative or quantitative? Explain your answer.
   b. Determine the mean, median and mode of the mass of the loaves of bread for each day of the week. Give your answers correct to one decimal place and summarise your findings in a table.
   c. Based on the data, do you think that this supplier is providing bread within the South African regulations? Justify your answer.

2. The heights of 11 girls in a netball team are measured in centimetres. The data set is as follows: $\{151,\ 171,\ 153,\ 147,\ 142,\ 167,\ 146,\ 157,\ 156,\ 157,\ 158\}$
   a. Calculate the mean, median and mode of the data.
   b. Is the data skewed? If so, how?
   c. A twelfth player is added to the squad who is exceptionally tall at $183\ \text{cm}$. Recalculate the mean, median and mode for the new dataset and explain any changes (or not) to these measures.
   d. Is the new data set skewed? If so, how?

The full solutions are at the end of the unit.

Example 2.2

A group of 13 friends each have some marbles. They work out that the mean number of marbles they have is 12. Then seven friends leave with an unknown number (x) of marbles. The remaining six friends work out that the mean number of marbles they have left is 15.5. How many marbles did the seven friends take with them?

Solution

We can calculate the total number of marbles the group of 13 had.

$$\begin{aligned}\bar{x} = \frac{x_1 + x_2 + \ldots + x_{13}}{13} &= 12 \\ \therefore x_1 + x_2 + \ldots + x_{13} &= 12 \times 13 \\ &= 156\end{aligned}$$

After the seven leave, the mean number of marbles is 15.5. Therefore, we can calculate the new total number of marbles.

$$\begin{aligned}\bar{x} = \frac{x_1 + x_2 + \ldots + x_6}{6} &= 15.5 \\ \therefore x_1 + x_2 + \ldots + x_6 &= 15.5 \times 6 \\ &= 93\end{aligned}$$

Therefore, the seven friends took $156 - 93 = 63$ marbles with them.

Exercise 2.2

1. A group of 27 employees have a mean monthly income of $\text{R}17\ 510$. Three employees resign and the mean monthly income drops to $\text{R}16\ 113$. What was the mean monthly income of the three employees who resigned?
2. While doing a fuel economy test, a driver makes 16 test drives. The mean amount of fuel consumed during each test drive was $7.72\ \ell$. If the four least economical journeys are removed from the data set, the mean fuel consumption drops to $6.96\ \ell$ per test drive.
   a. What was the mean fuel consumption of these four test drives?
   b. If three of the 12 most efficient test drives each consumed $8\ \ell$ of fuel, would you say the data is skewed and, if so, how?
3. Find a set of eight ages less than or equal to 10 for which the mean age is 4.75, the modal age is two and the median age is four years.

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- There are three simple ways to represent the 'middle' of a quantitative data set – the mean, the median

and the mode.
- The mean is the sum of all the values divided by the number of values.
- The median is the middle value when the values are arranged from smallest to biggest.
- The mode is the value that appears most often (with the highest frequency).
- If the value of the calculation of mean − median is very close to 0, the data set is symmetrical.
- If the value of the calculation of mean − median is greater than 0 (or positive), the data is skewed right or is positively skewed.
- If the value of the calculation of mean − median is less than 0 (or negative), the data is skewed left or is negatively skewed.

Unit 2: Assessment

Suggested time to complete: 30 minutes

1. The ages of 20 cyclists in the Cape Argus Cycle race were recorded. Calculate the mean, median and modal age.
$\{31,\ 42,\ 28,\ 38,\ 67,\ 43,\ 45,\ 51,\ 33,\ 53,\ 29,\ 42,\ 26,\ 34,\ 35,\ 56,\ 33,\ 43,\ 46,\ 41\}$
2. A group of 15 salespeople were surveyed on the total value of their sales of a product in the previous month. The data is given below. Answer the following questions based on this data.
$$\text{R13 346},\ \text{R14 341},\ \text{R14 416},\ \text{R24 512},\ \text{R36 973},\ \text{R12 014},\ \text{R43 852},\ \text{R40 915},$$
$$\text{R16 536},\ \text{R82 366},\ \text{R17 340},\ \text{R28 361},\ \text{R130 011},\ \text{R14 815},\ \text{R24 836}$$
   a. What is the mean, median and mode of this data?
   b. Is the data skewed? If so, how? Explain your answer.
   c. If it came to light that the highest value was actually misreported and should have been reported as R13 011 instead, would you still consider the data to be skewed? Explain your answer.
3. Four friends each have some marbles. They work out that the mean number of marbles they have is 10. One friend leaves with four marbles. What is the new mean number of marbles of the three remaining friends?

The full solutions are at the end of the unit.

Unit 2: Solutions

Exercise 2.1

1.
   a. The data is quantitative. It consists of numerical values of the mass of loaves of bread.
   b.

Measure	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
Mean	801.5	797.0	801.4	805.4	802.5	801.4	800.3
Median	801.8	796.1	802.9	800.8	805.95	800.45	800.2
Mode	796.2 and 802.4	No mode	No mode	826	807.9	No mode	No mode

   c. The regulations state that each loaf must be between $760\ \text{g}$ and $880\ \text{g}$. On each day, the mean and median were very close to each other and to 800 indicating that the data is not significantly skewed negative or positive. Where modes are present, these are close to, or greater than, 800. The summary data does not specifically indicate or identify any outliers, but the summary data does seem to suggest that the manufacturer is well within the regulations.

Looking at the raw data, one can see that all values do fall within the stipulated range.

2. $\{151,\ 161,\ 153,\ 147,\ 142,\ 167,\ 146,\ 157,\ 156,\ 157,\ 158\}$
   a. Mean: $155\ \text{cm}$
      Median: $142, 146, 147, 151, 153, \boxed{156}, 157, 157, 158, 167, 171$. The median is $156\ \text{cm}$.
      Mode: 3. $142, 146, 147, 151, 153, 156, \boxed{157, 157}, 158, 167, 171$. The mode is $157\ \text{cm}$.
   b. The mean and median are fairly close to each other. That the median is less than the mean indicates that the data is positively skewed. The fact that the mode is 157 indicates that there are some players significantly below the mean height.
   c. Mean: $157.3\ \text{cm}$
      Median: $142, 146, 147, 151, 153, \boxed{156, 157}, 157, 158, 167, 171$. Therefore, the median is $156.5\ \text{cm}$.
      Mode: $142, 146, 147, 151, 153, 156, \boxed{157, 157}, 158, 167, 171$. The mode is $157\ \text{cm}$.
   d. In this case, even with this new outlier, the mean and median are much closer together, indicating less overall skew than previously. The mean, median and mode are also all much more similar. Hence even though there is now a clear outlier, this value seems to balance out the heights that are significantly below the mean.

Back to Exercise 2.1

Exercise 2.2

1. Total earnings before employees left:
$$\begin{aligned}\bar{x} = \frac{x_1 + x_2 + \ldots + x_{27}}{27} &= 17\ 510\\ \therefore x_1 + x_2 + \ldots + x_{27} &= 17\ 510 \times 27\\ &= 472\ 770\end{aligned}$$
Total earnings after employees left:
$$\begin{aligned}\bar{x} = \frac{x_1 + x_2 + \ldots + x_{24}}{24} &= 16\ 113\\ \therefore x_1 + x_2 + \ldots + x_{24} &= 16\ 113 \times 24\\ &= 386\ 712\end{aligned}$$
Total earnings of three employees:
$472\ 770 - 386\ 712 = 86\ 058$
Mean earnings of three employees:
$$\bar{x} = \frac{86\ 058}{3} = 28\ 686$$
Therefore, the three employees who left earned a mean monthly salary of $\text{R}28\ 686$.

2.
   a. Total fuel consumption of all test drives:
$$\begin{aligned}\bar{x} = \frac{x_1 + x_2 + \ldots + x_{16}}{16} &= 7.72\\ \therefore x_1 + x_2 + \ldots + x_{16} &= 7.72 \times 16\\ &= 123.52\end{aligned}$$
Total fuel consumption excluding four least efficient test drives:
$$\begin{aligned}\bar{x} = \frac{x_1 + x_2 + \ldots + x_{12}}{12} &= 6.96\\ \therefore x_1 + x_2 + \ldots + x_{12} &= 6.96 \times 12\\ &= 83.52\end{aligned}$$
Total fuel consumption of four least efficient test drives:

$123.52 - 83.52 = 40$
Mean fuel consumption of the four least efficient test drives:
$\bar{x} = \frac{40}{4} = 10$
Therefore, the four least efficient test drives consumed an average of $10\ \ell$ of fuel.

b. Yes, the data is skewed. It is negatively skewed. The mean of the four least efficient test drives is quite a bit greater than the overall mean of the data and three of the remaining 12 most efficient drives are higher than the mean of these 12 test drives as well.

3. A set of eight ages less than or equal to 10 for which the mean age is 4.75 , the modal age is 2 and the median age is 4 years.There are eight values in the set: □, □, □, □, □, □, □, □

The median is four: □, □, □, $\boxed{3}$, $\boxed{5}$, □, □, □
The mode is two: □, $\boxed{2}$, $\boxed{2}$, $\boxed{3}$, $\boxed{5}$, □, □, □

Therefore, the first value must be one: $\boxed{1}$, $\boxed{2}$, $\boxed{2}$, $\boxed{3}$, $\boxed{5}$, □, □, □

The mean is 4.75. Therefore, the total ages must be $4.75 \times 8 = 38$.

Therefore, the total of the last three ages must be $38 - 1 - 2 - 2 - 3 - 5 = 25$ and none of them can be the same (the mode is two).

Therefore, the data set must be $\boxed{1}$, $\boxed{2}$, $\boxed{2}$, $\boxed{3}$, $\boxed{5}$, $\boxed{6}$, $\boxed{9}$, $\boxed{10}$ or $\boxed{1}$, $\boxed{2}$, $\boxed{2}$, $\boxed{3}$, $\boxed{5}$, $\boxed{7}$, $\boxed{8}$, $\boxed{10}$

Back to Exercise 2.2

Unit 2: Assessment

1. Mean:
$$\bar{x} = \frac{31 + 42 + 28 + 38 + 67 + 43 + 45 + 51 + 33 + 53 + 29 + 42 + 26 + 34 + 35 + 56 + 33 + 43 + 46 + 41}{20}$$
$= 40.8$
Median: $26, 28, 29, 31, 33, 34, 35, 38, \boxed{41, 42}, 42, 43, 43, 45, 46, 51, 53, 56, 67$. Therefore, the median is 41.5.

Mode: $33, 42$ and 43 are each repeated twice. Therefore, there are three modal ages (the data is 'tri-modal').

a. Mean: $\bar{x} = \frac{\text{R}514\ 634}{15} = \text{R}34\ 308.93$
Median:
R12 014, R13 346, R14 341, R14 416, R14 815, R16 536, R17 340, $\boxed{\text{R24 512}}$,
R24 836, R28 361, R3636 973, R40 915, R43 852, R82 366, R130 011
The median value is R24 512.Mode: There is no mode

b. Yes, the data is skewed. It is positively skewed because the median is substantially less than the mean. Therefore, there are a few, but extreme, values to the right of the mean.

c. Mean: $\bar{x} = \frac{\text{R}397\ 634}{15} = \text{R}26\ 508.93$
Median:
R12 014, R13 011, R13 346, R14 341, R14 416, R14 815, R16 536, $\boxed{\text{R17 340}}$,
R24 512, R24 836, R28 361, R36 973, R40 915, R43 852, R82 366
The median value is R17 340.With the correction of the data value, the mean value is reduced by about R8 000 to R26 508.93. This is similar to the original median value. However, the new median value is R17 340. Therefore, the median is still less than the mean although by a slightly smaller amount. Therefore, the data is still positively skewed but, perhaps, not as skewed.

2. Four friends:

$$\bar{x} = \frac{x_1 + x_2 + \ldots + x_4}{4} = 10$$

$$\therefore x_1 + x_2 + \ldots + x_4 = 10 \times 4 = 40$$

Three remaining friends will have $40 - 4 = 36$ marbles.Therefore, the new mean is:

$\bar{x} = \frac{36}{3} = 12$ marbles

Back to Unit 2: Assessment

Media Attributions

Unit 3: Measures of dispersion of ungrouped data

DYLAN BUSA

Unit 3: Measures of dispersion of ungrouped data

By the end of this unit you will be able to:

- Calculate the range, inter-quartile range and semi-inter-quartile range.
- Calculate percentiles and quartiles.

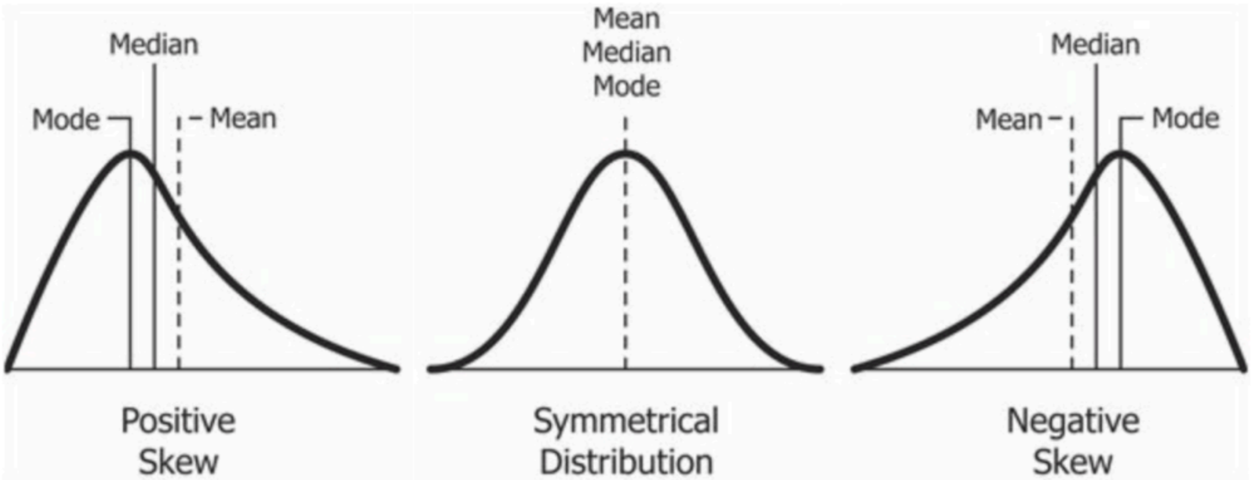
What you should know

Before you start this unit, make sure you can:

Calculate the mean, median and mode of a data set. If you need help with this, review Unit 2 in this subject outcome.

Introduction

We saw in the previous unit that a data set is not always nicely and evenly or symmetrically distributed about the mean. Sometimes it is skewed, either positively or negatively. We also saw that we can tell if a data set is skewed by looking at the value of mean – median.



$\text{mean} - \text{median} > 0$ $\quad$ $\text{mean} - \text{median} = 0$ $\quad$ $\text{mean} - \text{median} < 0$

Figure 1: Data distributions

While the value of mean – median might tell us whether a data set is skewed or not, it is not a very good method of working out by how much the data is skewed or how the data is distributed. The measures of central tendency do not fully describe a data set and can often be misleading without some indication of how spread out or variable the data is. For this, we need to look at how the data is actually spread out or **dispersed**. To do this, we use several **measures of dispersion**.

But what is dispersion anyway? Let's investigate one such measure, the range, with an activity.

The range

Activity 3.1: Measure dispersion

Time required: 10 minutes

What you need:

- a pen or pencil
- a piece of paper

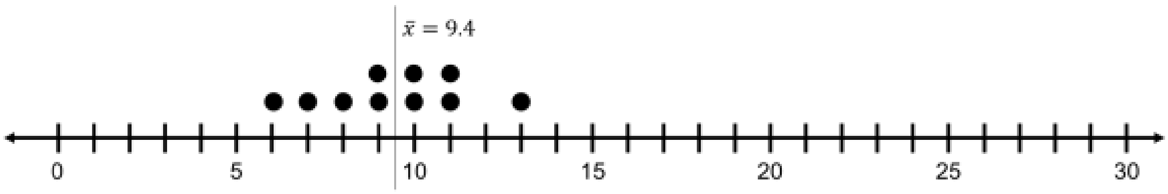
What to do:

Consider these two sets of data:
Data set A: $\{8,\ 6,\ 7,\ 11,\ 13,\ 10,\ 11,\ 9,\ 9,\ 10\}$
Data set B: $\{3,\ 1,\ 8,\ 16,\ 12,\ 1,\ 17,\ 13,\ 21,\ 18,\ 2,\ 7,\ 9,\ 12,\ 1\}$

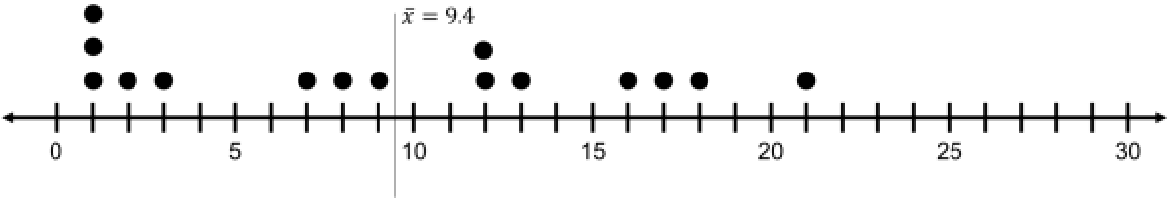
1. For each data set, calculate the mean.
2. What do you notice about the means? Do you think this means that the data sets are the same?
3. What are the smallest and biggest values in each data set?
4. What is the difference between the biggest and smallest values in each data set?
5. Represent each data set on a number line by drawing a dot above a number on the line to represent each element in the set. If two entries have the same value, draw one dot above the other. Also indicate the mean with a vertical line.
6. What do you notice about how spread out the data in each set is? How does this correspond to your answer to question 4?

What did you find?

1. $\bar{x}_{\text{A}} = \frac{94}{10} = 9.4 \qquad \bar{x}_{\text{B}} = \frac{141}{15} = 9.4$
2. Both means are the same. This does not, however, mean that the data sets are the same. For one, Data set A has ten values, while Data set B has 15.
3. Data set A: smallest value is 6 and biggest value is 13.
   Data set B: smallest value is 1 and biggest value is 21.
4. Data set A: $13 - 6 = 7$
   Data set B: $21 - 1 = 20$
5. Data set A:



Data set B:



6. The data in set B is far more spread out than in set A. It makes sense then that the difference between the biggest and smallest value in set B was greater than in set A.

The difference that we calculated between the biggest and smallest value in the data set in Activity 3.1 is called the **range** of the data set, and it is one of the measures of dispersion that we use to describe a data set. We saw that the data in set B was more spread out, or dispersed, than the data in set A, even though their means were the same. Therefore, a measure of dispersion (like the range) helps us to describe a data set more accurately, together with the measures of central tendency we learnt about in Unit 2.

By its nature, the range is very sensitive to outliers.

Take note!

Range

The range of a data set is the difference between the maximum and minimum values in the set.

Percentiles

Consider the following situation.

Walter and Thabisang were first year mathematics students who applied for a tutor job at a local community college. One of the criteria for being awarded the job was the applicant's rank in their university class.

Thabisang was ranked 30th in her Mathematics class at university while Walter was ranked 15th in his Mathematics class at another university.

At the moment it would seem that Walter had a higher ranking. However, we have no idea how many other students were in either applicant's class and, therefore, we cannot really compare their rankings. More information is needed to arrive at an informed conclusion.

How do their ranks compare if we include the fact that there were 50 students in Walter's class and 150 students in Thabisang's class?

If Thabisang was 30th out of 150 students, this means that she was in the top 20% of students. $\left(\frac{30}{150} \times 100 = 20\%\right)$. 80% of the students were ranked below her.

If Walter was 15th out of 50 students, this means that he was in the top 30% of students. $\left(\frac{15}{50} \times 100 = 30\%\right)$. 70% of the students were ranked below him.

So, even though Thabisang's ranking seemed lower, in the context of her bigger class, she was actually better ranked. She was in the top 20% of students while Walter was only in the top 30 of students. Which student would you hire?

Because 80% of students performed worse than she did, we say that Thabisang was in the 80th percentile. Walter was in the 70th percentile because 70% of students were ranked lower than him.

Percentiles divide sets of data into 100 equal parts. 100% is the basis of measure, hence the name percentile. Percentiles give us an excellent way to see how one value compares to other values in the same dataset.

Can you see that the median is really the same thing as the 50th percentile? Exactly half (50%) of the values lie above it and half lie below it.

Here is another data set: $14.2,\ 13.9,\ 19.8,\ 10.3,\ 13.0,\ 11.1$. If we arrange it in ascending order, we can find the rank of each value and its percentile (see Table 1).

Value	Rank	Percentile
10.3	1	0
11.1	2	20
13.0	3	40
13.9	4	60
14.2	5	80
19.8	6	100

Table 1: Value, rank and percentile

Because there are six values, it is easy to divide the data into six equal groups (one value per group). This means that 10.3 is at the zero percentile. There are 0% of values below it and 100% of values above it.

11.1 is at the 20th percentile. There is one value smaller than it (one of the remaining five values or 20%) and four values greater (four of the remaining five values or 80%). 14.2 is at the 80th percentile. There is one value greater than it and four values smaller.

The zero percentile is always the smallest value in a data set. The 100th percentile is always the biggest value in a data set.

We can determine the rank (r) of a value in a data set of n values at any percentile (p) using the formula $r = \frac{p}{100}(n-1)+1$.

In the above case we can work out what value is at the 60th percentile as follows:

$$\begin{aligned} r &= \frac{60}{100}(6-1)+1 \\ &= 4 \end{aligned}$$

The fourth value (13.9) is at the 60th percentile.

Take note!

Percentile

The $p\text{th}$ percentile is the value, v, that divides a data set into two parts, such that p percent of the values in the data set are less than v, and $100 - p$ percent of the values are greater than v. Percentiles can only lie in the range $0 \leq p \leq 100$.

The rank of the value at the $p\text{th}$ percentile can be calculated using $r = \frac{p}{100}(n-1)+1$.

Example 3.1

Determine the quartiles of the following data set.
$\{27,\ 45,\ 11,\ 13,\ 9,\ 15,\ 31,\ 17,\ 16,\ 40,\ 12,\ 16,\ 3,\ 11\}$

Solution

Quartiles, as the word suggests, means percentiles that cut the data into four groups where each group contains the same number of values. Therefore, the quartiles are the 25th, 50th and 75th percentiles.

First, we need to sort the dataset into increasing order.
$3,\ 9,\ 11,\ 11,\ 12,\ 13,\ 15,\ 16,\ 16,\ 17,\ 27,\ 31,\ 40,\ 45$

Next, we can find the rank of the value at the 25th percentile. There are 14 values in the dataset. Hence $n = 14$.

$$\begin{aligned} r &= \frac{p}{100}(n-1)+1 \\ &= \frac{25}{100}(14-1)+1 \\ &= 4.25 \end{aligned}$$

The rank is given as a fraction. Therefore, the 25th percentile lies between the fourth and the fifth value. When this happens, we take the value halfway between these values i.e. the mean. So, the 25th percentile is $\frac{11+12}{2} = 11.5$

50th percentile:

$$\begin{aligned} r &= \frac{p}{100}(n-1)+1 \\ &= \frac{50}{100}(14-1)+1 \\ &= 7.5 \end{aligned}$$

The 50th percentile is between the seventh and eighth value: $\frac{15+16}{2} = 15.5$

75th percentile:

$$
\begin{aligned}
r &= \frac{p}{100}(n-1)+1 \\
&= \frac{75}{100}(14-1)+1 \\
&= 10.75
\end{aligned}
$$

The 75th percentile is between the 10th and 11th value: $\dfrac{17+27}{2} = 22$

Exercise 3.1

Determine the deciles (the percentiles that divide a dataset into ten groups i.e. the 10th, 20th, 30th, ... percentiles).

$20, 29, 32, 35, 46, 51, 54, 60, 68, 68, 72, 76, 78, 83, 86, 89, 91, 92, 98, 101, 109, 114, 117, 118, 123, 126, 130, 135, 139, 144$

The full solutions are at the end of the unit.

Inter-quartile range

In Example 3.1, we found the quartiles of a dataset – the 25th, 50th and 75th percentiles. We often make use of these quartiles to describe a dataset and its characteristics and so give these quartiles special names. We call the 25th percentile the first or lower quartile ($Q1$), the median or 50th percentile the second quartile ($Q2$), and the 75th percentile the third or upper quartile ($Q3$).

We can get a sense of how tightly or widely dispersed a dataset is by calculating not just the range (the difference between the biggest and smallest value or between the zero and 100th percentiles), but also the difference between $Q1$ and $Q3$ (called the **inter-quartile range**) and comparing the answers we get.

The inter-quartile range gives us the range of the middle half of the data. We can also find the middle of this range, a measure called the **semi-inter-quartile range**.

Example 3.2

A high school has two cricket teams: a junior and a senior team. The junior team consists of 17 players (including reserves) and the senior team consists of 16 players (including reserves). The mass of each team member is given below. Use the data to answer the questions that follow.

Junior team masses (kg)
$\{56,\ 60,\ 67,\ 45,\ 51,\ 53,\ 64,\ 49,\ 56,\ 48,\ 42,\ 51,\ 64,\ 52,\ 64,\ 49,\ 50\}$

Senior team masses (kg)
$\{88,\ 81,\ 53,\ 62,\ 83,\ 68,\ 70,\ 62,\ 91,\ 78,\ 64,\ 74,\ 73,\ 54,\ 62,\ 62\}$

1. Calculate the range of both datasets.
2. Calculate the inter-quartile range of both datasets.

3. Calculate the semi-inter-quartile range of both datasets.
4. Which dataset is more dispersed?

Solutions

1. Junior team: $42,\ 45,\ 48,\ 49,\ 49,\ 50,\ 51,\ 51,\ 52,\ 53,\ 56,\ 56,\ 60,\ 64,\ 64,\ 64,\ 67$
   The range is $67 - 42 = 25$
   Senior team: $53,\ 54,\ 62,\ 62,\ 62,\ 62,\ 64,\ 68,\ 70,\ 73,\ 74,\ 78,\ 81,\ 83,\ 88,\ 91$
   The range is $91 - 53 = 38$
2. Junior team:
   $Q1$:
   $$r = \frac{25}{100}(17-1)+1$$
   $$= 5$$
   Therefore $Q1 = 49$
   $Q3$:
   $$r = \frac{75}{100}(17-1)+1$$
   $$= 13$$
   Therefore, $Q3 = 60$
   Therefore the inter-quartile range is $Q3 - Q1 = 60 - 49 = 11$

   Senior team:
   $Q1$:
   $$r = \frac{25}{100}(16-1)+1$$
   $$= 4.75$$
   Therefore $Q1 = \dfrac{62+62}{2} = 62$
   $Q3$:
   $$r = \frac{75}{100}(16-1)+1$$
   $$= 12.25$$
   Therefore, $Q3 = \dfrac{78+81}{2} = 79.5$
   Therefore the inter-quartile range is $Q3 - Q1 = 79.5 - 62 = 17.5$
3. Junior team: semi-inter-quartile range is $\dfrac{11}{2} = 5.5$
   Senior team: semi-inter-quartile range is $\dfrac{17.5}{2} = 8.75$
4. By all measures the senior team data is more dispersed. It has both a greater range as well as a greater inter-quartile range. Therefore, not only is there a greater difference between the largest and smallest values but the middle 50% of the data is also more spread out over a greater range.

Exercise 3.2

Class A and Class B both wrote a test out of 50 marks. The results of each class are given. Use the data to answer the questions that follow.

Class A: $\{36,\ 39,\ 49,\ 18,\ 36,\ 24,\ 38,\ 36,\ 28,\ 27,\ 30,\ 42,\ 48,\ 45,\ 39,\ 21,\ 29,\ 31,\ 34\}$
Class B: $\{34,\ 34,\ 19,\ 27,\ 36,\ 26,\ 39,\ 39,\ 29,\ 39,\ 35,\ 46,\ 41,\ 35,\ 29,\ 25,\ 35,\ 37,\ 31,\ 39,\ 42,\ 28\}$

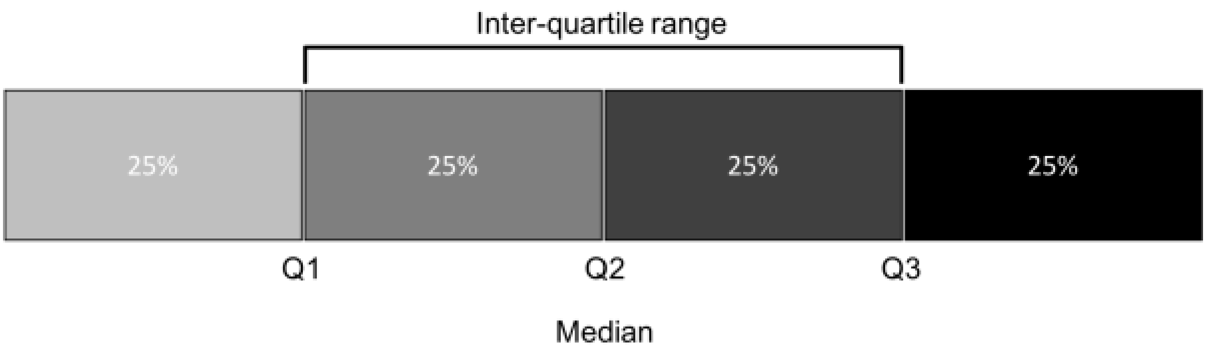
1. Calculate the mean and median for each class.
2. Calculate $Q1$ and $Q3$ for each class.
3. Calculate the inter-quartile range for each class.
4. What would you have had to score to be in the 80th percentile in each class?
5. Which class did better? Explain your answer.

The full solutions are at the end of the unit.

Take note!

Quartiles

- $Q1$: first or lower quartile (25th percentile)
- $Q2$: second quartile (the median or 50th percentile)
- $Q3$: third or upper quartile (75th percentile)
- Inter-quartile range: $Q3 - Q1$
- Semi-inter-quartile range: $\dfrac{Q3 - Q1}{2}$



Summary

In this unit you have learnt the following:

- Measures of dispersion tell us how spread out or dispersed the values in a dataset are.
- The range is the difference between the biggest value and the smallest value.
- A percentile is the value below which a given percentage of scores fall. The 40th is that value in a data set below which 40% of the values lie and above which 60% of the values lie.
- The rank of the value at the $p\text{th}$ percentile for a dataset of n values can be calculated using $r = \dfrac{p}{100}(n-1)+1$.
- Quartiles divide a dataset into four equal groups. The quartiles are:
  - $Q1$: first or lower quartile (25th percentile)

- $Q2$: second quartile (the median or 50th percentile)
- $Q3$: third or upper quartile (75th percentile).

- The inter-quartile range is the difference between $Q3$ and $Q1$, and represents the range in which the middle 50% of the data lie.
- The semi-inter-quartile range is the middle of the inte-rquartile range.

Unit 3: Assessment

Suggested time to complete: 40 minutes

1. A group of 20 students count the number of phone calls they have each made in the past month. This is the data they collect:
   $\{11,\ 8,\ 17,\ 13,\ 9,\ 12,\ 2,\ 6,\ 15,\ 7,\ 14,\ 15,\ 1,\ 6,\ 6,\ 13,\ 19,\ 9,\ 6,\ 19\}$
   Calculate the range of values in the data set.
2. A company wanted to evaluate the training programme in its factory. They gave the same task to trained and untrained employees and timed each one in seconds.
   Trained: $\{121,\ 137,\ 131,\ 135,\ 130,\ 128,\ 130,\ 126,\ 132,\ 127,\ 129,\ 120,\ 118,\ 125,\ 134\}$
   Untrained: $\{135,\ 142,\ 126,\ 148,\ 145,\ 156,\ 152,\ 153,\ 149,\ 145,\ 144,\ 134,\ 139,\ 140,\ 142\}$
   a. Find the mean of each dataset.
   b. Find the median for each dataset.
   c. Find the inter-quartile range for both sets of data.
   d. Did the training programme work?
3. There are 14 men working in a factory. Their ages are:
   $22,\ 25,\ 33,\ 35,\ 38,\ 48,\ 55,\ 55,\ 55,\ 55,\ 55,\ 56,\ 59,\ 64$
   a. If three men had to be retrenched, but the median had to stay the same, show the ages of the three men you would retrench.
   b. Find the mean age of the men in the factory using the original data.

The full solutions are at the end of the unit.

Unit 3: Solutions

Exercise 3.1

The data has already been ordered.
$$*20l29,\ 32,\ 35,\ 46,\ 51,\ 54,\ 60,\ 68,\ 68,\ 72,\ 76,\ 78,\ 83,\ 86,\ 89,\ 91,$$
$$92,\ 98,\ 101,\ 109,\ 114,\ 117,\ 118,\ 123,\ 126,\ 130,\ 135,\ 139,\ 144$$
p_0: 29
$$p_{10}: r = \frac{10}{100}(29-1)+1 = 3.8$$

We get a rank of 3.8 so we need to find the mean of the third and fourth values in the ordered data set. The third value (rank $=3$) is 35 and the fourth value (rank $=4$) is 46.

$\therefore p_{10} = \dfrac{35 + 46}{2} = 40.5$

$p_{20}: r = \dfrac{20}{100}(29 - 1) + 1 = 6.6$

We get a rank of 6.6 so we need to find the mean of the sixth and seventh values in the ordered data set. The sixth value (rank $= 6$) is 54 and the seventh value (rank $= 7$) is 60.

$\therefore p_{20} = \dfrac{54 + 60}{2} = 57$

$p_{30}: r = \dfrac{30}{100}(29 - 1) + 1 = 9.4 \;\; \therefore p_{30} = \dfrac{68 + 72}{2} = 70$

$p_{40}: r = \dfrac{40}{100}(29 - 1) + 1 = 12.2 \;\; \therefore p_{40} = \dfrac{78 + 83}{2} = 80.5$

$p_{50}: r = \dfrac{50}{100}(29 - 1) + 1 = 15 \;\; \therefore p_{50} = 89$

$p_{60}: r = \dfrac{60}{100}(29 - 1) + 1 = 17.8 \;\; \therefore p_{60} = \dfrac{92 + 98}{2} = 95$

$p_{70}: r = \dfrac{70}{100}(29 - 1) + 1 = 20.6 \;\; \therefore p_{70} = \dfrac{109 + 114}{2} = 111.5$

$p_{80}: r = \dfrac{80}{100}(29 - 1) + 1 = 23.4 \;\; \therefore p_{80} = \dfrac{118 + 123}{2} = 120.5$

$p_{90}: r = \dfrac{90}{100}(29 - 1) + 1 = 26.2 \;\; \therefore p_{90} = \dfrac{130 + 135}{2} = 132.5$

p_{100}: 144

Back to Exercise 3.1

Exercise 3.2

Class A: $\{36,\ 39,\ 49,\ 18,\ 36,\ 24,\ 38,\ 36,\ 28,\ 27,\ 30,\ 42,\ 48,\ 45,\ 39,\ 21,\ 29,\ 31,\ 34\}$
Class B: $\{34,\ 34,\ 19,\ 27,\ 36,\ 26,\ 39,\ 39,\ 29,\ 39,\ 35,\ 46,\ 41,\ 35,\ 29,\ 25,\ 35,\ 37,\ 31,\ 39,\ 42,\ 28\}$

1. Class A:

   Mean: $\bar{x} = \dfrac{650}{19} = 34.21$

   Median: $18, 21, 24, 27, 28, 29, 30, 31, 34, \boxed{36}, 36, 36, 38, 39, 42, 45, 48, 49$

   The median mark is 36.

   Class B:

   Mean: $\bar{x} = \dfrac{745}{22} = 33.86$

   Median: $19, 25, 26, 27, 28, 29, 29, 31, 34, 34, \boxed{35, 35}, 35, 36, 37, 39, 39, 39, 39, 41, 42, 46$

   The median is 35.

2. Class A:

   Class A: $Q1 = \dfrac{25}{100}(19 - 1) + 1 = 5.5$

   Therefore $Q1 = \dfrac{28 + 29}{2} = 28.5$

   $Q3 = \dfrac{75}{100}(19 - 1) + 1 = 14.5$

   Therefore $Q3 = \dfrac{39 + 39}{2} = 39$

   Class B:

   Class B: $Q1 = \dfrac{25}{100}(22 - 1) + 1 = 6.25Q3 = \dfrac{75}{100}(22 - 1) + 1 = 16.75$

   Therefore $Q1 = \dfrac{29 + 29}{2} = 29$

$Q3 = \frac{75}{100}(22 - 1) + 1 = 16.75$
Therefore $Q3 = \frac{39 + 39}{2} = 39$

3. Class A inter-quartile range (IQR): $Q3 - Q1 = 39 - 28.5 = 10.5$
Class B inter-quartile range (IQR): $Q3 - Q1 = 39 - 29 = 10$
4. Class A:
$r = \frac{80}{100}(19 - 1) + 1 = 15.4$
$p_{80} = \frac{39 + 42}{2} = 40.5$
If you scored more than 40.5 out of 50 you would have been in the 80th percentile.

Class B:
$r = \frac{80}{100}(22 - 1) + 1 = 17.8$
$p_{80} = \frac{39 + 39}{2} = 39$
If you scored more than 39 out of 50 you would have been in the 80th percentile.
5. Class A had a higher mean. The two classes IQRs were more or less the same. Class A had a higher 80th percentile. Therefore Class A did marginally better than Class B.

Back to Exercise 3.2

Unit 3: Assessment

1. Order the data: $1,\ 2,\ 6,\ 6,\ 6,\ 6,\ 7,\ 8,\ 9,\ 9,\ 11,\ 12,\ 13,\ 13,\ 14,\ 15,\ 15,\ 17,\ 19,\ 19$
Range: $19 - 1 = 18$
2.
   a. Trained:
   $\bar{x} = \frac{1\ 923}{15} = 128.2$

   Untrained:
   $\bar{x} = \frac{2\ 150}{15} = 143.3$
   b. Trained: $118, 120, 121, 125, 126, 127, 128, \boxed{129}, 130, 130, 131, 132, 134, 135, 137$
   The median is 129 seconds
   Untrained: $126, 134, 135, 139, 140, 142, 142, \boxed{144}, 145, 145, 148, 149, 152, 153, 156$
   The median is 144 seconds
   c. Trained:
   Class A: $Q1 = \frac{25}{100}(15 - 1) + 1 = 4.5$
   Therefore $Q1 = \frac{125 + 126}{2} = 125.5$
   $Q3 = \frac{75}{100}(15 - 1) + 1 = 11.5$
   Therefore $Q3 = \frac{131 + 132}{2} = 131.5$
   Trained IQR: $131.5 - 125.5 = 6$

   Untrained:
   Class A: $Q1 = \frac{25}{100}(15 - 1) + 1 = 4.5$

Therefore $Q1 = \dfrac{139 + 140}{2} = 139.5$

$Q3 = \dfrac{75}{100}(15 - 1) + 1 = 11.5$

Therefore $Q3 = \dfrac{148 + 149}{2} = 148.5$

Trained IQR: $148.5 - 139.5 = 9$

d. The programme did work. Not only was the mean time of the trained workers less than the untrained workers but the distribution of times was also smaller, meaning that the trained workers were all performing more consistently.

3. Data is already ordered: $22,\ 25,\ 33,\ 35,\ 38,\ 48,\ 55,\ 55,\ 55,\ 55,\ 55,\ 56,\ 59,\ 64$

   a. The median is currently 55. To keep the median at 55, one must retrench one person younger and one person older than the median, and a third at the median age. One combination could be the oldest and youngest workers as well as one of the workers aged 55. You might have given a different answer that also keeps the median at 55

   $\cancel{22},\ 25,\ 33,\ 35,\ 38,\ 48,\ 55,\ 55,\ 55,\ 55,\ \cancel{55},\ 56,\ 59,\ \cancel{64}$

   $25, 33, 35, 38, 48, \boxed{55}, 55, 55, 55, 56, 59$

   b. $\bar{x} = \dfrac{655}{14} = 46.79$

Back to Unit 3: Assessment

Media Attributions

SUBJECT OUTCOME XIII

STATISTICAL AND PROBABILITY MODELS: REPRESENT DATA EFFECTIVELY

Subject outcome 4.2

Represent data effectively.

Learning outcomes

- Represent data effectively, choosing appropriately from:
  - Construction of frequency distribution/tally chart
  - Bar and compound bar graphs
  - Construction of the stem and leaf plot
  - Histograms (grouped data)
  - Frequency polygons
  - Pie charts
  - Line and broken line graphs.

Unit 1: Data representation

By the end of this unit you will be able to:

- Construct a frequency distribution/tally chart.
- Construct a stem and leaf plot.
- Construct a pie chart.

Unit 2: Bar graphs and histograms

By the end of this unit you will be able to:

- Construct a bar graph.
- Construct a compound bar graph.
- Construct a histogram.

Unit 3: Frequency polygons and line graphs

By the end of this unit you will be able to:

- Calculate measures of central tendency and percentiles on grouped data.
- Construct a frequency polygon.
- Construct a line graph.

Unit 1: Data representation

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Unit 1: Data representation

By the end of this unit you will be able to:

- Construct a frequency distribution/tally chart.
- Construct a stem and leaf plot.
- Construct a pie chart.

What you should know

Before you start this unit, make sure you can:

- Define the types of data.
- Differentiate between grouped and ungrouped data.
- Create a frequency table.

Introduction

They say that a picture is worth a thousand words and the same is often true in statistics. Graphs and other visual depictions are routinely used to communicate data and the findings of various pieces of research.

However, beware! Sometimes, the way that data is presented graphically is knowingly or unknowingly misleading. We watched a video about how graphs can be misleading in Unit 1 of the previous subject outcome. Here is another one with some real-life examples.

Misleading Graphs Real Life Examples (Duration: 05.24)

Frequency distributions

One of the most common starting points for organising and representing data is a frequency distribution. We learnt in Unit 1 of Subject outcome 4.1 that frequency distributions or frequency tables can be used to organise data about a group of ladies' favourite flowers from the raw data (see Figure 1) to a table that is easier to read (Table 1).

Roses	Tulips	Roses
Daffodils	Asters	Daffodils
Tulips	Roses	Roses
Asters	Roses	Tulips
Tulips	Roses	Asters
Asters	Tulips	Roses
Asters	Daffodils	Tulips
Roses	Asters	Daffodils
Asters	Roses	Asters
Roses	Daffodils	Tulips

Figure 1: Raw data from a survey of 30 ladies about their favourite flowers

Flower type	Tally	Frequency
Roses	卌 卌	10
Daffodils	卌	5
Tulips	卌 II	7
Asters	卌 III	8
Total	30	

Table 1: Frequency table showing the favourite flowers of 30 ladies

In the case of the flowers, there were only four different types mentioned in the survey. In addition, the data was qualitative, so it was a simple matter of listing the four different flowers and then counting the frequency of each.

However, what if the data is quantitative, possibly continuous, and there are far more values in the data set like the data set below which represents the heights (in centimetres) of a group of 90 students.

$165, 148, 158, 150, 160, 165, 150, 156, 155, 164, 162, 160, 158, 148, 158, 140, 146, 160, 148, 152, 139, 165, 148, 160, 156, 158, 170, 155, 160, 148, 155, 158, 179, 170, 158, 161, 155, 160, 163, 178, 138, 172, 170, 156, 160, 160, 171, 140, 160, 170, 175, 148, 170, 177, 155, 167, 154, 160, 170, 155, 136, 179, 150, 167, 148, 160, 164, 167, 157, 165, 163, 140, 162, 178, 160, 170, 163, 162, 165, 175, 165, 152, 147, 180, 148, 170, 165, 167, 165$

Because the chances of any one height measured being reported more than once or twice is quite small, it would be silly to simply list all the values and count their frequency. In this case, we need to group the data into intervals or classes and then count how many values lie in each interval. Table 2 shows what the resulting frequency distribution for this raw data might look like.

Class limits	Frequency
$136 - 144$	6
$145 - 153$	15
$154 - 162$	33
$163 - 171$	27
$172 - 180$	9

Table 2: Grouped data frequency table of the heights of 90 *students*

Take note!

The data in its raw form is called **ungrouped data**. It has not been organised in any way and is merely a list of the individual measurements. The data can be organised into **intervals**, and then presented in the form of a frequency distribution. We call this **grouped data**.

In Unit 3 of this Subject outcome, we will learn how to use this frequency distribution to create a histogram. For now, let's focus on learning how to create a grouped frequency distribution.

Step 1:
The first step in creating a grouped frequency distribution is to work out what the range of the data is. Can you do this? What is the range of the raw data above?

Remember, to find the range we must subtract the smallest value from the largest value in the data set. This is $180 - 136 = 44$.

Step 2:
The second step is to decide on the number of groups or classes you want. There are no hard and fast rules for this but, in general, you should not have fewer than five classes or more than 15 classes. You can experiment with the number of classes until you find one that you think fits the data best. Often, step 3 will help you decide on the number of classes to have. With a range of 44, about five or six classes probably makes sense.

Step 3:
In this step you find the class width; in other words, the difference between the lower class limit and the upper class limit. It is often a great idea to have an odd class width so that the midpoint of your class is a whole number. You will see why in Unit 3 when we draw histograms.

For now, if we were to have six classes, our class width would be $\frac{\text{range}}{\text{classes}} = \frac{44}{6} = 7.33$. We always round up to the next whole number. So, the class width would be eight. This is not too bad, but it is not an odd number. Let's try five classes.

Now our class width would be $\frac{\text{range}}{\text{classes}} = \frac{44}{5} = 8.8$. Rounding up to the next whole number gives us a width of nine which is odd. Let's go for five classes each with a width of nine.

Step 4:

Now it is time to create our frequency distribution table. The first column contains our class limits. The first class starts at the smallest value (136) and we count up nine to get to the upper class limit. Therefore, the first upper class limit is 144 (see Table 3). This class will contain all the values from 136 up to 144.

The second lower class limit is 145 and we count up another nine to get to 153. Notice that there is a difference of nine (the class width) between each of the lower limits and upper class limits (see Table 3).

Class limits	Tally	Frequency
$136 - 144$		
$145 - 153$		

Table 3: Frequency distribution of heights of students in centimetres

See if you can complete the class limits on your own before moving on to Step 5.

Table 4 shows all the class limits.

Class limits	Tally	Frequency
$136 - 144$		
$145 - 153$		
$154 - 162$		
$163 - 171$		
$172 - 180$		

Table 4: Frequency distribution of heights of students in centimetres

Step 5:
Now that we have our class limits, we can go through the data set and count how many values lie in each class. In Table 4 we have included a tally column to make this counting a little easier, but this is not strictly required, and we normally do not show frequency distributions with a tally column.

The first value in the data set is 165 so we put a tally mark in the $163 - 171$ class row (see Table 5). The next value is 148 so we put a tally mark in the $145 - 153$ class row. Continue like this through the rest of the values. To make counting the tally marks easier at the end, the fifth tally mark in a class is a cross through the previous four like this $\cancel{||||}$ or 𝍸.

Class limits	Tally	Frequency
$136 - 144$		
$145 - 153$	|	
$154 - 162$		
$163 - 171$	|	
$172 - 180$		

Table 5: Frequency distribution of heights of students in centimetres

See if you can complete the frequency distribution on your own before reading on.

Did you know?

There is evidence of tally marks being used over $20\ 000$ years ago. Tally sticks (see Figure 2) were also used thousands of years ago to aid counting and memory.

Figure 2: Ancient tally sticks

Table 6 shows the completed frequency distribution. Were you able to complete it correctly?

Class limits	Tally	Frequency
$136 - 144$	𝍸 |	6
$145 - 153$	𝍸 𝍸 𝍸	15
$154 - 162$	𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 |||	33
$163 - 171$	𝍸 𝍸 𝍸 𝍸 𝍸 ||	27
$172 - 180$	𝍸 ||||	9

Table 6: Frequency distribution of heights of students in centimetres

As noted above, we usually do not present the frequency distribution with any tally marks. Table 7 shows the final frequency distribution.

Class limits	Frequency
$136 - 144$	6
$145 - 153$	15
$154 - 162$	33
$163 - 171$	27
$172 - 180$	9

Table 7: Frequency distribution of heights of students in centimetres

We can see, just by looking at the frequency of values in each interval, that the data follows a fairly normal distribution (most values are in the middle of the range). It is not significantly skewed positive or negative. This is not something we could ever have seen from the raw data.

Take note!

In the scenario above, all our values were rounded off to whole numbers. If you look at Table 7 again, you will see that the first class ends at 144 and the next class begins at 145. This was acceptable in this scenario because there were no fractional values that fell between any of the class limits.

However, most of the time with continuous data, this will not be the case and you will need to make sure that your classes are continuous, in other words that there are no gaps between them. The way to do this is to define your classes with inequalities as shown below.

Class limits	Frequency
$136 \leq x < 145$	
$145 \leq x < 154$	
$154 \leq x < 163$	
$163 \leq x < 172$	
$172 \leq x < 181$	

In this case, you would include all values equal to and bigger than 136 **up to, but not including**, 145 in the first interval.

Exercise 1.1

A group of learners counts the number of pairs of shoes each group member has. The data they collected is as follows:

$\{3,\ 6,\ 5,\ 14,\ 20,\ 4,\ 7,\ 14,\ 8,\ 5,\ 11,\ 19,\ 17,\ 16,\ 16,\ 13,\ 9,\ 6,\ 9,\ 13,\ 21,\ 7,\ 11,\ 9,\ 13,\ 5,\ 7\}$

Create a frequency distribution for the data using five intervals.

The full solutions are at the end of the unit.

Stem and leaf plots

Another quick way to represent numeric or quantitative data is to use a stem and leaf plot. Stem and leaf plots, like frequency distributions, give you a quick way to see the 'shape' of the data. Is it normally distributed? Is it significantly negatively or positively skewed? Stem and leaf plots give you a way to summarise all the data in a set in one simple representation.

Note

Stem and leaf plots are very quick and easy to create. Watch this video called "Stem & Leaf Plots" to find out how.

Stem & Leaf Plots (Duration: 02.53)

Example 1.1

Farmer Dlamini has a dairy with 40 cows. He recorded the quantity of milk (in litres) that each cow produced in a week in Table 8.

41	84	89	67	87	89	86	54
50	46	43	89	88	71	42	69
67	68	75	49	71	68	70	54
72	56	65	77	53	56	43	63
67	83	68	88	67	69	80	82

Table 8: Quantity of milk (in litres) produced per cow in one week

Draw a stem and leaf plot for this data.

Solution

Our stem will represent tens and the leaves will represent units. First list the stems.

$$\begin{array}{c|l} 4 & \\ 5 & \\ 6 & \\ 7 & \\ 8 & \end{array}$$

Now work through the data and add each value to the plot. The first value is 41 and is represented as follows:

$$\begin{array}{c|l} 4 & 1 \\ 5 & \\ 6 & \\ 7 & \\ 8 & \end{array}$$

The second value is 84 and is represented as follows:

4	1
5	
6	
7	
8	4

Work through the rest of the values. Don't forget to repeat duplicate values.

4	1 6 3 2 9 3
5	4 0 4 6 3 6
6	7 9 7 8 8 5 3 7 8 7 9
7	1 5 1 0 2 7
8	4 9 7 9 6 9 8 3 8 0 2

Order the leaves in each stem from smallest to largest.

4	1 2 3 3 6 9
5	0 3 4 4 6 6
6	3 5 7 7 7 7 8 8 8 9 9
7	0 1 1 2 5 7
8	0 2 3 4 6 7 8 8 9 9 9

Finally, remember to add in a key so others know how to read your plot.

4	1 2 3 3 6 9
5	0 3 4 4 6 6
6	3 5 7 7 7 7 8 8 8 9 9
7	0 1 1 2 5 7
8	0 2 3 4 6 7 8 8 9 9 9

4 | 1 means 41

Exercise 1.2

Draw a stem and leaf plot of the following data.

1.5	2.4	2.3	2.4	1.8	1.4	1.5	4.5
2.7	3.9	1.3	1.8	1.9	3.2	3.3	3.4
4.8	2.6	1.9	1.9	4.2	3.0	3.9	3.2

The full solutions are at the end of the unit.

Pie charts

Pie charts are a common way to represent data graphically. They are also easy to make with spreadsheet software like Microsoft Excel or Google Sheets. However, it is important that you can draw pie charts by hand with a protractor. Work through Activity 1.1 to learn how.

Activity 1.1: Draw a pie chart

Time required: 15 minutes

What you need:

- a pen or pencil
- a piece of paper
- a protractor

What to do:

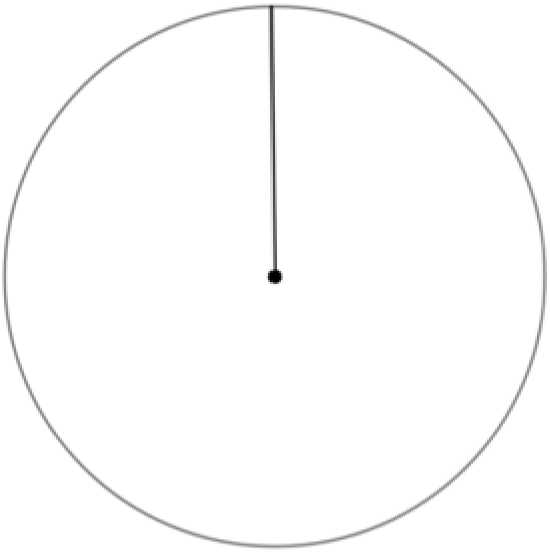
The table below shows the total population of each part of the world. We are going to draw a pie chart to represent this data. Notice how the population data has been organised from greatest to smallest. When drawing a pie chart, it is always best to arrange your categories in descending order.

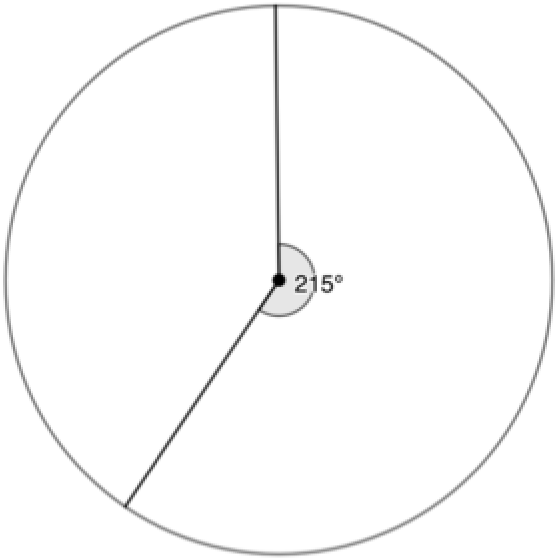
Region	Population (in millions), 2017
Asia	4 478
Africa	1 247
Europe	739
Latin America and Caribbean	648
North America	363
Oceania	40

1. Draw a circle on your piece of paper. It should have a radius of about 10 cm. Now draw a line from the centre to the circumference straight up. This is where we will start measuring our pie slices from.
2. To calculate what proportion of the entire pie each region needs to occupy, we have to work out the total population first. Do this calculation.

3. Now work out the angle that you will need to draw to represent the Asia region by dividing Asia's population by the total and then multiplying this proportion by $360°$. Round off to the nearest degree.
4. Measure this angle in a clockwise direction from the first line you drew and mark off this area to represent the Asia region.
5. Calculate the angles required to represent each of the other regions and measure and draw in these angles (in a clockwise direction), rounding off to the nearest degree.
6. Label each slice of your pie chart.
7. Calculate the total percentage of the total population in each region and add these figures below each label.
8. Give your pie chart a title. If you like, you can colour or shade in each slice to make them more distinct from each other.

What did you find?

1. 
2. The total population is $4\ 478 + 1\ 247 + 739 + 648 + 363 + 40 = 7\ 515$.
3. Asia angle: $\frac{4\ 478}{7\ 515} \times 360° = 215°$
4.



5. Africa angle: $\frac{1\ 247}{7\ 515} \times 360^\circ = 60^\circ$

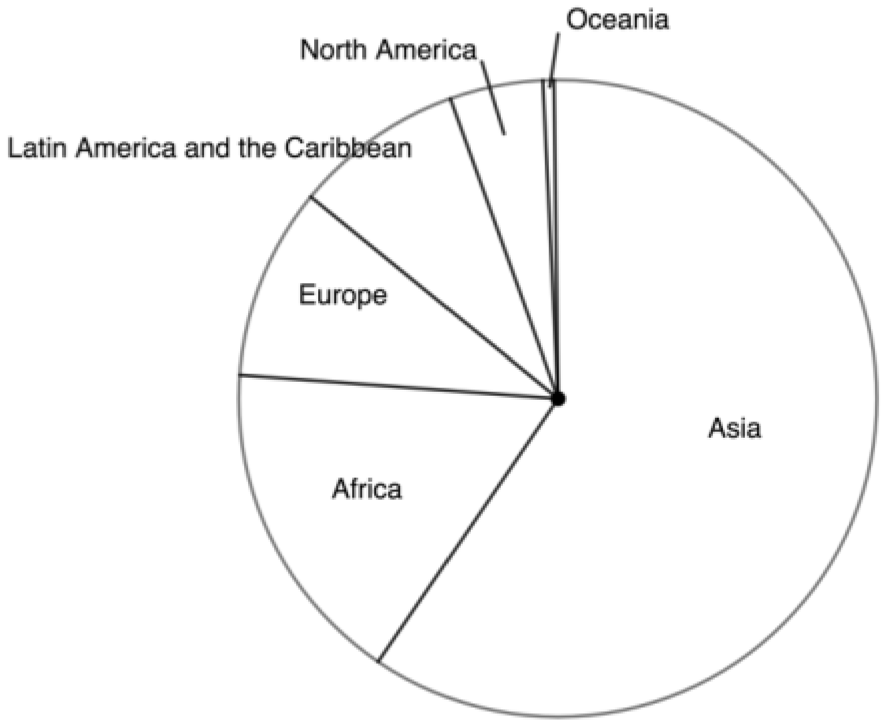
   Europe angle: $\frac{739}{7\ 515} \times 360^\circ = 35^\circ$

   Latin America and the Caribbean angle: $\frac{648}{7\ 515} \times 360^\circ = 31^\circ$

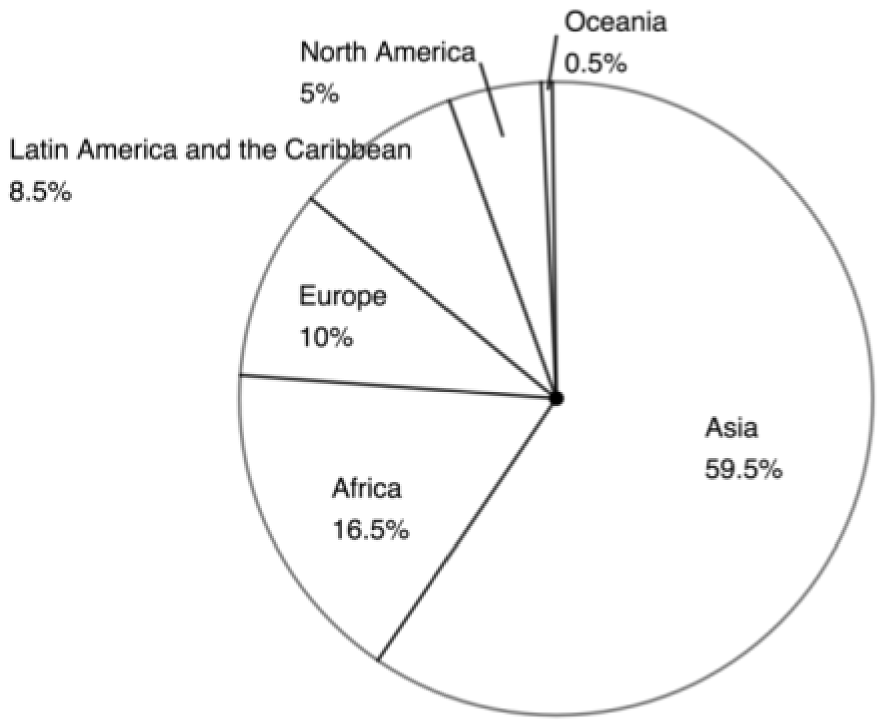
   North America angle: $\frac{363}{7\ 515} \times 360^\circ = 17^\circ$

   Oceania angle: $\frac{40}{7\ 515} \times 360^\circ = 2^\circ$

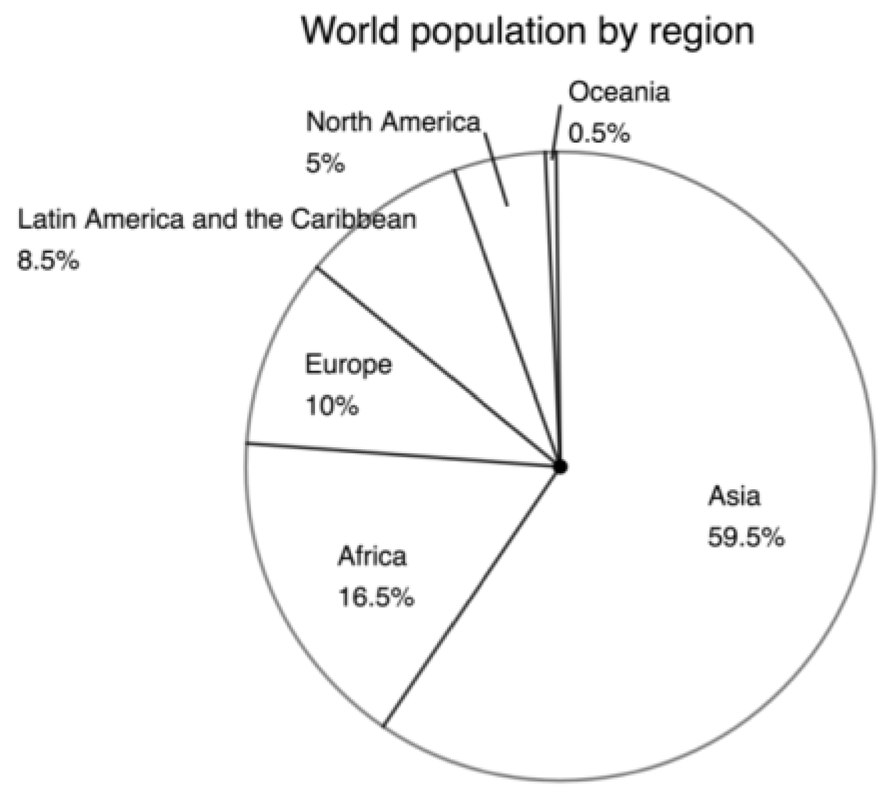
6.


Oceania
North America
Latin America and the Caribbean
Europe
Asia
Africa

7.


Oceania
0.5%
North America
5%
Latin America and the Caribbean
8.5%
Europe
10%
Asia
59.5%
Africa
16.5%

8.



Note

If you still need more help in understanding how to complete this activity, watch the video called "Pie Charts" which works through this same example.

Pie Charts (Duration: 06.02)

If you would like to watch another example of how to create a pie chart, watch this video called "Pie"

Pie Charts and Protractors (Duration: 05.06)

Exercise 1.3

Construct a pie chart of the following data. Show all your calculations. You may refer to the frequency distribution at the beginning of this unit to help you.

Roses	Tulips	Roses
Daffodils	Asters	Daffodils
Tulips	Roses	Roses
Asters	Roses	Tulips
Tulips	Roses	Asters
Asters	Tulips	Roses
Asters	Daffodils	Tulips
Roses	Asters	Daffodils
Asters	Roses	Asters
Roses	Daffodils	Tulips

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to create a grouped frequency distribution using intervals or classes to group numerical data.
- How to represent data using a stem and leaf plot.
- How to represent data using a pie chart.

Unit 1: Assessment

Suggested time to complete: 40 minutes

Question 1 adapted from the NC(V) Mathematics Second Paper November 2012 Question 1.2.

1. GQ Magazine held a poll to determine the favourite DJs of NC(V) learners. The results of the first 30 responses are tabulated below.

CHOICE OF DJ		
DJ Mbuso	DJ S'bu	DJ Mbuso
DJ China Man	DJ Cleo	DJ S'bu
DJ S'bu	DJ Fresh	DJ Cleo
DJ Mbuso	DJ Cleo	DJ Mbuso
DJ Fresh	DJ Mbuso	DJ Fresh
DJ Cleo	DJ China Man	DJ Cleo
DJ Fresh	DJ Cleo	DJ Fresh
DJ S'bu	DJ S'bu	DJ Mbuso
DJ Cleo	DJ Cleo	DJ S'bu
DJ China Man	DJ Mbuso	DJ Cleo

Use the information in the table to answer the following questions.

a. Complete the following frequency distribution

FREQUENCY DISTRIBUTION TABLE: CHOICE OF DJ		
DJ	TALLY	FREQUENCY
DJ Fresh		
DJ S'bu	~~||||~~ /	6
DJ China Man		
DJ Cleo		
DJ Mbuso		
	Total:	

b. Which DJ is most liked by these learners?

c. Use the completed table from question a. to construct a pie chart to represent the data.

Question 2 adapted from the NC(V) Mathematics Second Paper November 2012 Question 1.4.

2. The following data set shows the number of eggs laid by Mama Mchunu's hens.

4	17	9	9
12	15	14	11
15	6	24	4

Create a stem and leaf plot of the data.

3. A competition was held where students needed to guess the number of sweets in a jar. The closest guess would win the jar of sweets. The following guesses were recorded.

156	148	140	112	134	133	138	129	158	162
123	116	138	144	156	142	147	136	137	167
143	115	147	133	146	172	141	117	153	135
147	122	123	114	165	157	170	127	167	175
137	135	125	140	162	135	134	136	143	165

Create a grouped frequency distribution for this data using a class width of 10.

The full solutions are at the end of the unit.

Unit 1: Solutions

Exercise 1.1

The range of the data is $21 - 3 = 18$. We need to create five intervals or classes.
$\frac{\text{range}}{\text{classes}} = \frac{18}{5} = 3.6$. Round this up to 4.

Class limits	Frequency
$3 \leq x < 7$	7
$7 \leq x < 11$	7
$11 \leq x < 15$	7
$15 \leq x < 19$	3
$19 \leq x < 23$	3

Back to Exercise 1.1

Exercise 1.2

1	3 4 5 5 8 8 9 9 9
2	3 4 4 6 7
3	0 2 2 3 4 9 9
4	2 5 8

1 | 3 means 1.3

Back to Exercise 1.2

Exercise 1.3

The frequency distribution:

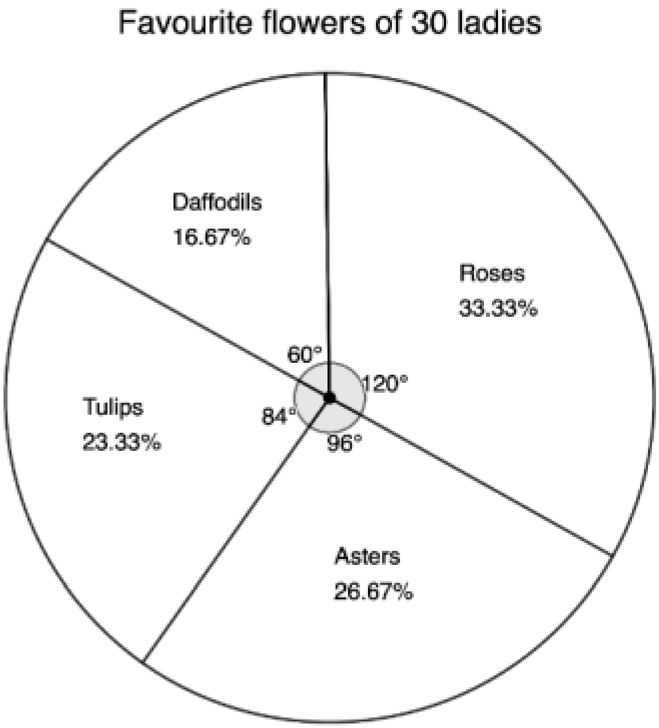
Flower type	Tally	Frequency
Roses	~~IIII~~ ~~IIII~~	10
Daffodils	~~IIII~~	5
Tulips	~~IIII~~ II	7
Asters	~~IIII~~ III	8
Total	30	

Roses angle: $\frac{10}{30} \times 360^\circ = 120^\circ$

Asters angle: $\frac{8}{30} \times 360^\circ = 96^\circ$

Tulips angle: $\frac{7}{30} \times 360^\circ = 84^\circ$

Daffodils angle: $\frac{5}{30} \times 360^\circ = 60^\circ$



Back to Exercise 1.3

Unit 1: Assessment

1.

   a.

DJ	Tally	Frequency
DJ Fresh	~~||||~~	5
DJ S'bu	~~||||~~ |	6
DJ China Man	|||	3
DJ Cleo	~~||||~~ ||||	9
DJ Mbuso	~~||||~~ ||	7
	Total	30

b. DJ Cleo is most liked.

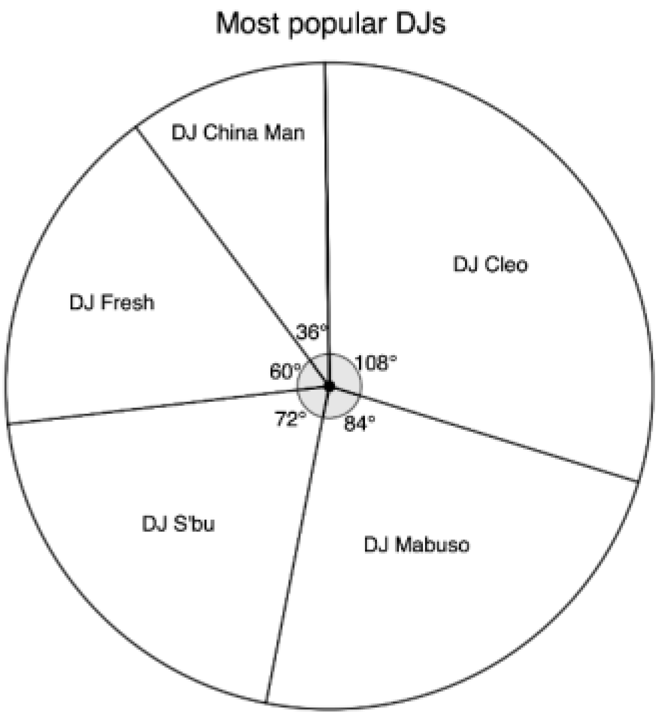
c. DJ Fresh: $\frac{5}{30} \times 360^\circ = 60^\circ$

DJ S'bu: $\frac{6}{30} \times 360^\circ = 72^\circ$

DJ China Man: $\frac{3}{30} \times 360^\circ = 36^\circ$

DJ Cleo: $\frac{9}{30} \times 360^\circ = 108^\circ$

DJ Mabuso: $\frac{7}{30} \times 360^\circ = 84^\circ$



2.

0	4 4 6 9 9
1	1 2 4 5 5 7
2	4

0 | 4 means 4 eggs

3. The class width must be 10. The lowest value is 112. Therefore, the first class will be $112 \leq x < 122$.

Class limits	Frequency
$112 \leq x < 122$	5
$122 \leq x < 132$	6
$132 \leq x < 142$	16
$142 \leq x < 152$	9
$152 \leq x < 162$	5
$162 \leq x < 172$	7
$172 \leq x < 182$	2

Back to Unit 1: Assessment

Media Attributions

Unit 2: Bar graphs and histograms

DYLAN BUSA

Unit 2: Bar graphs and histograms

By the end of this unit you will be able to:

- Construct a bar graph.
- Construct a compound bar graph.
- Construct a histogram.

What you should know

Before you start this unit, make sure you can:

- Create ungrouped and grouped frequency distributions. Refer to unit 1 in this Subject outcome if you need help with this.

Introduction

Bar graphs (or charts) and histograms are everywhere. They may even be the most common method of representing data in everyday life. They can be used to represent a wide variety of different kinds of data. They are also easy to create and easy to read.

Here are a few real examples of bar graphs (see Figures 1 and 2) and histograms (see Figures 3 and 4).

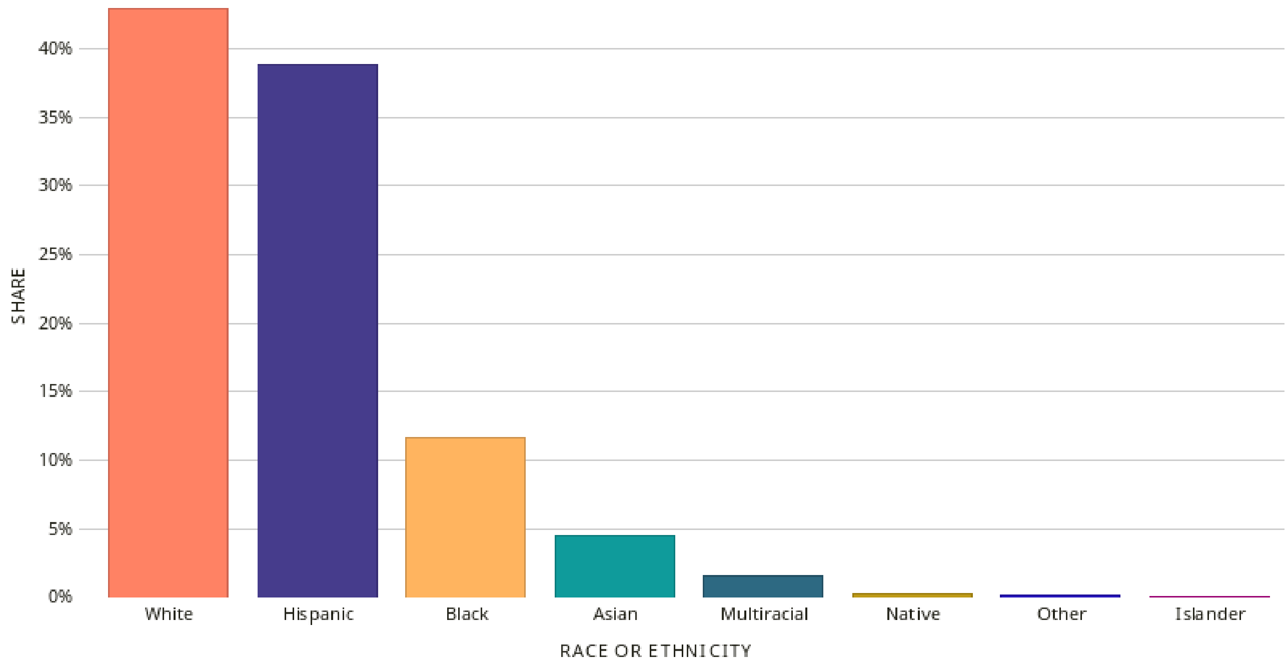


Figure 1: Bar graph showing the ethnicity of Texans (2015)

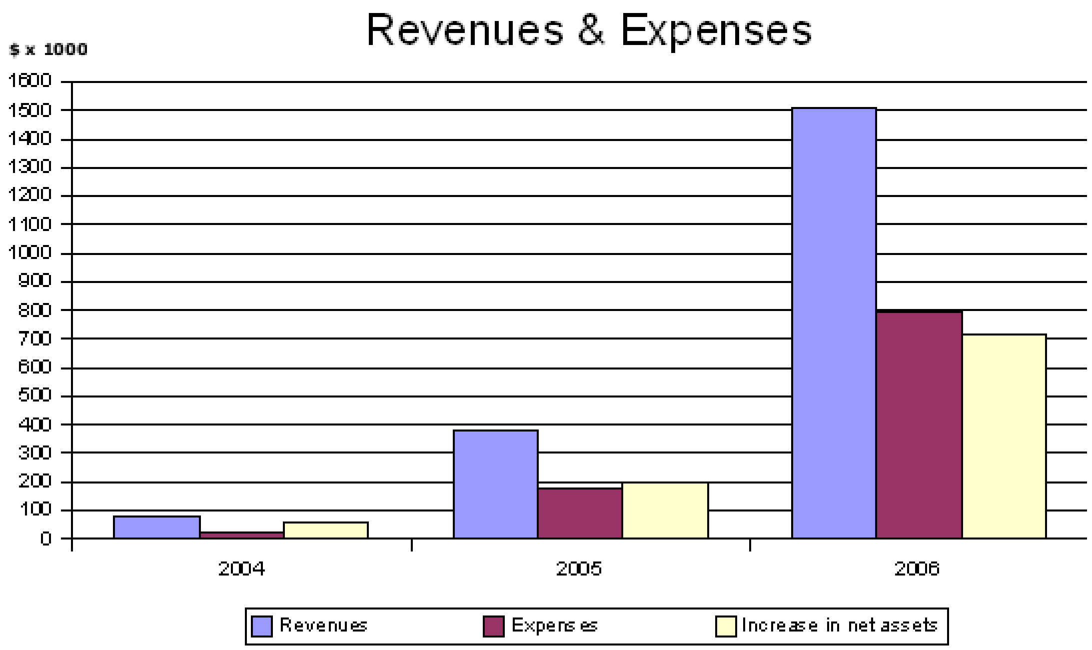


Figure 2: Bar graph showing revenues and expenses for 2004–2006

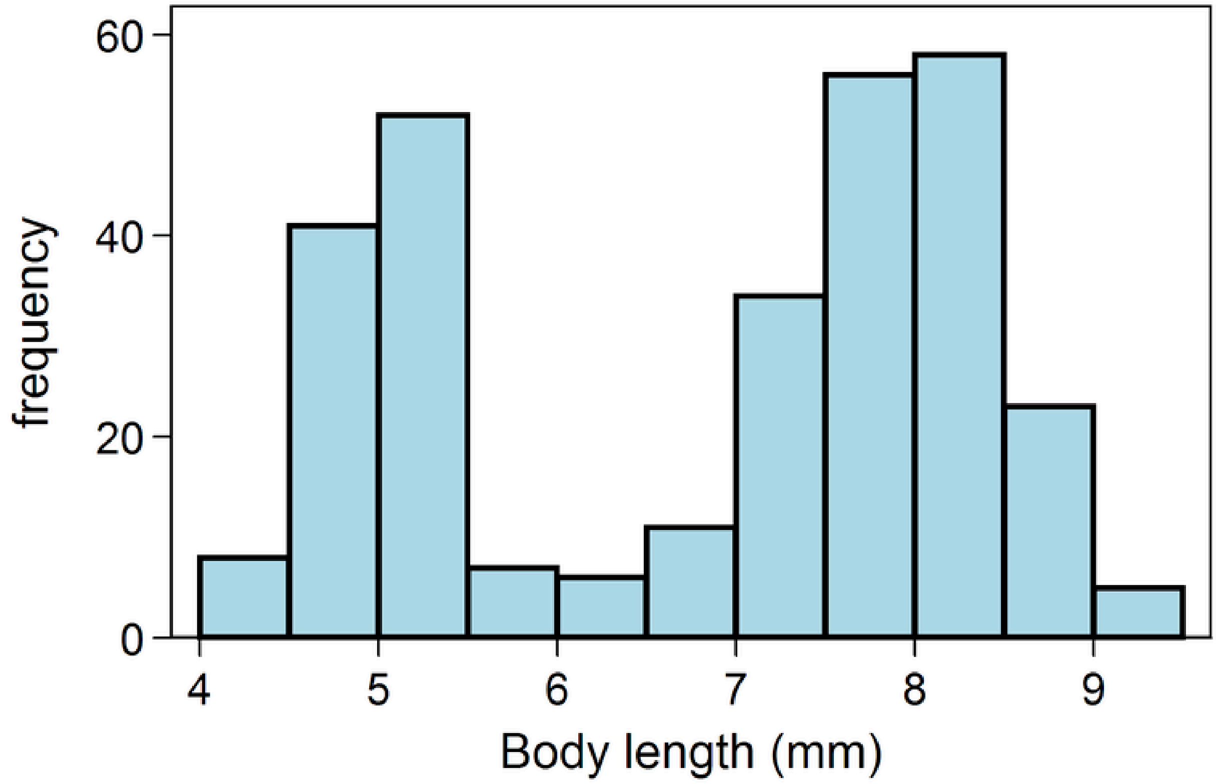


Figure 3: Histogram showing the length (in mm) of ants in a colony

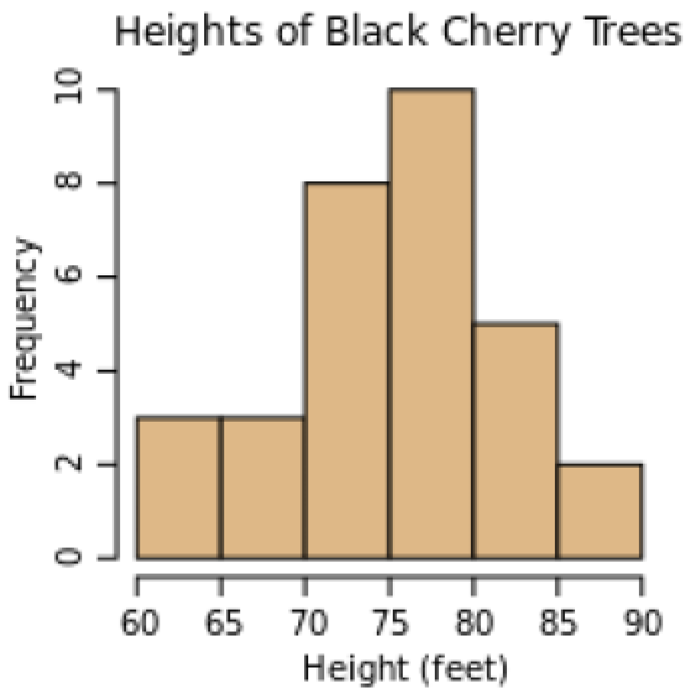


Figure 4: Histogram showing the height (in feet) of Black Cherry trees

At this point, you may be wondering what the differences between a bar graph and histogram are. On the surface they look the same. However, there are some important differences in the way in which they are drawn, and the kind of data they are used to represent.

Have a look at the bar graph and the histogram comparison in Figure 5.

1. In the bar graph, the bars do not touch. This is because bar graphs are used to represent categorical data and are a diagrammatic comparison of discrete variables. Histograms, on the other hand, represent a frequency distribution of continuous numerical variables. Because the data are continuous and there are no gaps in the intervals or classes of the frequency distribution, there can be no gaps between the bars.
2. In the case of a bar graph, it is quite common to rearrange the blocks, from highest to lowest or vice versa. But with a histogram, this cannot be done. The bars must be shown in the same order as the sequence of classes.
3. The width of the blocks in a histogram may or may not be same, while the width of the bars in a bar graph is always same. Generally, the bars of a histogram will be the same width because the width of the bar represents the width of the class and classes are usually the same width.

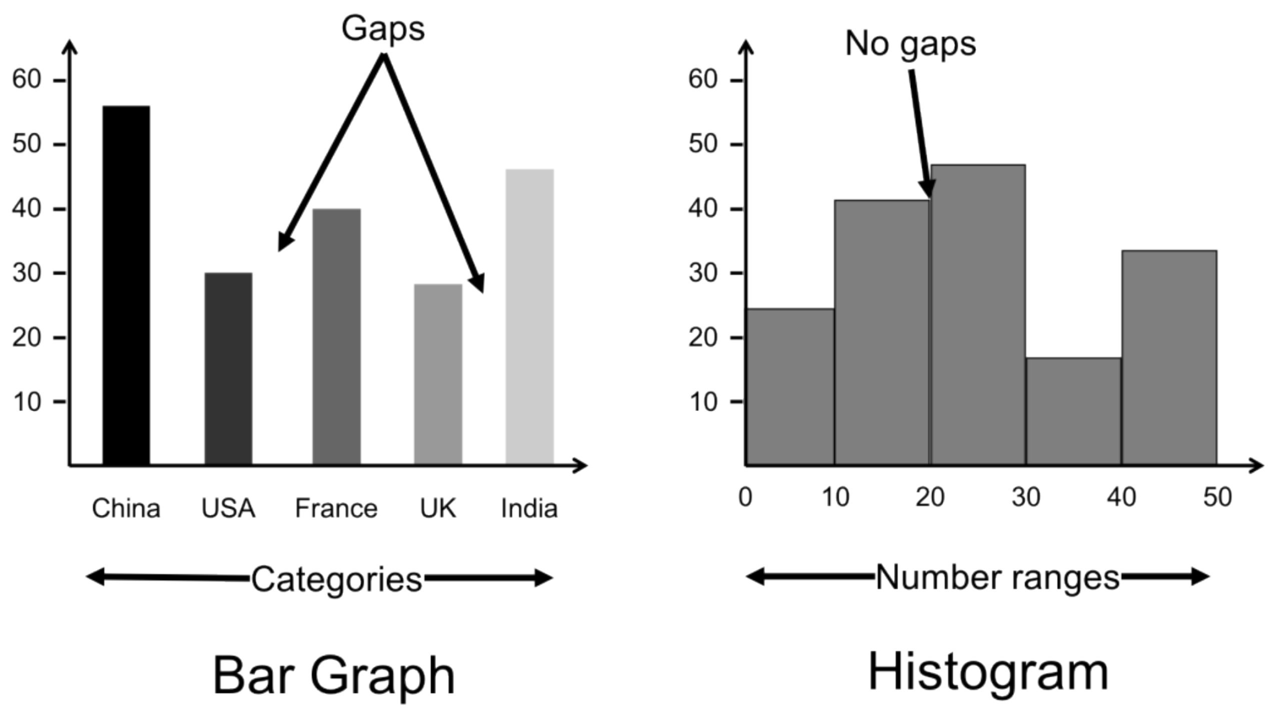


Figure 5: Key differences between bar graphs and histograms

Note

For more on the differences between bar graphs and histograms, watch the video called "How a histogram is different than a bar chart?".

How a histogram is different than a bar chart? (Duration: 01.54)

Simple bar graphs and compound bar graphs

Because bar graphs deal with categorical data, they are easy to draw. In the previous unit, we came across a set of survey data about which flowers a group of ladies favoured. Here is the raw data again.

Roses	Tulips	Roses
Daffodils	Asters	Daffodils
Tulips	Roses	Roses
Asters	Roses	Tulips
Tulips	Roses	Asters
Asters	Tulips	Roses
Asters	Daffodils	Tulips
Roses	Asters	Daffodils
Asters	Roses	Asters
Roses	Daffodils	Tulips

The first step in drawing a bar graph is to create an ungrouped frequency distribution. This we completed in the previous unit. The results are shown below in Table 1.

Flower type	Tally	Frequency
Roses	卌 卌	10
Daffodils	卌	5
Tulips	卌 ||	7
Asters	卌 |||	8
Total	30	

Table 1: Frequency distribution of favourite flowers among a group of 30 ladies

Next, we draw our axes placing the categories on the x-axis and, in this case, the frequencies on the y-axis. Lastly, we draw in each of the bars to represent the frequency of each category. Remember to always title your bar graphs. The completed bar graph is shown in Figure 6. Labels have been added above each bar to make the graph easier to read, but these are not essential.

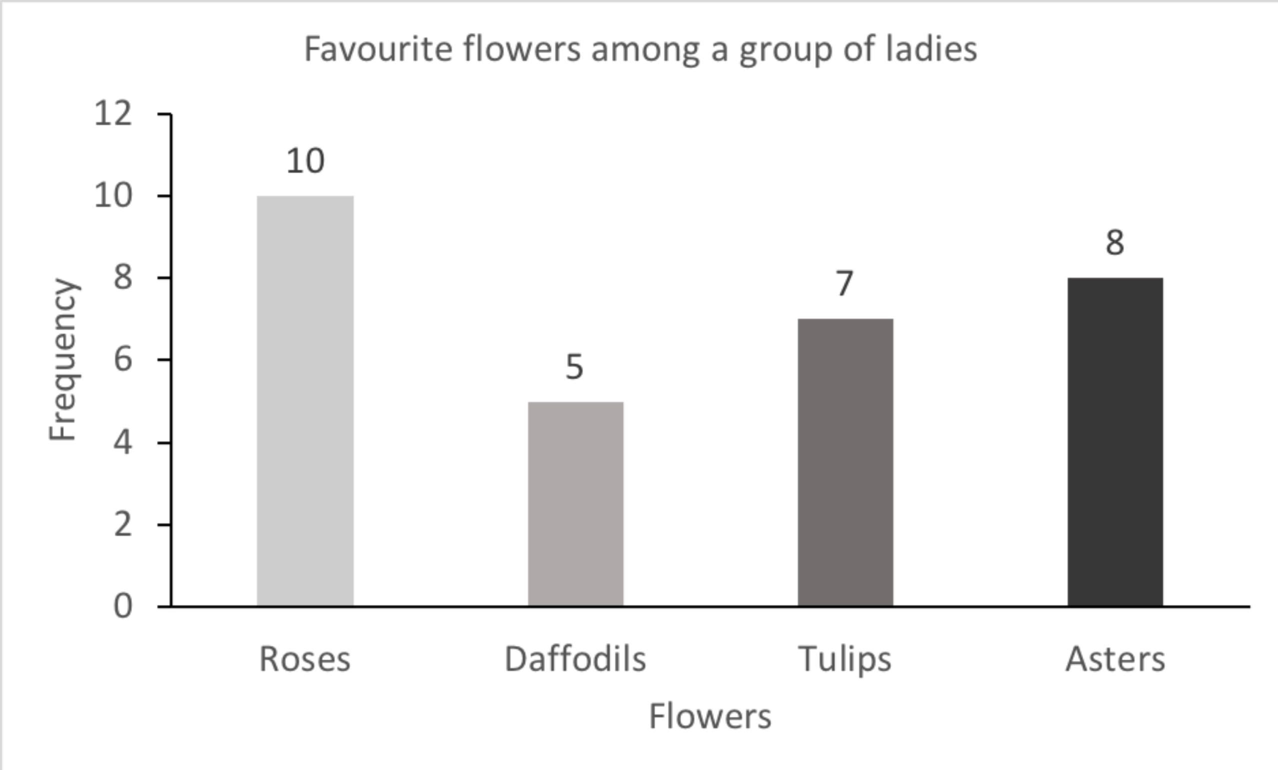


Figure 6: Favourite flowers among a group of ladies

A compound bar graph is very similar to an ordinary bar graph except that it represents multiple pieces of information in a single bar. The example in Figure 2 above shows the revenues, expenses and resultant increase in net assets for each of three years. In this case, each category (years) contained three subcategories. The bars representing the sub-categories can be touching but the categories must still be separated by a gap.

Another type of compound bar graph is like that shown in Figure 7. Here, instead of the bars being placed alongside each other, they are stacked on top of each other. It shows the comparative electrical power generation per month in Germany from solar and wind sources.

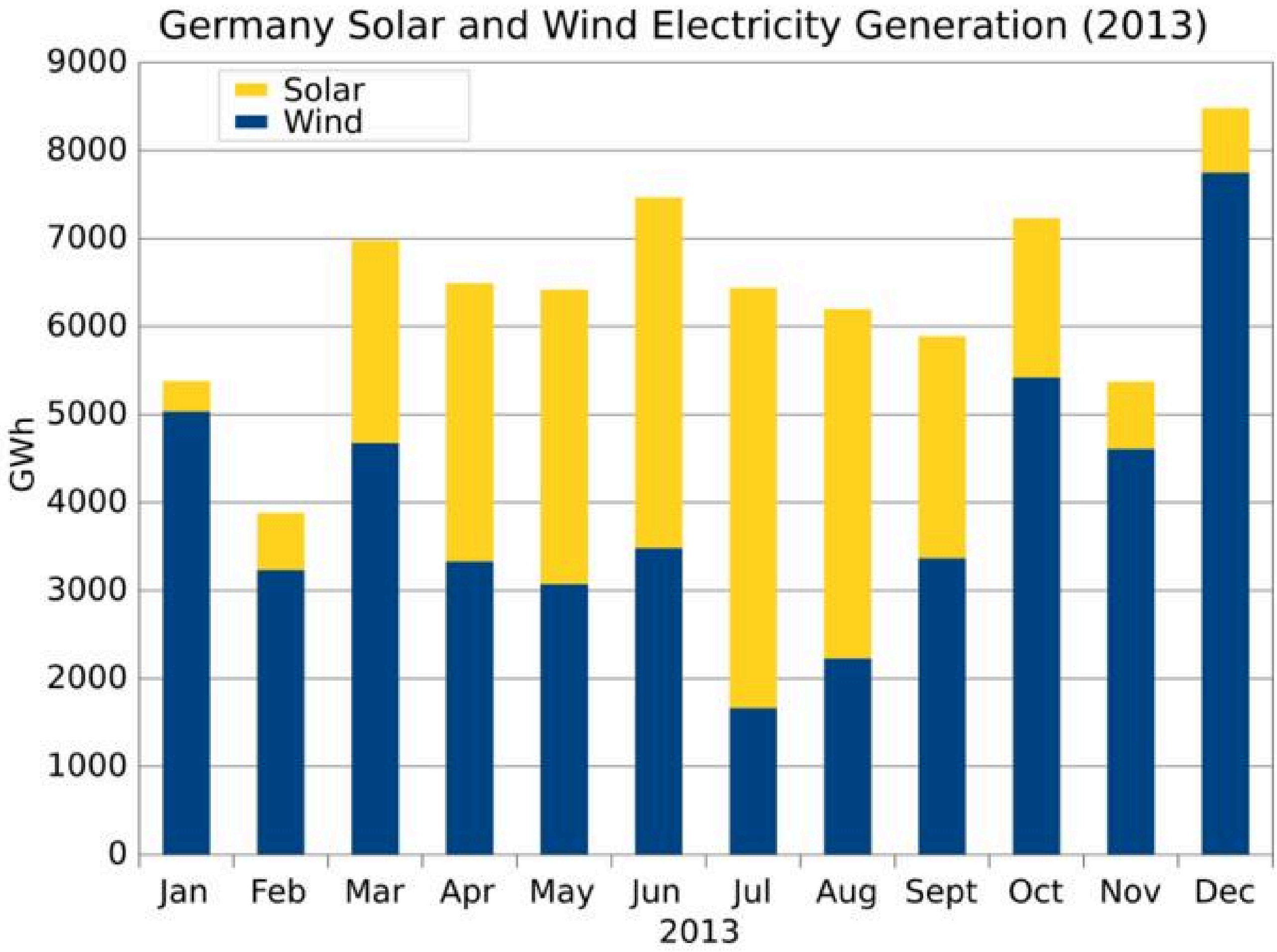


Figure 7: Wind and solar energy production (in GWh) per month in Germany in 2013

Whether the bars are placed alongside or stacked is a matter of whether it is more important for the graph to communicate the relative differences **within** categories or the combined differences **between** categories.

As with simple bar graphs, compound bar graphs are compiled from the data in frequency distributions. In almost all cases, however, compound bar graphs require a key to help readers understand what the different sub-categories are.

Note

Watch this excellent summary video on bar graphs called "Bar Graphs".

Bar Graphs (Duration: 04.05)

Exercise 2.1

GQ Magazine held a poll to determine the favourite DJs of NC(V) learners. The results of the first 30 responses are presented in the following frequency distribution. Draw a bar graph to represent this data.

DJ	Tally	Frequency
DJ Fresh	~~||||~~	5
DJ S'bu	~~||||~~ |	6
DJ China Man	|||	3
DJ Cleo	~~||||~~ ||||	9
DJ Mabuso	~~||||~~ ||	7
	Total	30

The full solutions are at the end of the unit.

Histograms

As we saw earlier, histograms are used to present grouped numerical data. In the previous unit, we created a grouped frequency distribution of the following data of the heights (in cm) of a group of 190 students. The resulting frequency distribution is shown in Table 2.

$165, 148, 158, 150, 160, 165, 150, 156, 155, 164, 162, 160, 158, 148, 158, 140, 146, 160, 148, 152, 139, 165, 148, 160, 156, 158, 170, 155, 160, 148, 155, 158, 179, 170, 158, 161, 155, 160, 163, 178, 138, 172, 170, 156, 160, 160, 171, 140, 160, 170, 175, 148, 170, 177, 155, 167, 154, 160, 170, 155, 136, 179, 150, 167, 148, 160, 164, 167, 157, 165, 163, 140, 162, 178, 160, 170, 163, 162, 165, 175, 165, 152, 147, 180, 148, 170, 165, 167, 165$

Class limits	Frequency
$136 \leq x < 145$	6
$145 \leq x < 154$	15
$154 \leq x < 163$	33
$163 \leq x < 172$	27
$172 \leq x < 181$	9

Table 2: Frequency distribution of the heights (in cm) of a group of 90 students

We follow a similar process as with the bar graph to produce the histogram. We plot our classes along the x-axis and the number or frequency of each class along the y-axis. Since there are five classes, the histogram will have five rectangles. The base of each rectangle is defined by its class. The height of each rectangle is determined by its frequency.

Figure 8 shows the completed histogram for this data. In this case, small labels have been placed above each bar to indicate its height. However, these are not essential.

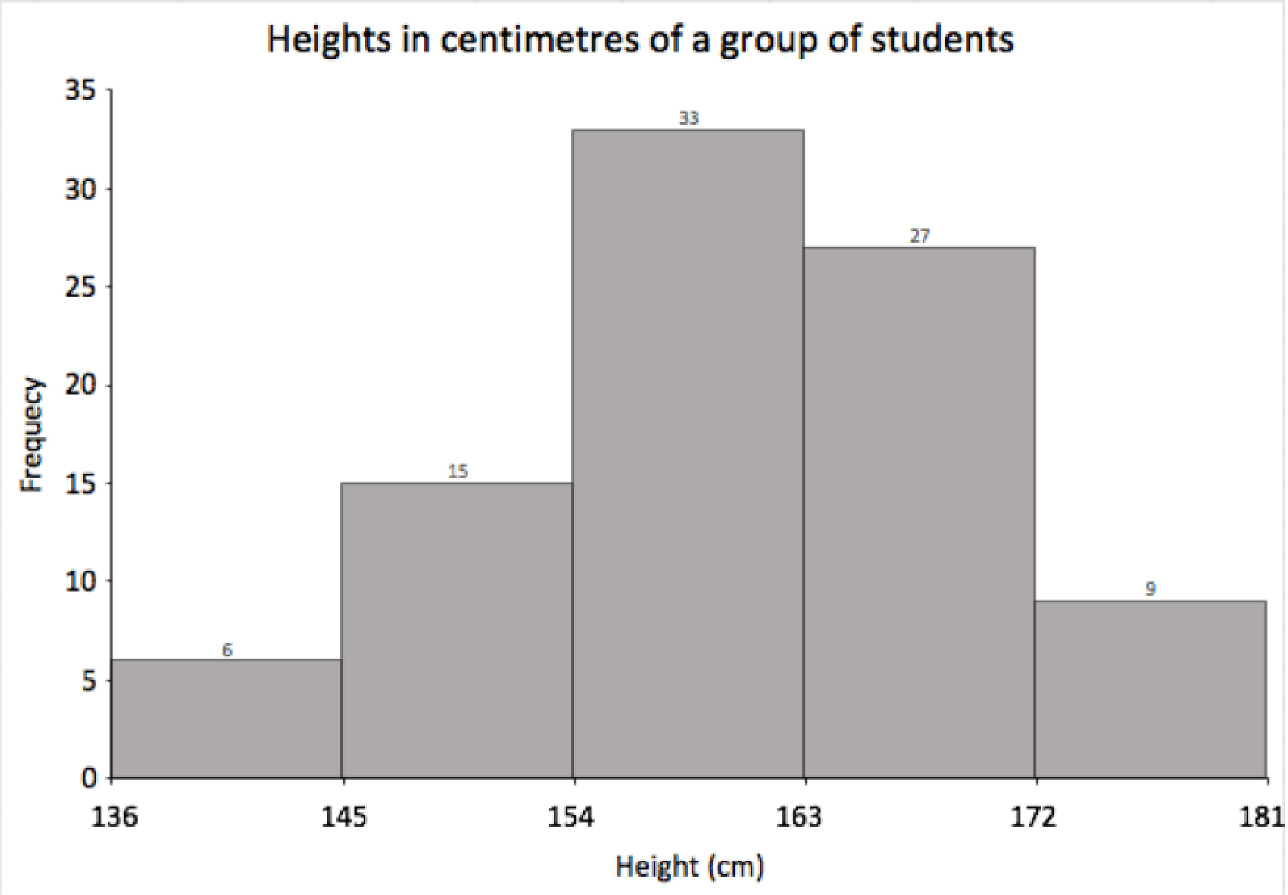
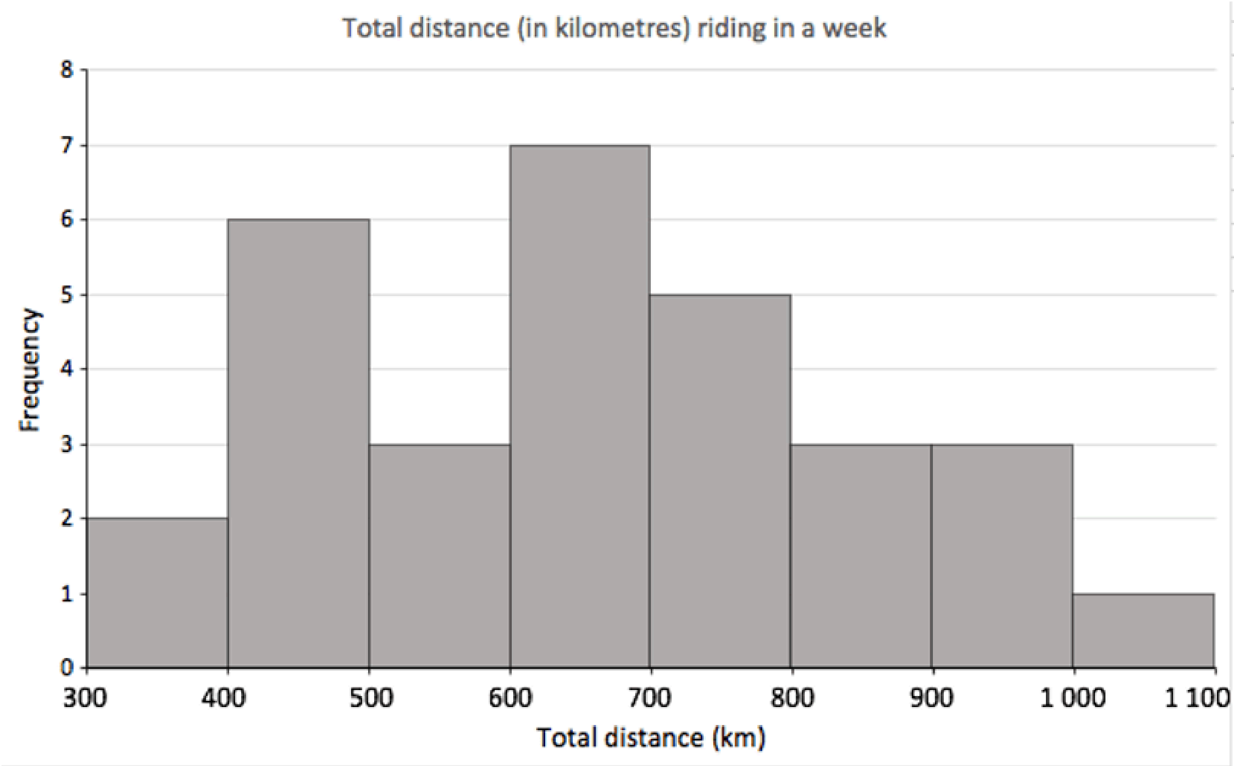


Figure 8: Histogram of the heights in centimetre of a group of students

Example 2.1

A group of cyclists calculated the total number of kilometres they each cycled in a given week. The histogram describes the data they collected.



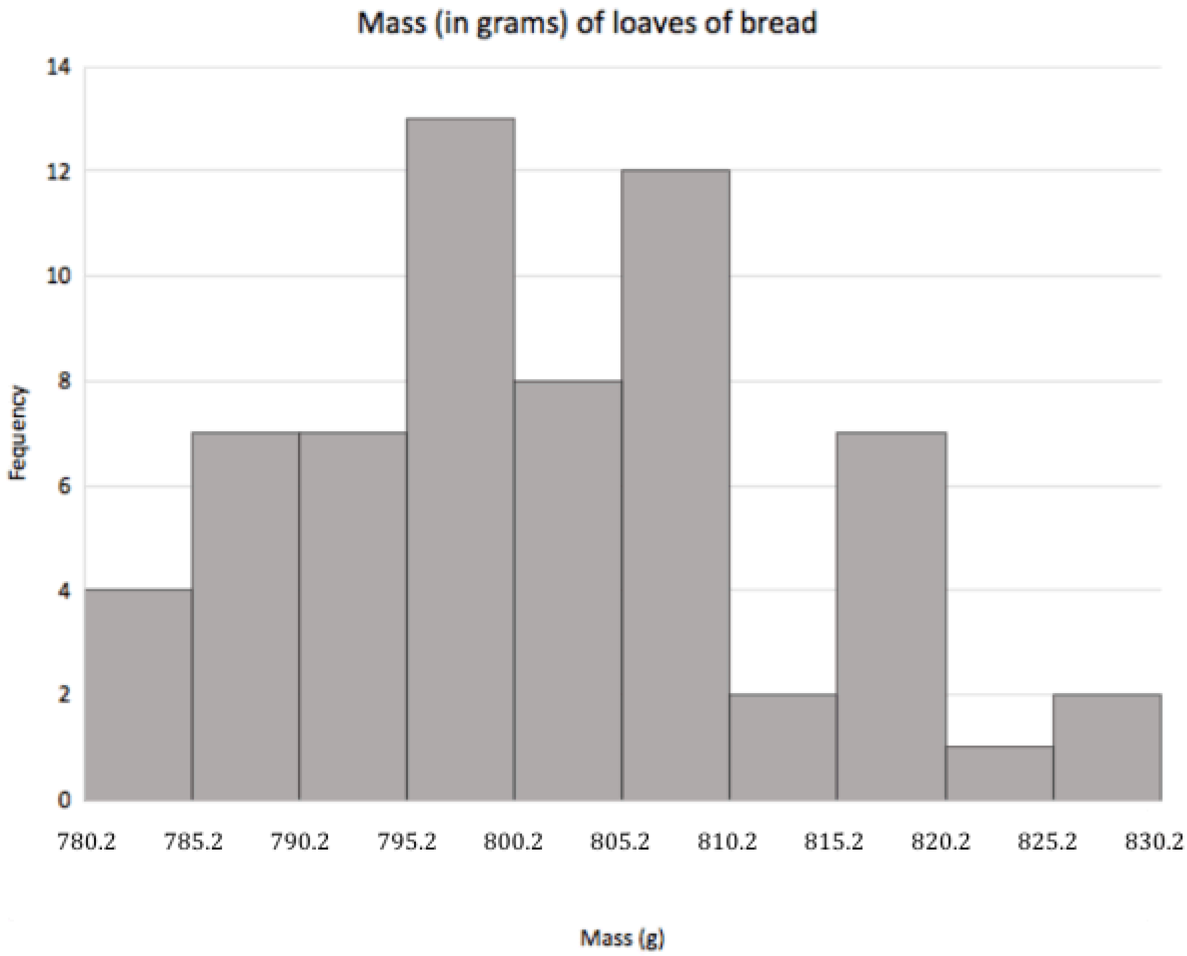
1. How many cyclists completed between 500 and 600 kilometres?
2. How many cyclists completed $500\ \text{km}$ or more?
3. What percentage of the cyclists completed $500\ \text{km}$ or more?
4. How many cyclists completed between 300 and 700 kilometres?
5. What would you say the mean number of kilometres completed by this group of cyclists is?

Solution

1. Three cyclists completed between 500 and 600 kilometres.
2. We need to add up all the cyclists that completed between 500 and $1\ 100$ kilometres. This is $3+7+5+3+3+1=22$.
3. To calculate the percentage of cyclists who completed $500\ \text{km}$ or more we need also know the total number of cyclists. We need to add the heights of all the bars: $2+6+3+7+5+3+3+1=30$.
   The percentage of cyclists who completed $500\ \text{km}$ or more was $\frac{22}{30}\times 100=73.33\%$.
4. $2+6+3+7=18$ cyclists completed between 300 and 700 kilometres?
5. The highest bar is the class $600 \leq x < 700$. This bar is also more or less in the middle of the distribution. Therefore, the mean is likely to be within this class. We can approximate the overall mean as the mean of this class i.e. $\bar{x}=\frac{600+700}{2}=650\ \text{km}$.

Exercise 2.2

The mass of loaves of bread coming off the production line were measured. The results are shown in the histogram below.



1. How many loaves of bread were measured in total?
2. How many loaves of bread had a mass greater than 785.2 g but less than or equal to 800.2 g?
3. What percentage of the loaves had a mass greater than 810.2 g?

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- What the difference is between a bar graph and a histogram.
- How to draw a bar graph.
- How to draw a compound bar graph.
- How to draw a histogram.

Unit 2: Assessment

Suggested time to complete: 30 minutes

1. A survey was conducted among a group of commuters. One of the questions was what cell phone network they preferred. The results are shown below.

Cell C	Telkom	Cell C	Telkom	MTN
Telkom	Cell C	MTN	MTN	MTN
Vodacom	Vodacom	Telkom	Vodacom	Telkom
Vodacom	Vodacom	Cell C	Vodacom	Vodacom
Vodacom	MTN	Vodacom	Cell C	MTN

   a. Create a tally table or frequency distribution of this data.
   b. Construct a bar graph of this data.
   c. What was the most popular cell phone network?
   d. What was the least popular cell phone network?

2. The results of a survey asking families how much they spent on food in the previous week was conducted. The results are shown below.

R480	R496	R484	R468	R390	R512
R564	R462	R487	R429	R413	R498
R346	R431	R298	R494	R474	R451
R257	R438	R523	R501	R437	R470
R378	R468	R583	R478	R378	R380

   a. Complete the following frequency distribution for the data.

Class limits	Frequency
$257 \leq x < 312$	
$312 \leq x < 367$	
$367 \leq x < 422$	
$422 \leq x < 477$	
$477 \leq x < 532$	
$532 \leq x < 587$	

   b. Use this completed frequency distribution to construct a histogram.
   c. Use your histogram to answer the following questions.
      i. How many families spent less than $\mathrm{R}312$ on food in the previous week?
      ii. How many families spent $\mathrm{R}477$ or more on food in the previous week?

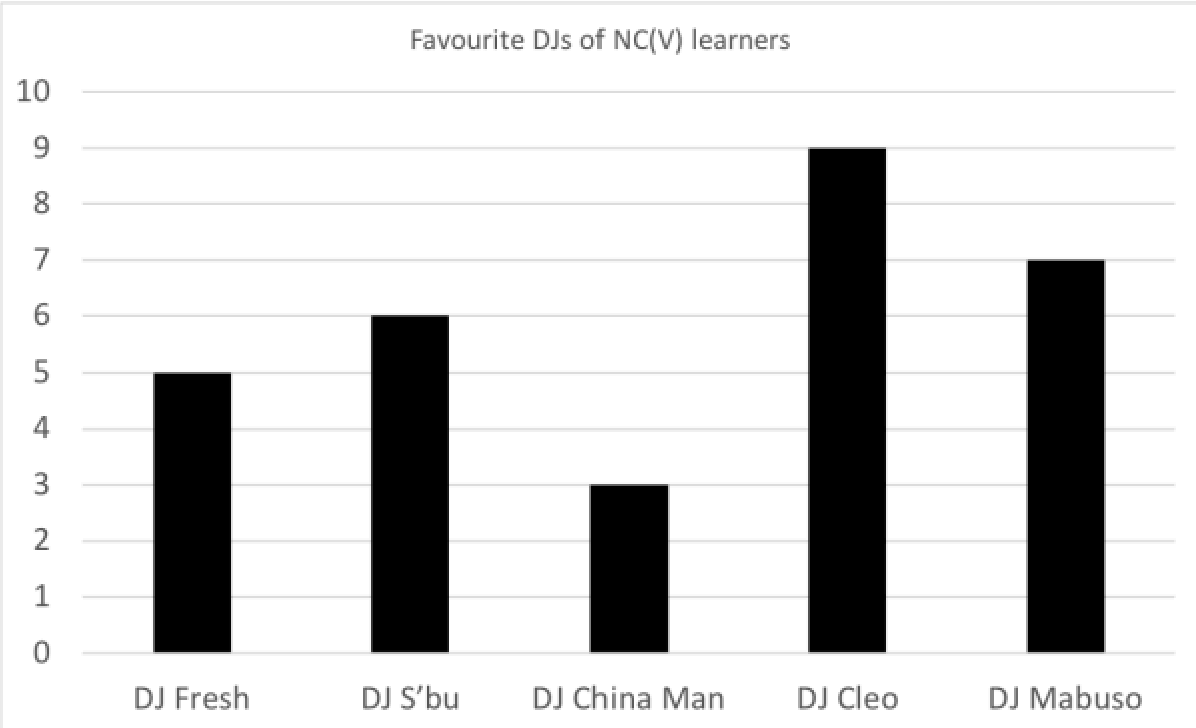
iii. How many families spent between $\mathrm{R}367$ and $\mathrm{R}532$ on food in the previous week?

iv. What percentage of families spent $\mathrm{R}532$ or more on food in the previous week?

The full solutions are at the end of the unit.

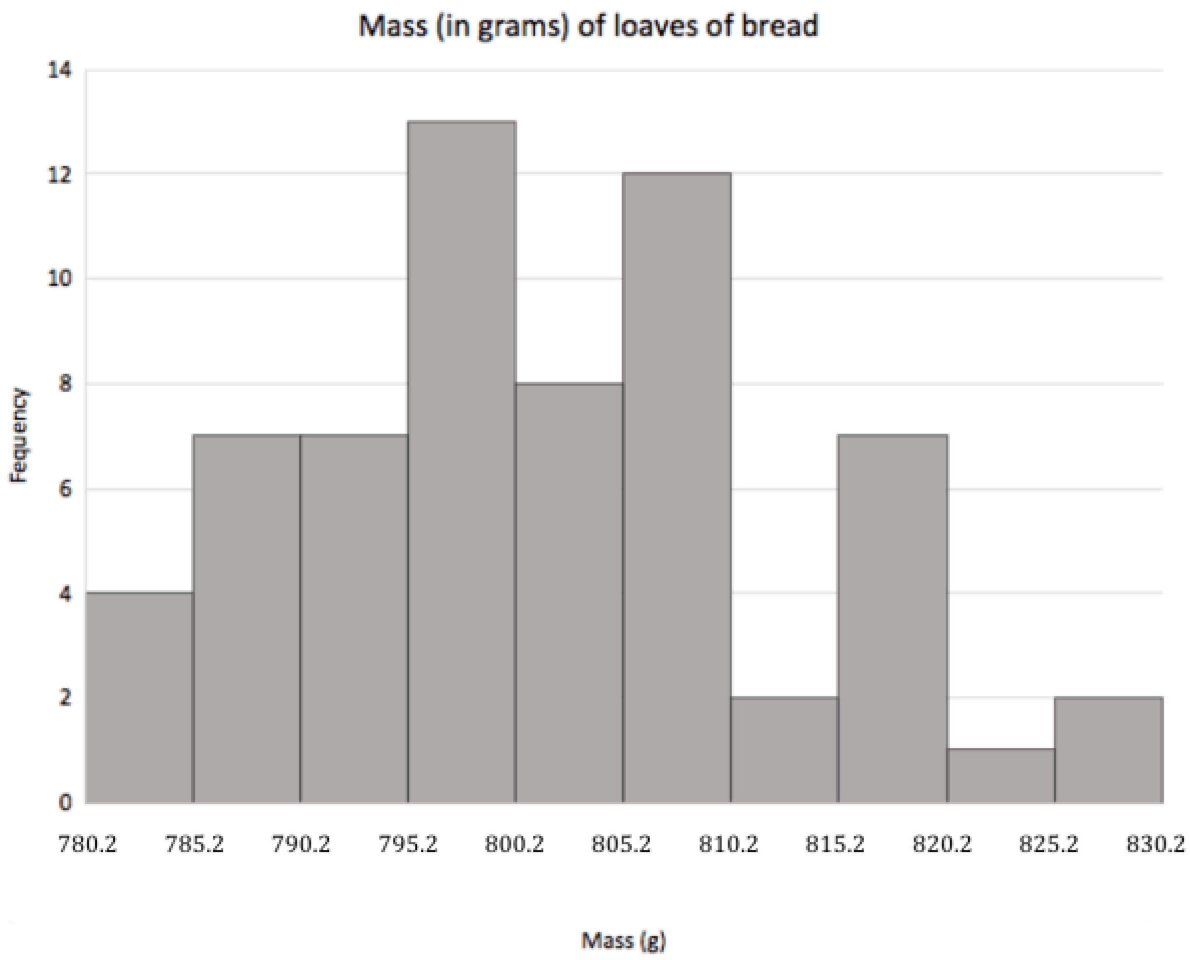
Unit 2: Solutions

Exercise 2.1



Back to Exercise 2.1

Exercise 2.2



1. 63 loaves were measured in total
2. $7+7+13=27$
3. $\frac{2+7+1+2}{63} \times 100 = \frac{12}{63} \times 100 = 19.05\%$ of the loaves had a mass greater than 810.2 g.

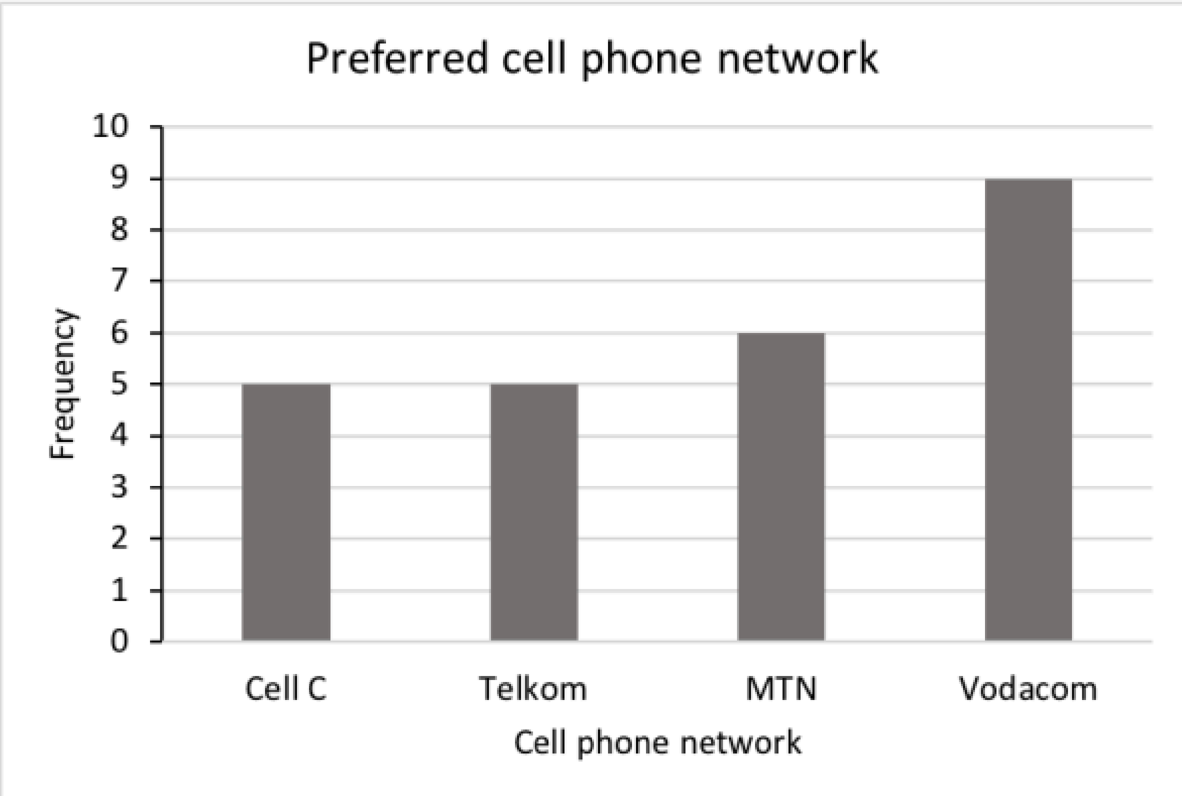
Back to Exercise 2.2

Unit 2: Assessment

1.
   a.

Cell phone network	Frequency
Cell C	5
Telkom	5
MTN	6
Vodacom	9

b.



c. Vodacom

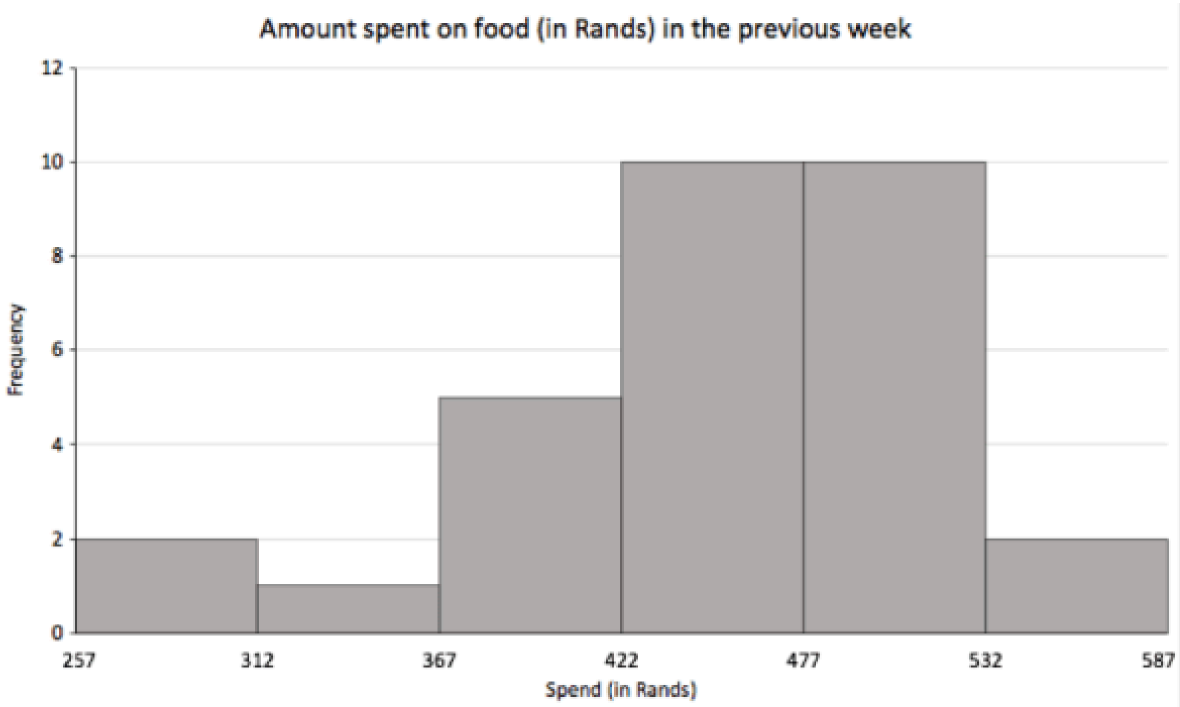
d. Both Cell C and Telkom were the least popular.

2.

a.

Class limits	Frequency
$257 \leq x < 312$	2
$312 \leq x < 367$	1
$367 \leq x < 422$	5
$422 \leq x < 477$	10
$477 \leq x < 532$	10
$532 \leq x < 587$	2

b.



c.

i. Two families.

ii. 12 families.

iii. 25 families.

iv. Total families is 30. Percentage of families spending $\text{R}532$ or more is $\frac{2}{30} \times 100 = 6.67\%$

Back to Unit 2: Assessment

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Unit 3: Frequency polygons and line graphs

DYLAN BUSA

Unit 3: Frequency polygons and line graphs

By the end of this unit you will be able to:

- Calculate measures of central tendency and percentiles on grouped data.
- Construct a frequency polygon.
- Construct a line graph.

What you should know

Before you start this unit, make sure you can:

- Calculate measures of central tendency and percentiles. Refer to unit 3 of Subject outcome 4.1 if you need help with this.
- Construct a bar graph. Refer to unit 2 of this Subject outcome if you need help with this.
- Construct a histogram. Refer to unit 2 of this Subject outcome if you need help with this.

Introduction

In this unit, we will continue to explore some other types of graphs that we can use to visualise data and make it easier to interpret. In particular we will look at how to create frequency polygons and line graphs. See Figures 1 and 2 for examples.

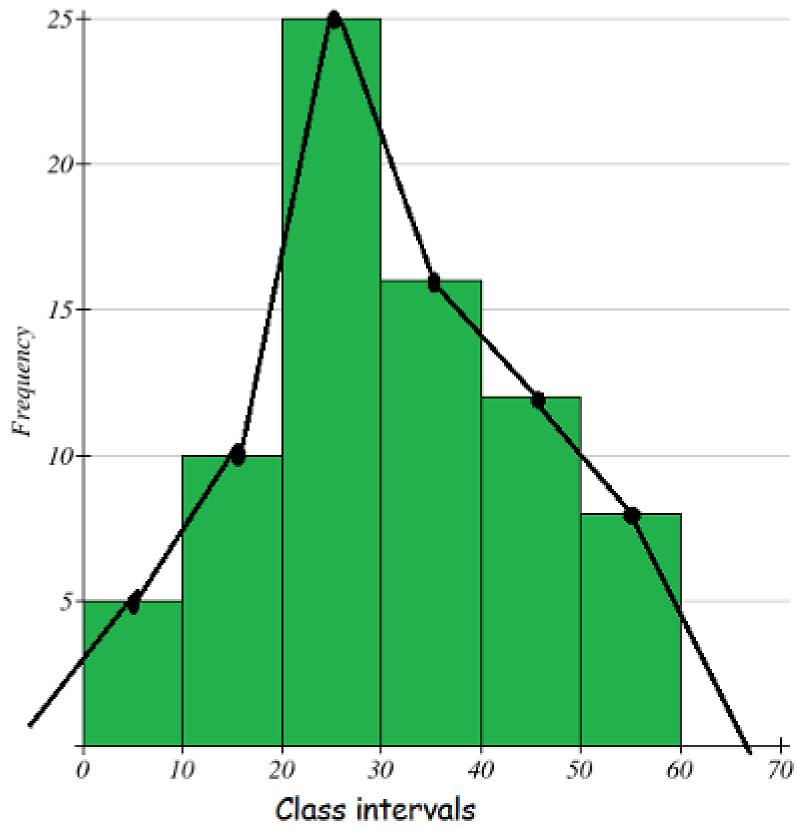


Figure 1: Example of a frequency polygon

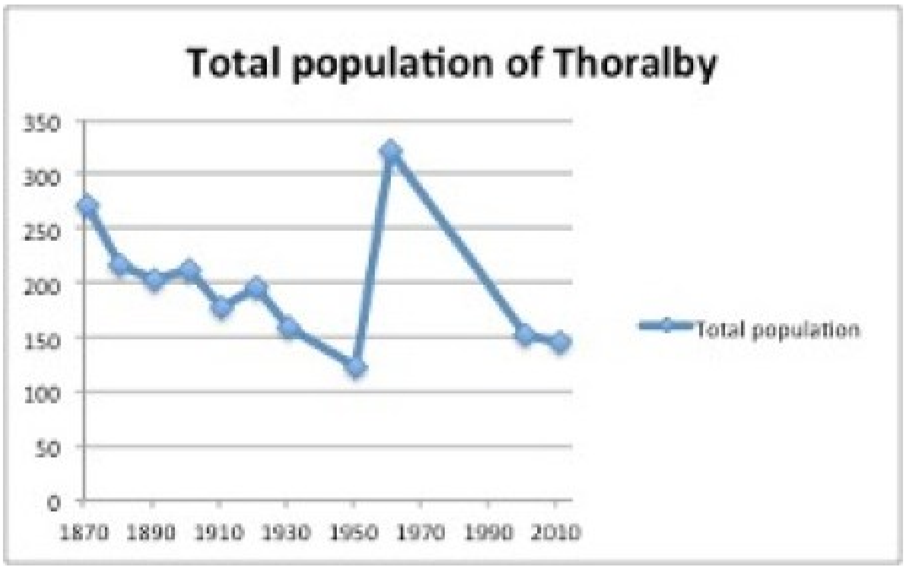


Figure 2: Example of a line graph

Measures of central tendency for grouped data

In Subject outcome 4.1 we learnt how to calculate the three main measures of central tendency:

- The mean – the 'average' value calculated by dividing the sum of all the values by the number of values.

- The median – the centre-most value of the data set.
- The mode – the value in the data set that occurs most often.

Up until now, we calculated these measures on raw data. However, these values can also be calculated using grouped data.

Example 3.1

The following frequency distribution shows the mass (in kilograms) of a troop of chimpanzees.

Mass (kg)	Count
$40 < t \leq 45$	7
$45 < t \leq 50$	10
$50 < t \leq 55$	15
$55 < t \leq 60$	12
$60 < t \leq 65$	6

1. Calculate the approximate mean of the data.
2. Why can we only calculate the approximate mean?
3. What is the median and the median group?
4. What is the mode and the modal group?

Solutions

1. Because we do not have each individual value, we treat each value as though it is at the midpoint of the class. Therefore, we assume that there are seven values of $\frac{40+45}{2} = 42.5$, 10 values of $\frac{45+50}{2} = 47.5$, and so on. Therefore, the approximate mean of the grouped data is as follows:
$$\begin{aligned}\bar{x} &= \frac{(42.5 \times 7) + (47.5 \times 10) + (52.5 \times 15) + (57.5 \times 12) + (62.5 \times 6)}{(7 + 10 + 15 + 12 + 6)} \\ &= \frac{297.5 + 475 + 787.5 + 690 + 375}{50} \\ &= 52.5\end{aligned}$$
2. We can only approximate the mean because we do not have all the exact values and we must assume that each of the values within each class falls at the midpoint of that class.
3. Just as the median is the middle value of ungrouped data, the median of grouped data is the midpoint of the middle class. In this case, the middle class is $50 < m \leq 55$ so the median is 52.5. The median group is $50 < m \leq 55$.
4. Just as the mode is the most frequent value of ungrouped data, the mode of grouped data is the midpoint of the class with the greatest frequency. In this case, the class with the greatest frequency is $50 < m \leq 55$ so the mode is 52.5. The modal group is $50 < m \leq 55$.

Exercise 3.1

The frequency distribution below shows the time taken (in minutes) for various games in a chess tournament to be completed.

Times (min)	Count
$35 < t \leq 45$	5
$45 < t \leq 55$	12
$55 < t \leq 65$	15
$65 < t \leq 75$	28
$75 < t \leq 85$	18
$85 < t \leq 95$	14
$95 < t \leq 105$	7

1. Find the mean of the data.
2. What is the modal group?
3. What is the median?

The full solutions are at the end of the unit.

Frequency polygons

A frequency polygon (see Figure 1) is a graph created by using straight lines to join the midpoints of each interval, or class in a frequency distribution. The height of each point represents the frequency of the class. The midpoint of a class is calculated by dividing the sum of the upper and lower boundaries by two.

A frequency polygon can also be created from a histogram. You will often see the frequency polygon superimposed over the histogram (see Figure 3).

Because a polygon is a closed shape, the beginning and end of the frequency polygon **must** touch the x-axis. To do this, we must add a class below and above the classes we have for the data, and must join the midpoints of these classes as well. Because these classes or intervals are empty (frequency is zero), these dots lie on the x-axis.

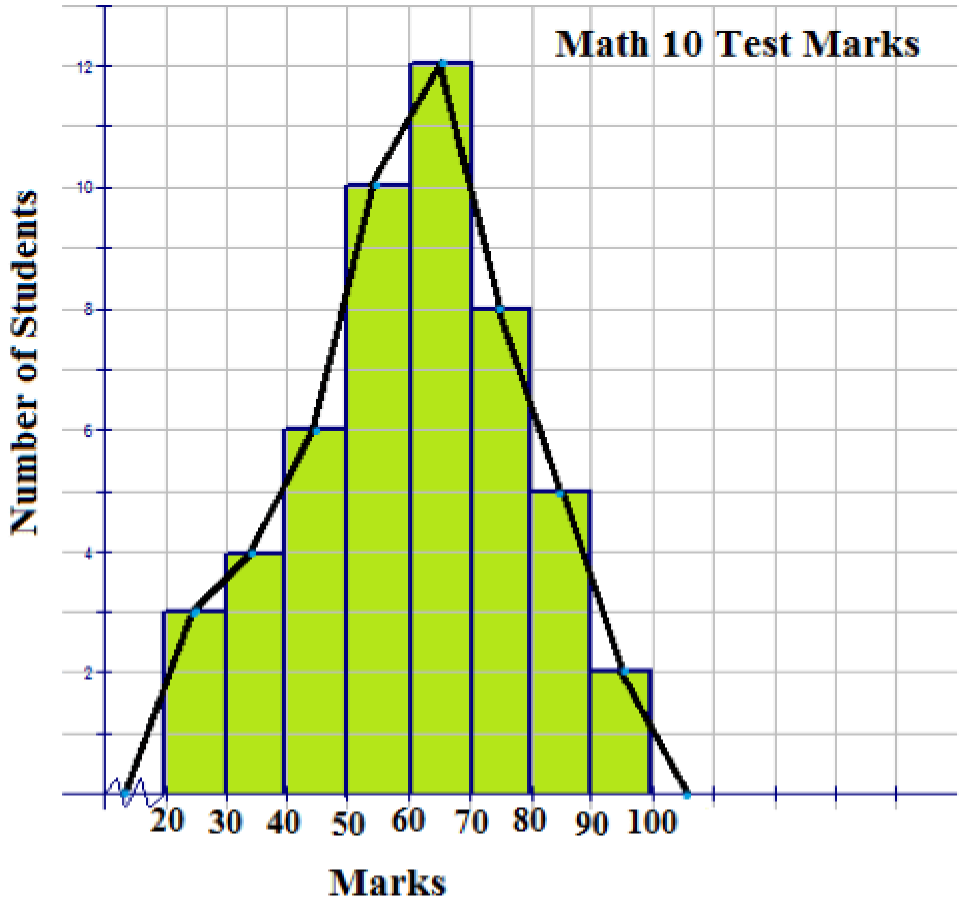


Figure 3: Example of a frequency polygon

Example 3.2

The NC(V) mathematics marks, out of 50 , for 35 learners are given below:

$46, 40, 12, 10, 47, 23, 26, 8, 29, 34, 37, 17, 40, 50, 18, 23, 33, 23, 24, 15, 35, 23, 19, 22, 28, 35, 27, 42, 29, 26, 46, 33, 27, 19, 28$

1. Complete the table below using the above marks.

Interval of scores	Frequency
$0-10$	
$11-20$	
$21-30$	
$31-40$	
$41-50$	

2. Construct a histogram for these data.

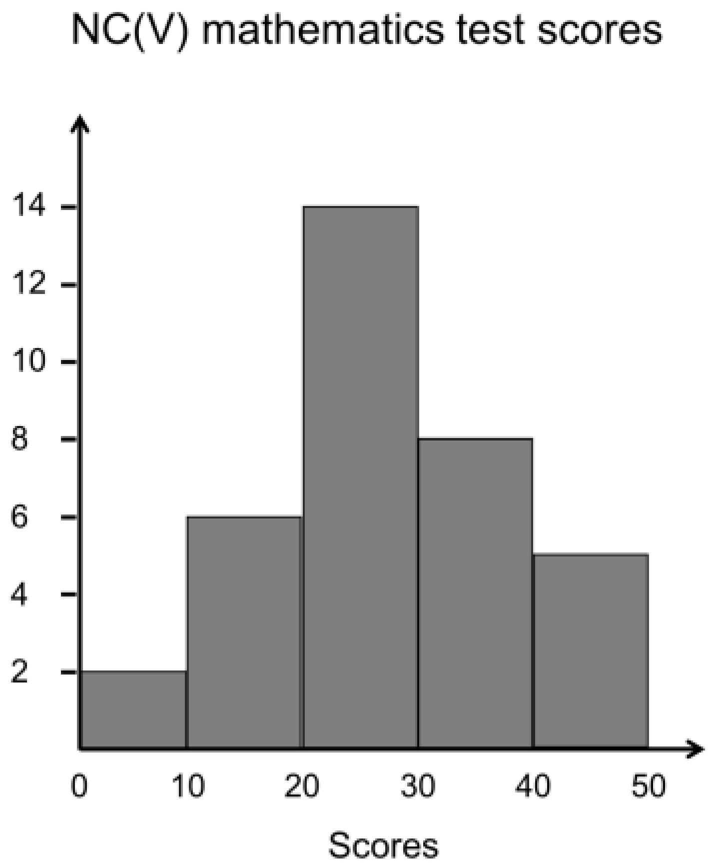
3. If the pass mark is 21 out of 50, what percentage of learners passed the test?
4. Construct the frequency polygon for these data.
5. Comment on any trends you may see in the graphs.

Solutions

1.

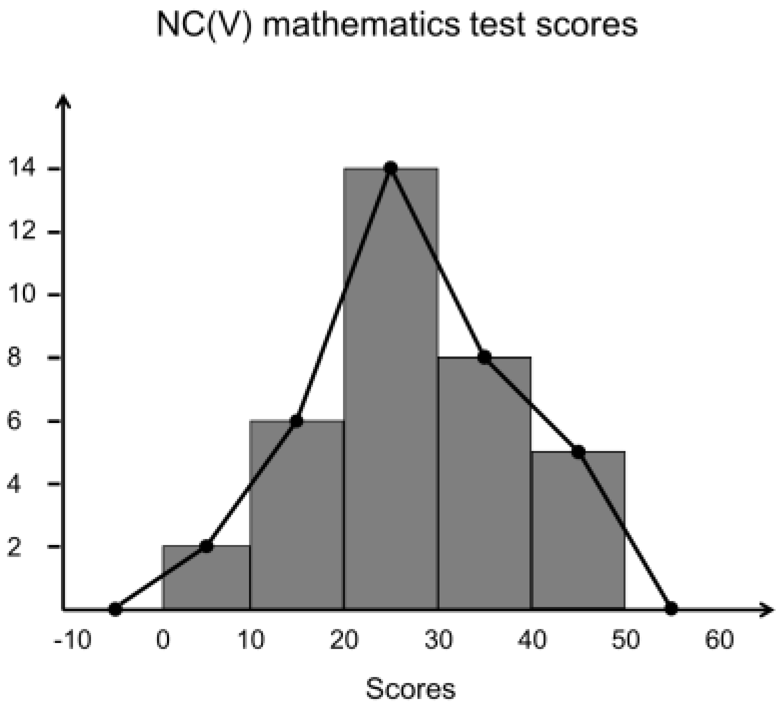
Interval of scores	Frequency
$0 - 10$	2
$11 - 20$	6
$21 - 30$	14
$31 - 40$	8
$41 - 50$	5

2.



3. A total of 35 learners wrote the test. A total of $14 + 8 + 5 = 27$ learners scored 21 out of 50 or more. Therefore, the percentage of learners who passed is $\frac{27}{35} \times 100 = 77.14\%$.
4. To construct the frequency polygon, we need to join dots at the midpoints of each of the classes or

intervals. However, we also need to close the polygon by adding dots at the midpoints of the intervals below and above the data intervals i.e. the intervals of $-10-0$ and $51-60$. The frequency polygon is drawn as follows:



5. The data follows a normal distribution but there is a slight skew to the right with more students scoring above 30 marks than would otherwise be expected.

Exercise 3.2

The following frequency distribution is for the heights (in cm) of a group of 90 students.

Class limits	Frequency
$136-144$	6
$145-153$	15
$154-162$	33
$163-171$	27
$172-180$	9

1. Create a frequency polygon for this data.
2. Calculate the mean of this data.

3. What is the median of this data?
4. What is the mode of this data?

The full solutions are at the end of the unit.

Line graphs

Another common way to display numerical (discrete and continuous quantitative) data is by using a line graph. You may be familiar with line graphs already. In many ways they are similar to frequency polygons but are not based on grouped data.

They are most often used to show changes or trends over time but what they show is the relationship between an **independent** variable (like time) and a **dependent** variable that depends on the independent variable.

In a line graph, each data value is represented by a point on the graph. The points are then connected by straight lines. The independent variable is listed along the horizontal, or x-axis, and the quantity or value of the data is listed along the vertical, or y-axis.

Note

Watch the short video called "Line Graphs: Lesson (Basic Probability and Statistics Concepts)" for a basic introduction to line graphs.

Line Graphs: Lesson (Basic Probability and Statistics Concepts) (Duration: 04.44)

Activity 3.1: Draw a line graph

Time required: 10 minutes

What you need:

- a pen or pencil
- a piece of paper
- a ruler

What to do:

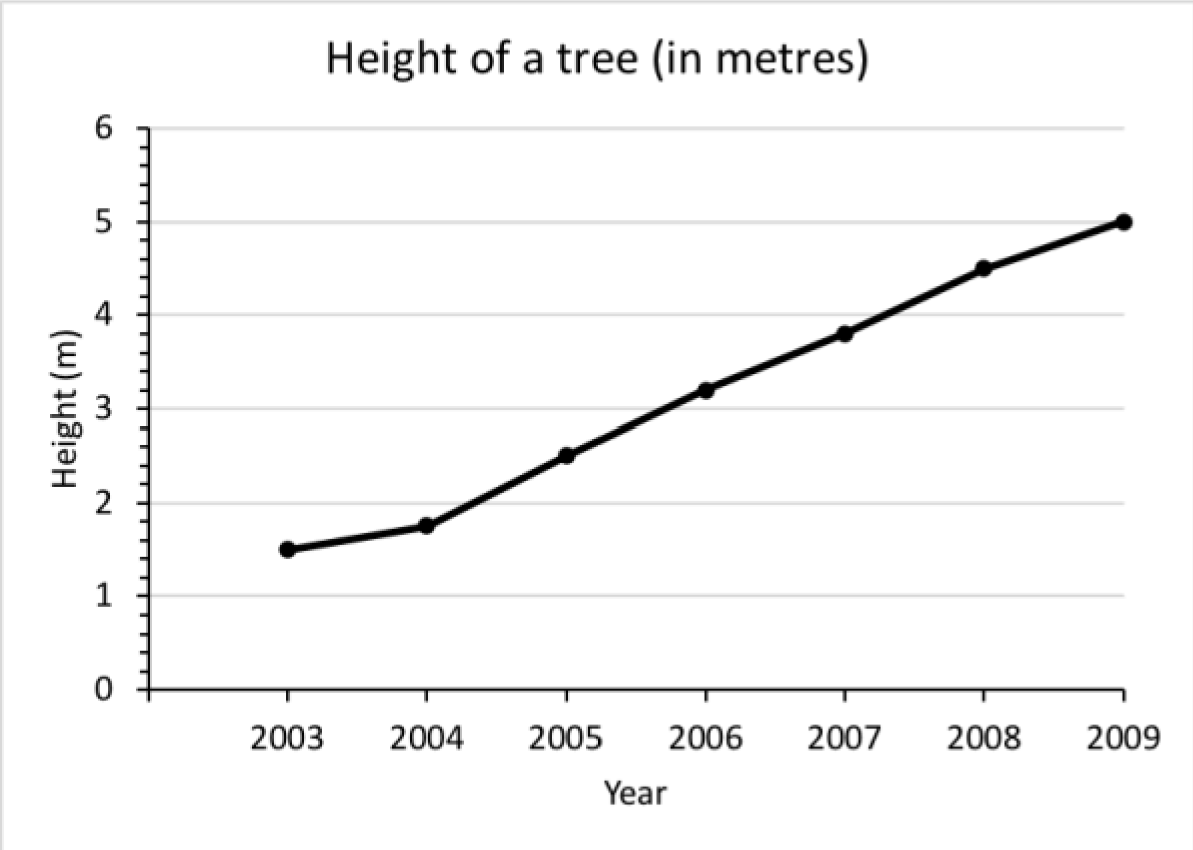
The table below shows the height of a certain tree (in metres) for each year between 2003 and 2009.

Year	Height of tree (m)
2003	1.5
2004	1.75
2005	2.5
2006	3.2
2007	3.8
2008	4.5
2009	5

1. Draw a set of axes. You only need to draw the positive portions of the x- and y-axes. Make each axis at least $20\ \text{cm}$ long.
2. Next, label your axes and add a title for your graph.
3. Now determine the units on the y-axis. Look at the smallest and largest values in the table and determine the best scale for this axis. Mark and label these points on your y-axis.
4. Evenly divide your x-axis among the years in the table and label these marks.
5. Now plot each of your data points as accurately as possible and draw straight lines between each of these points.
6. State any conclusions you can draw from your graph.
7. How does your line graph differ from a frequency polygon?

What did you find?

After working through points 1 to 5 this is what your line graph should look like.



6. The rate of growth of the tree has been steady since 2004. Between 2003 and 2004, its rate of growth was lower, possibly due to a lack of water or nutrients.
7. This line graph differs from a frequency polygon in two important ways. Firstly, there is no need for the graph to start and end on the x-axis. Secondly, the points on the graph represent actual data values rather than a frequency of values within a particular interval.

Note

If you have an internet connection, spend time playing with the wonderful line graph simulation called Understand and Create Line Graphs: Line Graphs. Here you can practise creating a line graph and answer some questions about it.

Exercise 3.3

The table below shows South Africa's real Gross Domestic Product (GDP) growth rate (in percentage terms) between 2007 and 2017.

Year	GDP Growth (%)
2007	5.4
2008	3.2
2009	−1.5
2010	3.0
2011	3.3
2012	2.2
2013	2.5
2014	1.8
2015	1.3
2016	0.6
2017	1.3

1. Draw a line graph of these data.
2. What was the general trend in South Africa's economic performance over this period?
3. In which year was the highest rate of growth?
4. What was the mean growth rate over this period?
5. Draw a horizontal line on your graph representing the mean growth rate.
6. In which year(s) was the growth rate below the mean?

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to calculate the mean, median and mode of grouped data.
- How to create a frequency polygon.
- How to create a line graph.

Unit 3: Assessment

Suggested time to complete: 30 minutes

Question 1 adapted from Everything Maths Grade 10 Exercise 10 -4.

1. The frequency distribution below shows the number of passengers that travel in Alfred's minibus taxi per week.

Class limits	Frequency
$400 \leq x < 500$	4
$500 \leq x < 600$	6
$600 \leq x < 700$	11
$700 \leq x < 800$	16
$800 \leq x < 900$	7
$900 \leq x < 1\,000$	1

   a. Create a frequency polygon of this data.
   b. What is the modal interval?
   c. How many weeks does this data cover?
   d. Calculate an estimate for the total number of passengers to travel in Alfred's taxi over the entire period?
   e. Give an estimate of the mean number of passengers per week.
   f. If it is estimated that every passenger travelled an average distance of 5 km, how much money would Alfred have taken in an average week if he charged $\text{R}3.50$ per kilometre?

2. The table gives the average fuel price per litre (in Rands) in South Africa for unleaded 95 octane petrol (ULP 95) between 2007 and 2017.

April	ULP 95
2007	R6.54
2008	R8.91
2009	R7.38
2010	R8.58
2011	R9.96
2012	R11.94
2013	R13.20
2014	R14.39
2015	R12.89
2016	R12.62
2017	R13.30

   a. Create a line graph to represent this data.
   b. What overall trend do you see in the data?

The full solutions are at the end of the unit.

Unit 3: Solutions

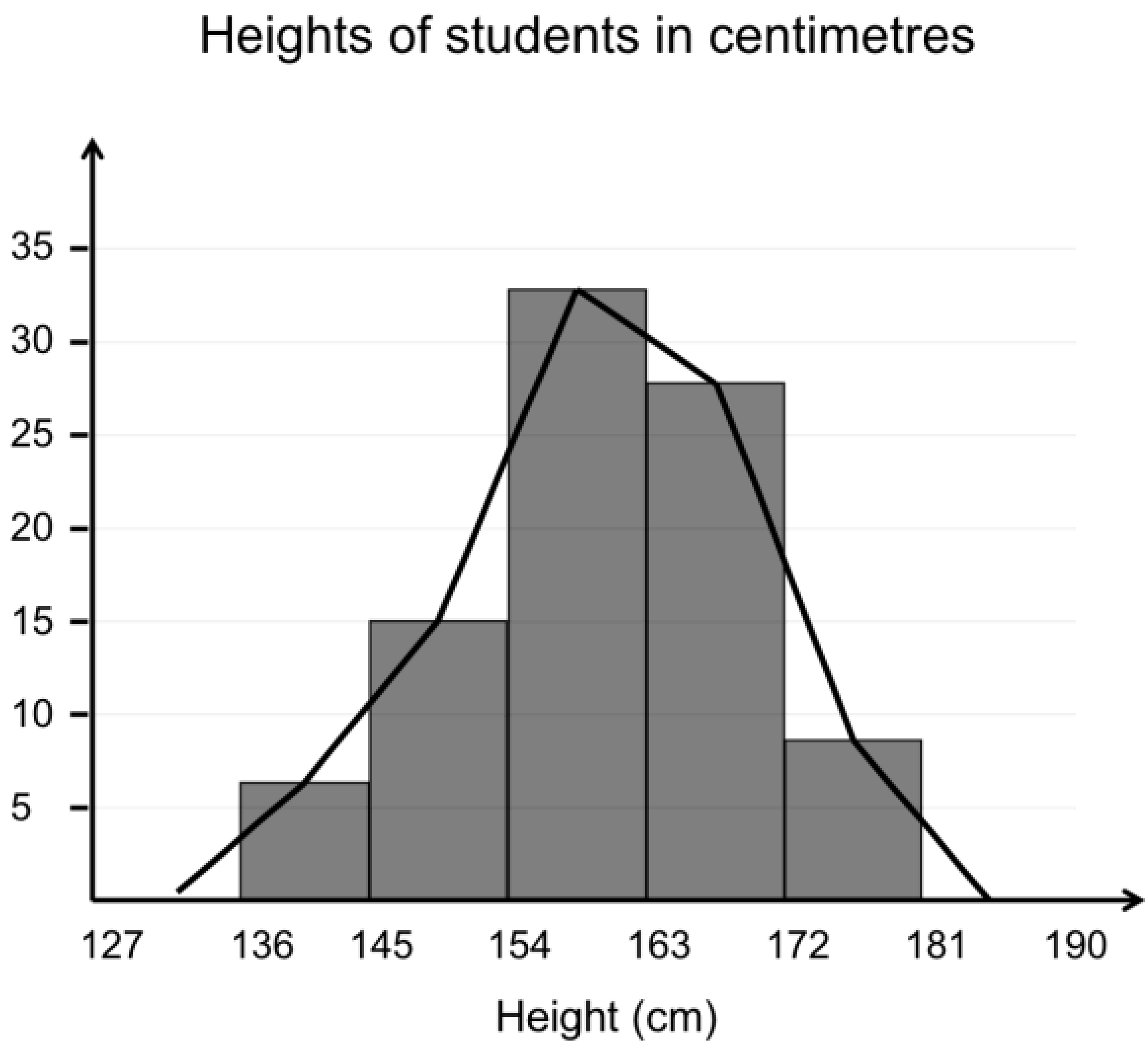
Exercise 3.1

1. We will need to calculate the mean of grouped data. To do this we assume that each value within a class or interval is at the midpoint of the interval.
$$\begin{aligned}\bar{x} &= \frac{(5 \times 40) + (12 \times 50) + (15 \times 60) + (28 \times 70) + (18 \times 80) + (14 \times 90) + (7 \times 100)}{99} \\ &= 71.31 \ \text{ min}\end{aligned}$$
2. The modal group is the group, class or interval with the highest count or frequency. This is $65 < t \leq 75$.
3. The median is calculated as the midpoint of the middle group, class or interval. The median class is $65 < t \leq 75$. Therefore, the median is 70.

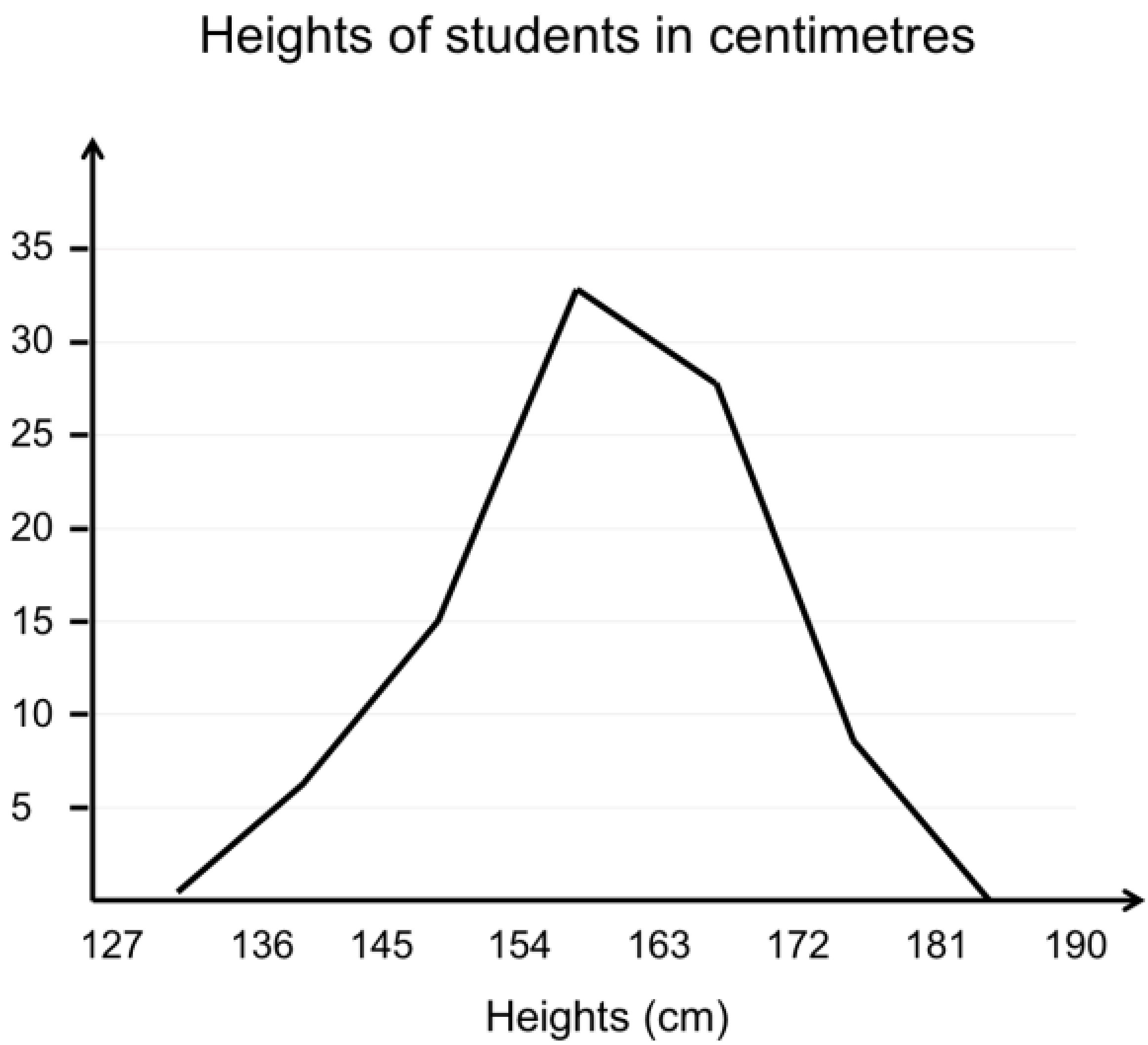
Back to Exercise 3.1

Exercise 3.2

1.


Heights of students in centimetres
35
30
25
20
15
10
5
127
136
145
154
163
172
181
190
Height (cm)

or

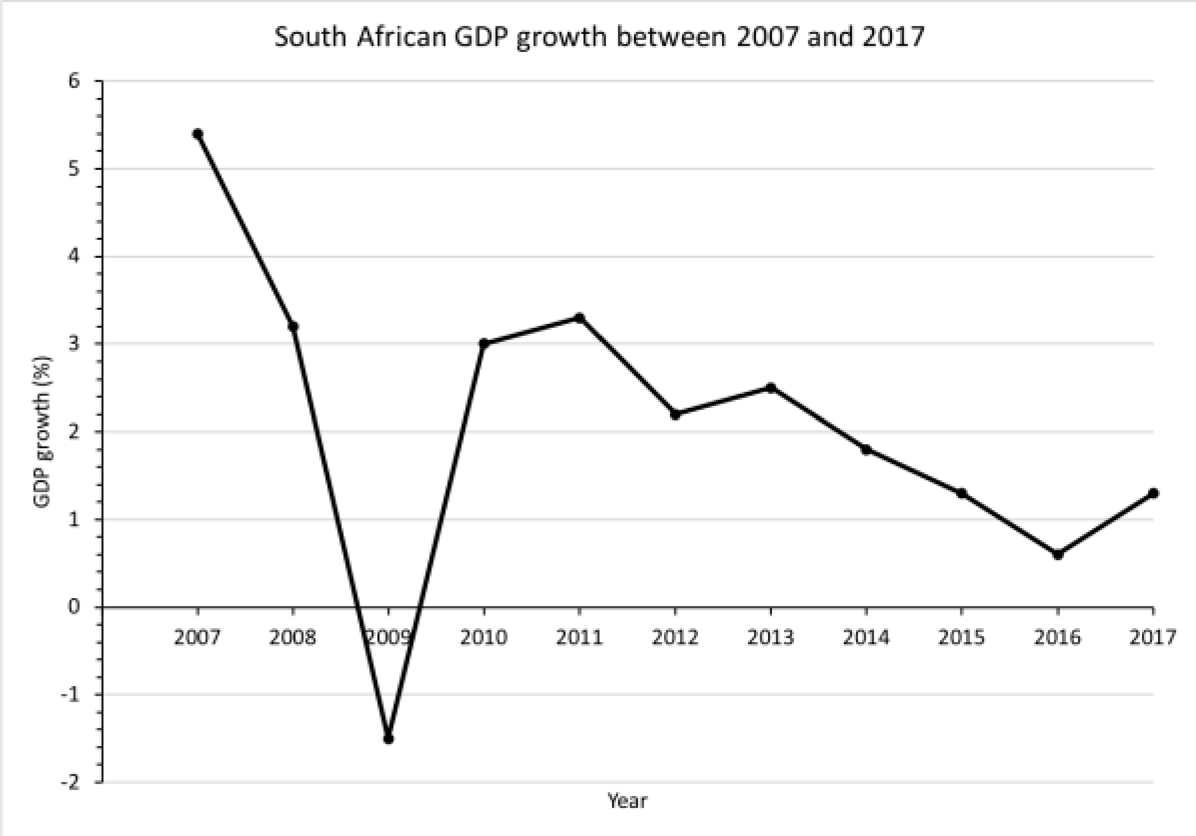


2. We do not have the original raw data so we will need to calculate the mean of the grouped data. To do this we assume that each value within a class or interval is at the midpoint of the interval.
$$\bar{x} = \frac{(6 \times 140) + (15 \times 149) + (33 \times 158) + (27 \times 167) + (9 \times 176)}{90}$$
$$= 159.8 \text{ cm}$$
3. The median is the midpoint of the median class or media interval. The median class is $154 - 162$. Therefore, the median is 158.
4. The mode is the midpoint of the modal class or modal interval. The modal interval is $154 - 162$. Therefore, the mode is 158.

Back to Exercise 3.2

Exercise 3.3

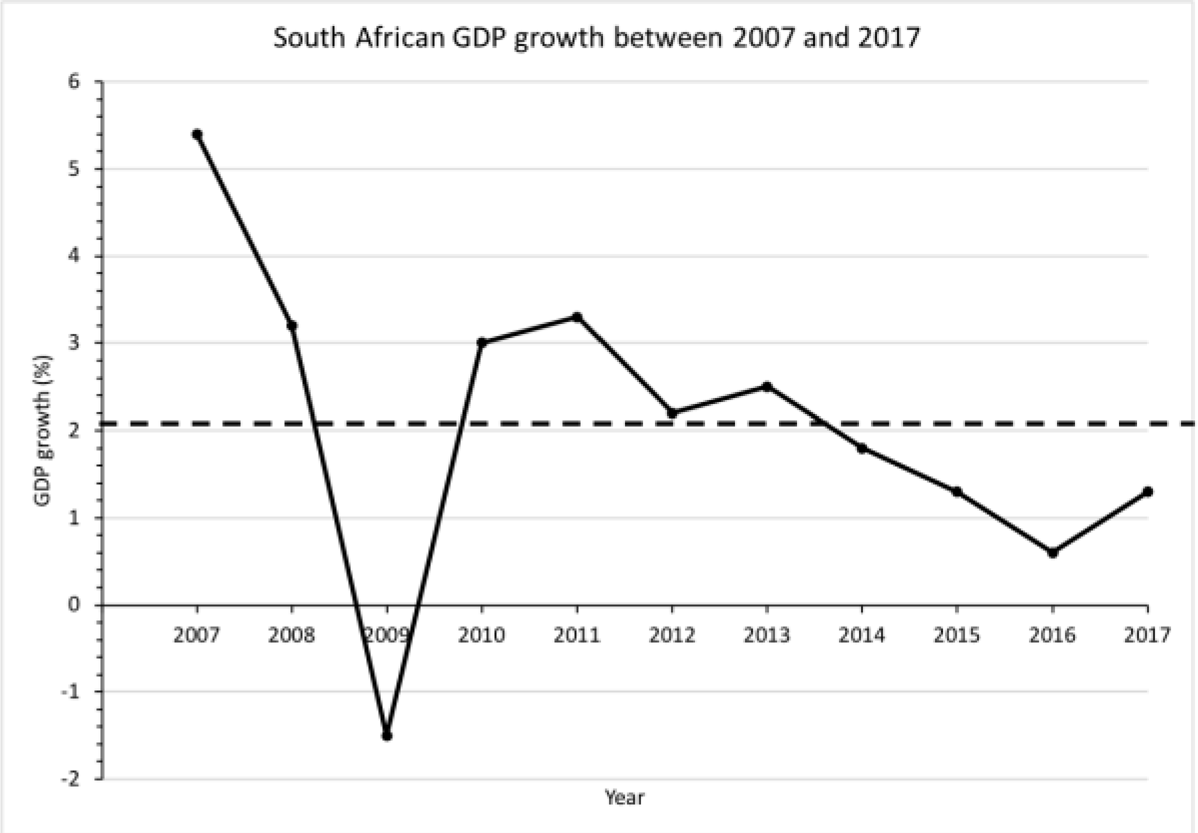
1. In this set of data there was a negative value. This means that the graph extends below the x-axis for a period.



2. The general trend was a decline in GDP growth over the period, especially between 2012 and 2016.
3. The highest growth rate was in 2007.
4. Mean growth rate:

$$\bar{x} = \frac{5.4 + 3.2 - 1.5 + 3.0 + 3.3 + 2.2 + 2.5 + 1.8 + 1.3 + 0.6 + 1.3}{11}$$
$$= 2.1$$

5.

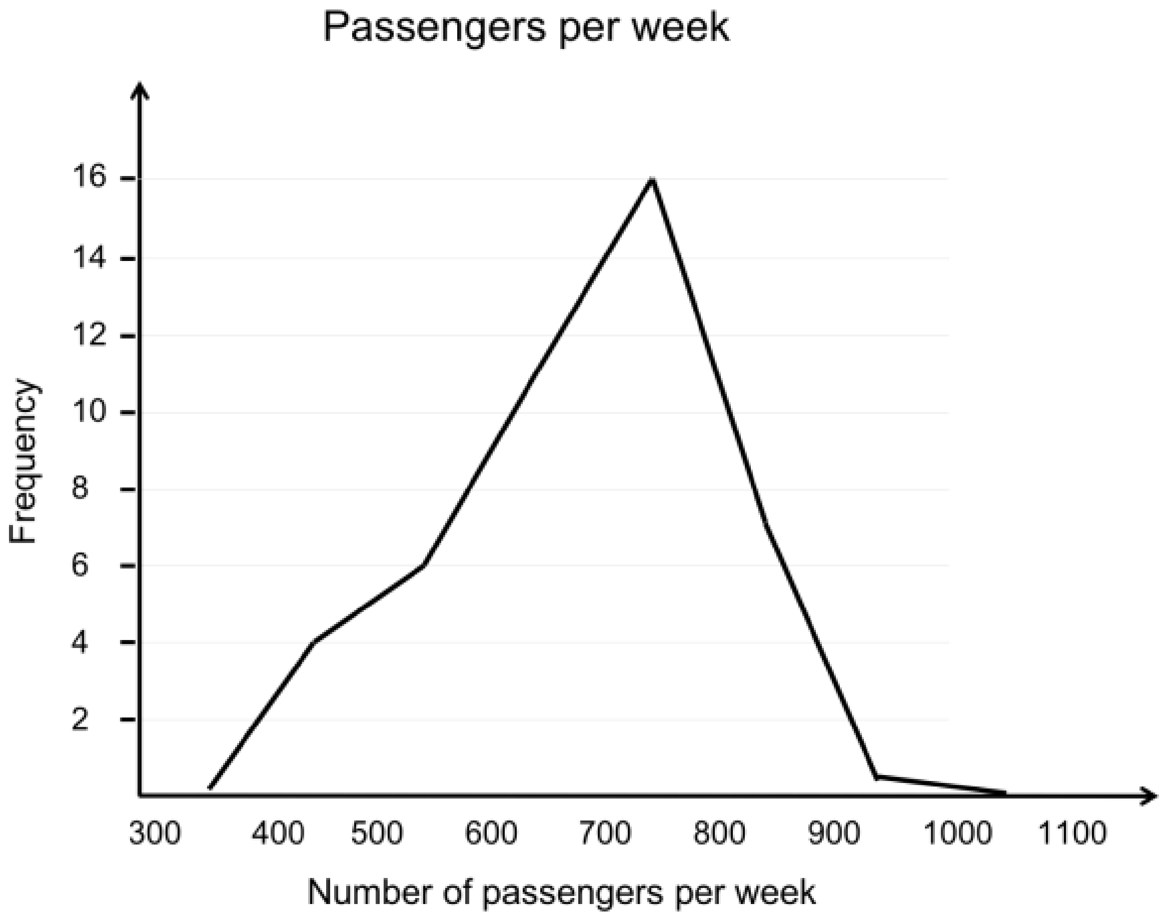


6. The growth rate was below the mean in 2009, and from 2014 to 2017.

Back to Exercise 3.3

Unit 3: Assessment

1.
    a.



b. The modal interval is $700 - 799$.

c. The data covers $4 + 6 + 11 + 16 + 7 + 1 = 45$ weeks.

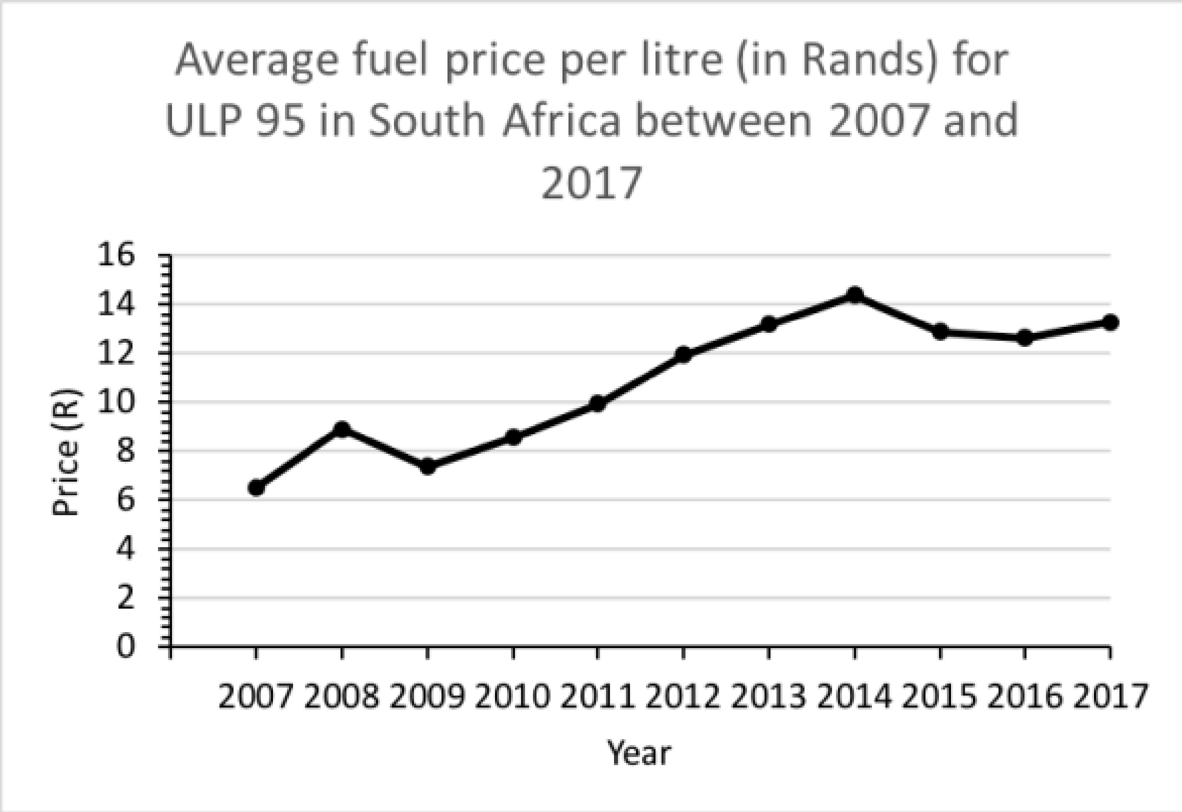
d. Estimated total number of passengers:
$(4 \times 450) + (6 \times 550) + (11 \times 650) + (16 \times 750) + (7 \times 850) + (1 \times 950) = 31\ 150$

e. $\bar{x} = \dfrac{31150}{45} = 692$ passengers per week

f. $692 \text{ passengers} \times 5 \text{ km } \times \text{R}3.50/\text{km} = \text{R}12\ 110/\text{week}.$

2.

a.



b. The overall trend is an increase in the price of ULP 95 over the period.

Back to Unit 3: Assessment

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SUBJECT OUTCOME XIV

FINANCIAL MATHS: PLAN AND MANAGE PERSONAL AND HOUSEHOLD FINANCES

Subject outcome 5.1

Plan and manage personal and household finances.

Learning outcomes

- Describe financial concepts related to personal finances, methods of financing and financial control.
- Draw up a projected personal and household monthly budget.
- Record actual income and expenditure over a period (one month, six months or twelve months) and compare to the projected budget.
- Identify and explain variances between actual and projected figures.
- Provide possible corrective methods of financial control.

Unit 1: Personal budgets

By the end of this unit you will be able to:

- Draw up an income/expenditure sheet describing plans for projected income and expenditure.
- Do calculations using computational tools efficiently and correctly and verify solutions in terms of the context.
- Draw up a personal and/or household budget.
- Record actual income and expenditure over a period and compare to the projected budget.
- Identify, discuss and explain variances and methods for financial control.

Unit 1: Personal budgets

DYLAN BUSA

Personal budgets

By the end of this unit you will be able to:

- Draw up an income/expenditure sheet describing plans for projected income and expenditure.
- Do calculations using computational tools efficiently and correctly and verify solutions in terms of the context.
- Draw up a personal and/or household budget.
- Record actual income and expenditure over a period and compare to the projected budget.
- Identify, discuss and explain variances and methods for financial control.

What you should know

Before you start this unit, make sure you can:

- Perform basic calculations on a calculator including addition, subtraction, multiplication division and finding percentages.

Introduction

How to manage money well is one of the fundamental lessons we all need to learn to be successful in life. Very few individuals are born into circumstances where they do not need to manage their money carefully and effectively.

There are a few fundamental but easy to understand principles that you should follow if you want to manage your money and make it work for you.

Principles of managing money

The first and most important of the principles of managing money is this: **A part of all you earn is yours to keep**. Another way of saying this is that you need to pay yourself first. A good rule of thumb is that you should save at least 10% of your **income**. Income is all the money that you earn. Obviously, you can save more than 10% but 10% should be the minimum.

But how is this possible, you may ask. I don't earn enough to save anything. Well, this leads to the second basic principle. **Control your expenses**. You need to make sure that you don't spend more than you earn. **Expenses** are all the things we spend our money on – from food, to clothes, to phones and everything in between. There is a difference between the things we need and the things we want. It can be hard to decide what we really need.

In order to control your expenses, you need to know in detail what they are. This is where a **personal budget** comes in. A budget is simply a list of all things you expect to spend money on and how much you expect to spend, and all the ways you expect to earn money and what you expect to earn. If the total you expect to spend is more than the total you expect to earn, you have a problem.

Now saving and keeping the money under your bed is not going to do you much good. You need to invest your savings so that your savings can grow, and your money can start to work for you. But it is important that you invest in things you understand, or that you seek the advice of people you trust who are good at investing. We will deal with how money grows through simple and compound interest in the next subject outcome.

Did you know?

Did you know that many of the principles of managing money well were first discovered by people in the ancient city of Babylon over $3\ 500$ years ago?

Watch the video called "The Richest Man in Babylon – Best Ideas Summary" to learn about the fundamentals of managing money well and growing your wealth.

The Richest Man in Babylon – Best Ideas Summary (Duration: 05.21)

Take note!

- **Income** – this is the money you receive regularly through work or investments.
- **Expenses** – this is the money you spend buying goods and services or repaying your debts.
- **Budget** – this is a list of all the income you expect to earn and the expenses you expect to have over a given period.

Activity 1.1: Where does my money go?

Time required: 20 minutes

What you need:

- a pen
- a piece of paper

What to do:

1. The first step in making a personal budget is to work out where all your money goes. Make a list of all the things you spent money on last month and how much you spent. Refer back to all your till slips (if you have them) or your bank statement if you used a debit or credit card. If you don't know what you spent your money on or how much you spent, this is a clear sign that you need to take control of your expenses. Start keeping track. Remember to include expenses like debt repayments and insurance.
2. Look through your list and divide your expenses into the things you **need** and the things you **want**. Be tough! Question every item. Do you really need KFC every week? Do you really need that online music download service?
3. Work out the total of all your expenses.
4. Now list all your income and how much each source is. Do you have a job? Do you get a stipend, wage, salary or an allowance? Do friends and family contribute to your expenses? Do you collect a government grant? Calculate the total of all your income.
5. Work out what 10% of all your income is and write this down as the amount you should save.
6. Now calculate what the difference is between what you earn, what you spend and what you should save. If you earn more than you spend and save, well done! You are doing a great job. Why not save even more? If you spend more than you earn, it is time to take a look at those expenses.
7. Look through all your expenses and decide what a reasonable average monthly amount would be. This is a great opportunity to remove some of your 'want' expenses. Your aim is to get your expenses and savings down to less than your income.

What did you find?

Obviously, your income and expenses are completely unique to you. Therefore, we will present an example of what each of the steps above might look like.

1. Here is a list of typical expenses for various people. Which ones apply to you? Which ones have we left out?

    - Housing
        - Rent or bond repayments
        - Home insurance
        - Electricity
        - Water
        - Property tax
    - Food
        - Groceries and household supplies
        - Take-aways and eating out
        - Other food expenses
    - Transportation
        - Public transportation and taxis
        - Petrol
        - Parking and tolls
        - Car maintenance
        - Car insurance
        - Car loan

- Other transportation expenses
- Health
  - Medicines
  - Health insurance
  - Other health expenses
- Personal and family
  - Childcare
  - Child support
  - Money given or sent to family
  - Clothing and shoes
  - Donations
  - Entertainment
  - Phone and communication
  - Other personal or family expenses
- Education
  - Tuition
  - Supplies
  - Student loans
- Financial
  - Bank fees
  - Loan repayments
- Other expenses

This is what a list of expenses might look like:

Expenses	**Total**
Groceries	R780
Transport	R450
Electricity	R255
Phone	R370
Hair products	R180
Rent	R800
Clothes	R460
Take-aways	R410

2. Identify which of your expenses could be categorised as 'wants'.

Expenses	**Total**
Needs	
Groceries	R780
Transport	R450
Electricity	R255
Phone	R370
Rent	R800
Wants	
Hair products	R180
Clothes	R460
Take-aways	R410

3. Calculate the total of all your expenses.

Expenses	Total
Needs	
Groceries	R780
Transport	R450
Electricity	R255
Phone	R370
Rent	R800
Wants	
Hair products	R180
Clothes	R460
Take-aways	R410
Total	**R3 705**

4. List your sources of income and your total income.

Income	Total
Stipend	R2 500
Allowance	R1 200
Total	**R3 700**

5. Show your savings goal based on your current levels of income.

Income	Total
Stipend	R2 500
Allowance	R1 200
Total	**R3 700**
Savings	**R370**

6. Calculate what money remains at the end of the month. In this example, we can see that more is spent than was earned. In this example, this person needs to cut R375 from their expenses.

Income	Total
Stipend	R2 500
Allowance	R1 200
Total	R3 700
Savings	R370
Expenses	**Total**
Needs	
Groceries	R780
Transport	R450
Electricity	R255
Phone	R370
Rent	R800
Wants	
Hair products	R180
Clothes	R460
Take-aways	R410
Total	R3 705
Income - Savings - Expenses	
$\mathbf{R3\ 700 - R370 - R3\ 705 = -R375}$	

7. In order to spend less money than is earned, the following changes could be made to this budget:

    - change to a cheaper cell phone package

- cut out all hair products
- reduce the monthly spend on clothing
- reduce the monthly spend on take-aways.

Below is the revised budget. Now there is a surplus of $\mathrm{R}445$ that can be saved for emergencies, saved for something like a car, or invested.

Income	**Total**
Stipend	R2 500
Allowance	R1 200
Total	R3 700
Savings	R370
Expenses	**Total**
Needs	
Groceries	R780
Transport	R450
Electricity	R255
Phone	~~R370~~ R250
Rent	R800
Wants	
~~Hair products~~	~~R180~~
Clothes	~~R460~~ R250
Take-aways	~~R410~~ R100
Total	~~R3 705~~ R2 885
Income - Savings - Expenses	
R3 700 − R370 − R2 885 = R445	

Activity 1.1 showed the basic budgeting process. Now obviously a budget is no good unless you are disciplined and keep to it. You are encouraged to create your own personal budget and to keep to it. Keep track

of what you spend each month. If you spend a bit more than you budgeted in one month, you know you need to cut back in the following month(s) to stay on track.

Did you know?

The $50/30/20$ rule can be an effective guideline with which to create a personal budget.

- Budget 50% of your income on things that you **need**.
- Budget 30% of your income on things that you **want**.
- Budget 20% of your income on **savings** and **debt repayments**.

Note

If you would like more information on personal budgets and how to create one, visit the following websites:

- Maths is Fun: Personal budget
- Consumer.gov: Making a budget

Example 1.1

Question taken from NC(V) Mathematics L2 Paper 1 February 2013

Kayelani is a painter who works 21 days per month. He works for eight hours per day, which is 168 hours per month and earns a nett wage of $\text{R}40$ per hour. He works away from home and rents a small flat. His monthly expenses are as follows:

- Rental of flat (including water and electricity): $\text{R}2\ 150$
- Transport: $\text{R}16$ per day
- Cell phone contract: $\text{R}145$
- Groceries: $\text{R}1\ 725.50$
- Clothing account: $\text{R}450.50$
- Entertainment: $\text{R}500$
- Laundry service: $\text{R}200$

Use the above information to fill in the income and expenditure sheet below.

INCOME AND EXPENDITURE: KAYELANI A PAINTER.	
INCOME	**AMOUNT**
Nett wage	
168 hours@ R40 per hour	
Nett monthly earnings	
EXPENDITURE	**AMOUNT**
Rental of flat	
Transport	
Cell phone	
Groceries	
Clothing	
Entertainment	
Laundry service	
TOTAL MONTHLY EXPENDITURE	
Amount left after all the expenses have been paid	

Solution:

We are told that Kayelani works 168 hours per month and earns a nett wage of $\text{R}40$ per hour. When we say **nett**, we mean AFTER all deductions have been made. In other words, he earns $\text{R}40$ per hour after any deductions. If you see the word **gross**, this means BEFORE any deductions like tax and Unemployment Insurance Fund (UIF).

So Kayelani's nett wage is $168 \times \text{R}40 = \text{R}6\ 720$. He does not make any additional income, so his nett monthly earnings are also $\text{R}6\ 720$.

Now all his expenses are monthly expenses except transport which is per day. Therefore, to work out his total monthly transport expense, we have to multiply this amount by the number of days he has to travel to work, i.e. 21. His total transport expense is $21 \times \text{R}16 = \text{R}336$.

Now we can complete the expenses portion of his income and expenditure sheet.

Here is the final completed sheet.

INCOME AND EXPENDITURE: KAYELANI A PAINTER.	
INCOME	**AMOUNT**
Nett wage	
168 hours@ R40 per hour	R6 720
Nett monthly earnings	R6 720
EXPENDITURE	**AMOUNT**
Rental of flat	R2 150
Transport	R336
Cell phone	R145
Groceries	R1 720.50
Clothing	R450.50
Entertainment	R500
Laundry service	R200
TOTAL MONTHLY EXPENDITURE	R5 502
Amount left after all the expenses have been paid	R1 218

Exercise 1.1

Sibongile has a personal budget as shown below.

	Monthly budget		January	February	March
Income	**Amount**				
Nett salary	R13 500,00				
Commission	R5 000,00				
Total income	**R18 500,00**				
Expenses					
Rent	R6 000,00				
Electricity	R650,00				
Water	R480,00				
Car repayment	R2 267,13				
Car insurance	R673,42				
Petrol	R800,00				
Groceries	R3 000,00				
Clothing	R1 500,00				
Cell phone	R500,00				
Life insurance	R600,00				
Entertainment	R850,00				
Total expenses	**R17 320,55**				
Surplus/deficit	**R1 179,45**				

Her income and expenses are as follows for the month of January.

INCOME

Nett salary: $\text{R}13\ 500$
Commission: 10% on sales of $\text{R}28\ 142.40$

EXPENSES

Rent: $\text{R}6\ 000$
Electricity: $\text{R}712$
Water: $\text{R}490$
Car repayment: $\text{R}2\ 767.13$
Car insurance: $\text{R}673.42$
Petrol: $\text{R}768$
Groceries: $\text{R}3\ 124.87$
Clothing: $\text{R}1\ 813.23$
Cell phone: $\text{R}490.90$
Life insurance: $\text{R}600$
Entertainment: $\text{R}874.45$

Answer the following questions.

1. Complete the income and expenses sheet for the month of January.
2. Did Sibongile stick to her budget?
3. In which areas was there a variance between her budget and her actual income and expenditure. In other words, where did she overspend and/or underspend?
4. What advice could you give to Sibongile to help her save at least 10% of her income?

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- The keys to financial success are to spend less than you earn, to save and to invest your savings.
- How to draw up a projected personal and household monthly budget.
- How to record actual income and expenditure over a period of time.
- Identify and explain variances between actual and projected figures.

Unit 1: Assessment

Suggested time to complete: 20 minutes

Question taken from NC(V) Mathematics L2 Paper 1 October 2012

Peter kept a record of all his earnings and all the payments he made during the month of August 2010. Use the information below to complete his income statement (also below).

INCOME

- Gross salary: $\text{R}15\ 500$ per month.
- Deductions from his salary:
    - UIF: 1% of the gross salary
    - Income tax: $\text{R}3\ 875$
    - Pension fund: $\text{R}110$

OTHER INCOME

- Birthday bonus: 10% of gross monthly salary
- Commission: 7% on his total sales of $\text{R}36\ 490$

EXPENSES

- House rent: $\text{R}4\ 500.10$
- Groceries: $\text{R}1\ 780.15$
- Car repayment: $\text{R}1\ 260$
- Cell phone: $\text{R}275$
- Petrol: $\text{R}850.30$
- Electricity: $\text{R}460$
- Water: $\text{R}69.17$
- Clothing accounts: $\text{R}1\ 200$
- Insurance: $\text{R}218.19$
- Entertainment: $\text{R}480$

INCOME AND EXPENDITURE: RECORD OF ALL EARNINGS FOR PETER	
INCOME	**AMOUNT**
Salary (after deductions)	
Bonus	
Commission	
Total income	
EXPENDITURE	**AMOUNT**
TOTAL EXPENDITURE	
SURPLUS/DEFICIT (income minus expenses)	

The full solutions are at the end of the unit.

Unit 1: Solutions

Exercise 1.1

1.

	Monthly budget	January
Income	**Amount**	
Nett salary	R13 500,00	R13 500,00
Commission	R5 000,00	R2 814,24
Total income	**R18 500,00**	**R16 314,24**
Expenses		
Rent	R6 000,00	R6 000,00
Electricity	R650,00	R712,00
Water	R480,00	R490,00
Car repayment	R2 267,13	R2 267,13
Car insurance	R673,42	R673,42
Petrol	R800,00	R768,00
Groceries	R3 000,00	R3 124,87
Clothing	R1 500,00	R1 813,23
Cell phone	R500,00	R490,90
Life insurance	R600,00	R600,00
Entertainment	R850,00	R874,45
Total expenses	**R17 320,55**	**R17 814,00**
Surplus/deficit	**R1 179,45**	**-R1 499,76**

2. Sibongile was not able to stick to her budget.
3. Sibongile did not earn as much commission as she had budgeted. Her income was, therefore, $\text{R}2\ 185.76$ below budget.

   Sibongile **overspent** on the following items by the following amounts:
   Electricity: $\text{R}62$
   Water: $\text{R}10$
   Groceries: $\text{R}124.87$
   Clothing: $\text{R}313.23$
   Entertainment: $\text{R}24.75$

   Sibongile underspent on the following items by the following amounts:
   Petrol: $\text{R}32$
   Cell phone: $\text{R}9.10$
4. 10% of her nett salary is $\text{R}1\ 350$. Therefore, she needs to cut at least $\text{R}1\ 350 - \text{R}1\ 179.45 = \text{R}170.55$ from her expenses. The best places to cut would be her clothing and entertainment budgets. Another idea is for Sibongile to cut her overall expenses by $\text{R}2\ 500 - \text{R}1\ 350 = \text{R}1\ 150$ so that she is not so reliant on making $\text{R}5\ 000$ commission every month. Any additional commission over $\text{R}2\ 500$ would go straight to savings.

Back to Exercise 1.1

Unit 1: Assessment

Deductions from salary:

UIF: $R15\ 500 \times 0.01 = R155$
Income tax: $R3\ 875$
Pension fund: $R110$

Salary after deductions (NETT salary):
$R15\ 500 - R155 - R3\ 875 - R110 = R11\ 360$

Bonus:
$R15\ 500 \times 0.1 = R1\ 550$

Commission:
$R36\ 490 \times 0.07 = R2\ 554.30$

INCOME AND EXPENDITURE: RECORD OF ALL EARNINGS FOR PETER	
INCOME	**AMOUNT**
Salary (after deductions)	R11 360.00
Bonus	R1 550.00
Commission	R2 554.30
Total income	R15 464.30
EXPENDITURE	**AMOUNT**
House rent	R4 500.10
Groceries	R1 780.15
Car repayment	R1 260.00
Cell phone	R275.00
Petrol	R850.30
Electricity	R 460.00
Water	R69.17
Clothing accounts	R1 200.00
Insurance	R218.19
Entertainment	R480.00
TOTAL EXPENDITURE	R11 092.81
SURPLUS/DEFICIT (income minus expenses)	R4 371.49

Back to Unit 1: Assessment

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SUBJECT OUTCOME XV

FINANCIAL MATHS: USE SIMPLE AND COMPOUND INTEREST TO EXPLAIN AND DEFINE A VARIETY OF SITUATIONS

Subject outcome 5.2

Use simple and compound interest to explain and define a variety of situations.

Learning outcomes

- Differentiate between simple and compound interest.
- Explain the advantages and disadvantages of using simple and compound interest in specific situations.
- Use and manipulate the simple growth formula $A = P(1 + in)$ to solve problems.
- Use and manipulate the compound growth formula $A = P(1 + i)^n$ to solve problems subject to only annual compounding being made.

Unit 1: Simple and compound interest

By the end of this unit you will be able to:

- Differentiate between simple and compound interest and extrapolate the advantages and disadvantages of each in specific situations.
- Calculate simple and compound interest over different periods at specific rates.
- Do calculations using computational tools efficiently and correctly and verify solutions in terms of the context.
- Use solutions to calculations effectively to define the changes that occur over a period.

Unit 1: Simple and compound interest

DYLAN BUSA

Simple and compound interest

By the end of this unit you will be able to:

- Differentiate between simple and compound interest and extrapolate the advantages and disadvantages of each in specific situations.
- Calculate simple and compound interest over different periods at specific rates.
- Do calculations using computational tools efficiently and correctly and verify solutions in terms of the context.
- Use solutions to calculations effectively to define the changes that occur over a period.

What you should know

Before you start this unit, make sure you can:

- Explain what is meant by income and expenditure. If you need help with this, review Unit 1 of Subject outcome 5.1.
- Solve linear equations. If you need help with this, review Unit 1 of Subject outcome 2.3.
- Solve literal equations. If you need help with this, review Unit 2 of Subject outcome 2.3.
- Describe what a straight line graph or **linear function** is. Refer back to Unit 1 in Subject outcome 2.1 if you need help with this.
- Describe what an **exponential function** is. Refer back to Unit 4 in Subject outcome 2.1 if you need help with this.

Introduction

In the previous subject outcome, we learnt about the importance of personal budgets and how these help us to follow two of the basic principles of good money management and wealth creation; namely, save at least 10% of your income, and control your expenses so that you do not spend more than you earn.

In that subject outcome, we mentioned that while saving is important, keeping the money under your bed is of little value. You need to make sure that your savings are growing and making more money for you. In other words, you need to invest your savings. It is important, however, that you invest wisely. Being able to analyse and calculate the expected rate of return of your investments is an important skill and to do this you need to understand what simple and compound interest are, and how they differ.

In the context of savings and investments, interest is the money that you earn (your fee) for lending your money to someone else. In the context of loans, interest is the money you have to pay to someone else (their fee) for them lending their money to you.

Note

Watch the video called “Intro to simple interest” to learn more about interest.

Intro to simple interest (Duration: 02.45)

Simple vs compound interest

An excellent way to discover what simple and compound interest really are is to see a practical example of each type of interest.

Activity 1.1: Simple vs compound interest

Time required: 30 minutes

What you need:

- a pen or pencil
- a piece of paper
- a calculator

What to do:

1. Imagine you invest $\text{R}1\ 000$ once-off in a savings account. The account pays 15% **simple interest** on your money each year. Simple interest means that you earn interest only on the money that you initially invest. How much money will be in your savings account after one year?
2. Calculate how much money will be in your savings account at the end of the second year at 15% simple interest per year. Remember that simple interest means that you only earn interest on your initial deposit.
3. Now copy the following table onto a blank piece of paper. Complete the simple interest column for 10 years calculating the total value of your investment at the end of each year.

Years	Simple interest (15%)	Compound interest (15%)
After 1 year		
After 2 years		
After 3 years		
After 4 years		
After 5 years		
After 6 years		
After 7 years		
After 8 years		
After 9 years		
After 10 years		

4. Now suppose that the same savings account paid **compound interest** rather than simple interest. Compound interest means that you earn interest on the money that you initially invest **as well as on any previous interest you have earned**. How much money will be in your savings account after one year at 15% compound interest?
5. Calculate how much money will be in your savings account at the end of the second year at 15% compound interest per year. Remember that compound interest means that you earn interest on your initial deposit as well as on any interest you have already earned.
6. Complete the compound interest column for 10 years calculating the total value of your investment at the end of each year.
7. Now compare the values at the end of each year for simple and compound interest. Which kind of interest is better for you as an investor? How long does it take for you to double your investment at both interest rates?
8. What is happening to the difference in the balance in the savings account under simple and compound interest? Why do you think this is the case?
9. On the same set of axes, draw graphs of the growth of each balance. What kind of function do you think each graph represents?
10. Calculate the difference in these balances after 20 years.

What did you find?

1. After one year, you would earn 15% of the initial deposit in interest. . So, at the end of year one, your balance would be $\text{R}1\ 000 + \text{R}150 = \text{R}1\ 150$.
2. At the end of the second year, you will also earn 15% but this will be 15% of your initial deposit only. Therefore, you will earn $\text{R}1\ 000 \times 0.15 = \text{R}150$ in interest. At the end of the second year, your balance would be $\text{R}1\ 150 + \text{R}150 = \text{R}1\ 300$.
3. Here is the table with the completed simple interest column. Each year you earn $\text{R}150$ in interest and so each year your balance increases by $\text{R}150$.

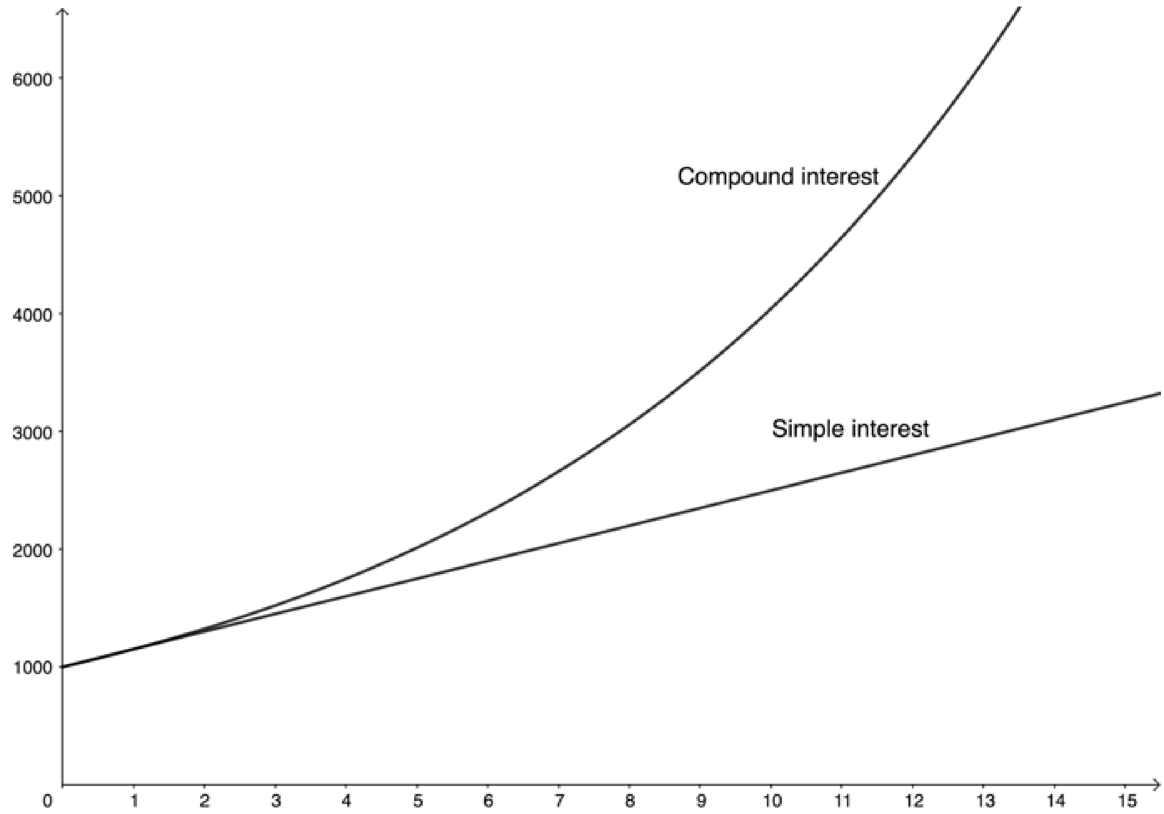
Years	Simple interest (15%)	Compound interest (15%)
After 1 year	R1 150	
After 2 years	R1 300	
After 3 years	R1 450	
After 4 years	R1 600	
After 5 years	R1 750	
After 6 years	R1 900	
After 7 years	R2 050	
After 8 years	R2 200	
After 9 years	R2 350	
After 10 years	R2 500	

4. After one year, you would earn 15% of the initial deposit in interest. . So, at the end of year one, your balance would be $\text{R}1\ 000 + \text{R}150 = \text{R}1\ 150$.
5. At the end of the second year, you will earn interest not only on your initial deposit but also on the interest you have already earned. Therefore, you will earn $\text{R}1\ 150 \times 0.15 = \text{R}172.5$. So, your balance at the end of the second year will be $\text{R}1\ 150 + \text{R}172.5 = \text{R}1\ 322.5$.
6. Here is the table with the completed compound interest column. Each year you earn 15% in interest on your balance from the previous year.

Years	Simple interest (15%)	Compound interest (15%)
After 1 year	R1 150	R1 150
After 2 years	R1 300	R1 322.50
After 3 years	R1 450	R1 520.88
After 4 years	R1 600	R1 749.01
After 5 years	R1 750	R2 011.36
After 6 years	R1 900	R2 313.06
After 7 years	R2 050	R2 660.02
After 8 years	R2 200	R3 059.02
After 9 years	R2 350	R3 517.88
After 10 years	R2 500	R4 045.56

7. Clearly, as an investor, compound interest is much better. You earn far more interest over time. Under simple interest, the investment doubles in value by the end of the seventh year. Under compound interest, it doubles in value after five years.
8. As time progresses, the difference between the simple interest and compound interest balances gets bigger and bigger. This is because the amount of interest earned under simple interest is fixed so the balance goes up by the same fixed amount each year. However, under compound interest, the amount of interest earned is always increasing and so the balance increases each year by an ever-increasing rate.

9.



The simple interest graph is a straight line or **linear function**. The compound interest graph is an **exponential function**. Refer back to Unit 1 and Unit 4 in Subject outcome 2.1 if you don't understand what these terms mean.

10. To calculate the simple interest balance after 20 years is reasonably simple. You will earn $\text{R}150$ in interest each year. Therefore, in 20 years you will earn a total of $\text{R}150 \times 20 = \text{R}3\ 000$ in interest. Your total balance after 20 years will be $\text{R}1\ 000 + \text{R}3\ 000 = \text{R}4\ 000$.

    Calculating the compound interest balance is a bit more difficult without a nice formula but it can be done by calculating the balance manually at the end of each year. If you do this, the balance after 20 years is $\text{R}16\ 366.54$. Therefore, the difference in the balances will be
    $\text{R}16\ 366.54 - \text{R}4\ 000 = \text{R}12\ 366.54$

After completing Activity 1.1 we can clearly see the differences between simple and compound interest and, for an investor, compound interest is significantly more beneficial, especially over the long term.

As a borrower, however, the opposite is true. If you borrow with a compound interest rate, you are not only charged interest on the initial loan but also on the interest that that loan attracts.

In most cases in the real world, both investments and loans are almost always calculated using compound interest.

Note

If you would like to practise doing simple and compound interest calculations, visit the wonderful online interest simulator.

You are able to change the initial amount of the investment / loan, the interest rate per year and the term (or the number of years).

You can zoom in and out of the graph as needed.

Take note!

Simple interest: You earn interest only on the initial investment. You are charged interest only on the amount of the loan.

Compound interest: You earn interest on the initial investment as well as on the interest you have earned. You are charged interest on the initial loan as well as on the interest that the loan attracts.

Did you know?

Usury

At certain times in the past (and in some places today) it has been illegal to make a person pay interest on a loan. It was, and still is, against some religious rules. If the interest that a person must pay is unfairly high, this is called **usury**. This is illegal in most places today.

Simple interest

Now that we know what simple interest is, let's explore it in a little more detail. There is a formula that we use to work out the value of an accumulated amount after an initial amount has been growing due to simple interest. That formula is below.

Simple interest growth formula:

$A = P(1 + in)$ where
A is the **accumulated amount**
P is the initial amount, called the **principal amount**
i is the rate of simple interest per year (always written as a decimal)
n is the number of years

Example 1.1

Jeff deposits $\mathrm{R}1\ 500$ into a savings account which pays a simple interest rate of 6.5% per annum (p.a.) for three years. How much will be in his account at the end of the investment term?

Solution:

The first thing we need to take note of is that we are dealing with simple interest. In this case we are given that $P = \mathrm{R}1\ 500$, $i = \frac{6.5\%}{100} = 0.065$ per year (p.a. is an abbreviation for 'per annum' and 'annum' is another word for year) and $n = 3$ years. We need to find the value of A.

$$\begin{aligned} A &= P(1 + in) \\ \therefore A &= 1\ 500(1 + 0.065 \times 3) \\ &= 1\ 792.50 \end{aligned}$$

$\mathrm{R}1\ 792.50$ will be in Jeff's account at the end of the investment term.

Note: It is important when calculating with the simple interest formula that you round off only your final answer, if necessary.

Example 1.2

Fatima borrows $\mathrm{R}3\ 450$ from her neighbour at an agreed simple interest rate of 11.2% p.a. She will pay back the loan in one lump sum at the end of four years. How much will she have to pay her neighbour?

Solution:

This is a simple interest problem.

$P = \mathrm{R}3\ 450$, $i = 0.112$ and $n = 4$ years

$$\begin{aligned} A &= P(1 + in) \\ \therefore A &= 3\ 450(1 + 0.112 \times 4) \\ &= 4\ 995.60 \end{aligned}$$

Fatima will need to pay back $\mathrm{R}4\ 995.60$ after four years.

Example 1.3

Big Burgers deposits $\mathrm{R}35\ 000$ into a bank account that pays a simple interest rate of 4.5% p.a. How many years must Big Burgers invest for to generate a total of $\mathrm{R}50\ 000$?

Solution:

This is a simple interest problem. However, in this case we need to find the number of years (n) of the investment. $P = \mathrm{R}35\ 000$, $i = 0.045$ and $A = \mathrm{R}50\ 000$.

$$
\begin{aligned}
A &= P(1+in)\\
\therefore \frac{A}{P} &= 1+in\\
\therefore \frac{A}{P}-1 &= in\\
\therefore n &= \frac{\frac{A}{P}-1}{i}\\
&= \frac{\frac{50\ 000}{35\ 000}-1}{0.045}\\
&= 9.52
\end{aligned}
$$

After 9.52 years the investment will have grown to $\text{R}50\ 000$.

Example 1.4

Makhize is planning to buy a new car in three years' time. He has $\text{R}42\ 850$ to save and expects he will need $\text{R}55\ 000$ for the new car. At what p.a. simple interest rate does he need to invest his money?

Solution:

This is a simple interest problem. However, in this case we need to find the interest rate p.a. (i) of the investment. $P = \text{R}42\ 850$, $n = 3$ and $A = \text{R}55\ 000$.

$$
\begin{aligned}
A &= P(1+in)\\
\therefore \frac{A}{P} &= 1+in\\
\therefore \frac{A}{P}-1 &= in\\
\therefore i &= \frac{\frac{A}{P}-1}{n}\\
&= \frac{\frac{55\ 000}{42\ 850}-1}{3}\\
&= 0.0945\\
&= 9.45\%
\end{aligned}
$$

He will need to invest his money at a simple interest rate of 9.45% p.a.

Exercise 1.1

1. An amount of $\text{R}7\ 850$ is invested in a savings account which pays simple interest at a rate of 6.23% per annum. Calculate the balance accumulated by the end of six years.
2. Joseph deposited $\text{R}5\ 724$ into a savings account on his son's fifth birthday. When his son turned 18,

the balance in the account had grown to $\mathrm{R}17\ 892.31$. If simple interest was paid, calculate the rate at which the money was invested.

3. Salim wants to invest $\mathrm{R}10\ 800$ at a simple interest rate of 8.25% p.a. How many years will it take for the money to grow to $\mathrm{R}33\ 000$? Round up your answer to the nearest year.

The full solutions are at the end of the unit.

Compound interest

As we saw in Activity 1.1, compound interest is extremely powerful, especially over the long term. This is why it is so important that everyone starts saving, even a little bit, as soon as possible.

$\mathrm{R}1\ 000$ invested at 10% compound interest per annum for 10 years will grow to $\mathrm{R}2\ 593.74$. The same amount invested at the same rate over 40 years will grow to $\mathrm{R}45\ 259.26$. One of the secrets to building wealth is to invest over the long term. The longer you invest, the quicker your money grows as shown in Figure 1.

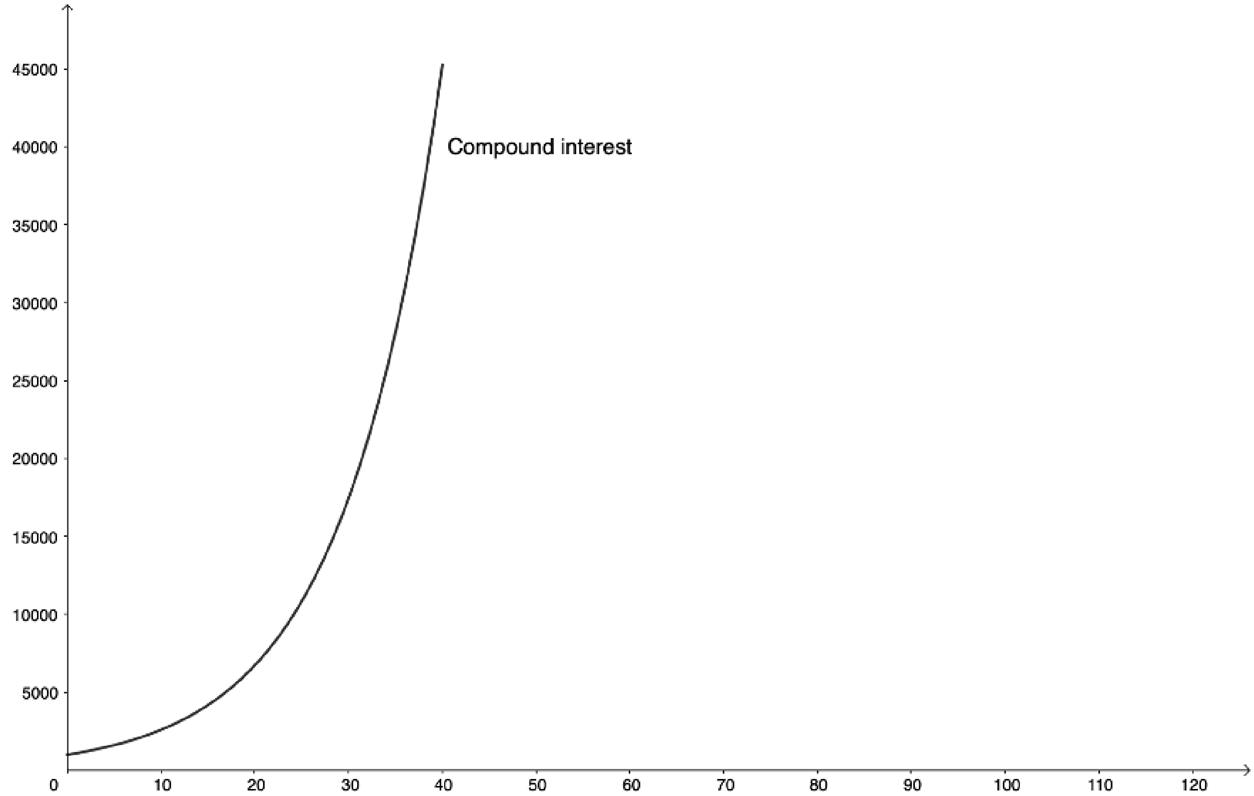


Figure 1: Growth of an investment of $\mathrm{R}1000$ *at* 10 *compound interest per annum*

Your savings will grow even faster if you keep adding, even little bits of money to your savings.

As for simple interest, we have a formula with which we can calculate the growth of an amount under compound interest. It is very similar to the simple interest formula except for one small detail.

Compound interest growth formula:

$A = P(1+i)^n$ where
A is the **accumulated amount**
P is the initial amount, called the **principal amount**
i is the rate of compound interest per year (always written as a decimal)
n is the number of years

Example 1.5

Mpho wants to invest $\text{R}25\ 000$ into an account that offers a compound interest rate of 6% p.a. How much money will be in the account at the end of five years?

Solution:

The first thing we need to take note of is that we are dealing with compound interest. In this case we are given that $P = \text{R}25\ 000$, $i = \dfrac{6\%}{100} = 0.06$ per year and $n = 5$ years. We need to find the value of A.

$$\begin{aligned} A &= P(1+i)^n \\ \therefore A &= 25\ 000(1+0.06)^5 \\ &= 33\ 455.64 \end{aligned}$$

$\text{R}33\ 455.64$ will be in Mpho's account at the end of the investment term.

Note: It is important when calculating with the compound interest formula that you round off only at your final answer, if necessary.

Example 1.6

Thabile has been given $\text{R}6\ 525$ for her thirteenth birthday. Rather than spending it, she has decided to invest it so that she can put down a deposit of $\text{R}13\ 000$ on a car on her eighteenth birthday. What compound interest rate does she need to achieve this growth?

Solution:

This is a compound interest problem. However, in this case we need to find the interest rate p.a. (i) of the investment. $P = \text{R}6\ 525$, $n = 5$ and $A = \text{R}13\ 000$.

$$\begin{aligned} A &= P(1+i)^n \\ \therefore \frac{A}{P} &= (1+i)^n \\ \therefore \sqrt[n]{\frac{A}{P}} &= 1+i \\ \therefore i &= \sqrt[n]{\frac{A}{P}} - 1 \\ &= \sqrt[5]{\frac{13\ 000}{6\ 525}} - 1 \\ &= 0.1478 \\ &= 14.78\% \end{aligned}$$

Thabile will need to invest her money at a compound interest rate of 14.78% p.a.

Did you know?

The rule of 70

One percent compound interest p.a. on $\text{R}100$ will cause this amount to double to $\text{R}200$ in about 70 years. That is called the **rule of 70**. This rule can be used to figure out how quickly anything that grows exponentially at a constant rate will double in size. Suppose an elephant grows 10% a year. If we divide 10 into 70 we get 7. So, the elephant will double in size in about seven years. If a colony of bacteria grows at a rate of 15% per day, the colony will double in size in about $\frac{70}{15} = 4.67$ days.

Exercise 1.2

1. An amount of $\text{R}4\ 378$ is invested in a savings account which pays a compound interest rate of 5.6% p.a. Calculate the balance accumulated by the end of seven years. As usual with financial calculations, round your answer to two decimal places, but do not round off until you have reached the solution.
2. Thobeka wants to invest some money at a compound interest rate of 9.2% p.a. How much money should she invest if she wants to reach a sum of $\text{R}35\ 000$ in 10 years' time? Round up your answer to the nearest rand.
3. Bongani invests $\text{R}7\ 650$ into an account which pays out a lump sum at the end of eight years. If he gets $\text{R}13\ 427.92$ at the end of the period, what compound interest rate did the bank offer him? Give the answer correct to one decimal place.

The full solutions are at the end of the unit.

Summary

In this unit you have learnt the following:

- How to differentiate between simple and compound interest.
- What the advantages and disadvantages of using simple and compound interest are in specific situations.
- How to use and manipulate the simple growth formula $A = P(1 + in)$ to solve problems.
- How to use and manipulate the compound growth formula $A = P(1 + i)^n$ to solve problems subject to only annual compounding being made.

Unit 1: Assessment

Suggested time to complete: 45 minutes

Questions 1, 2 and 3 taken from NC(V) Mathematics L2 Paper 1 February 2013

1. When Thembi was born her father invested $\text{R}5\ 000$ at an annual compound interest rate. On her eighteenth birthday the bank paid her $\text{R}26\ 292.89$. What was the bank's rate of compound interest?
2. Zack deposits $\text{R}30\ 000$ into a bank account that pays a simple interest rate of 7.5% p.a. How many years must he invest for to generate $\text{R}45\ 000$?
3. Sonto invested $\text{R}10\ 000$ at 15% simple interest per annum for four years. Nomsa invested the same amount at 15% compound interest per annum for four years.
   Calculate:
   a. The total amount Sonto will receive after four years.
   b. The interest Sonto will have earned after four years.
   c. The total amount Nomsa will receive after four years.
   d. The interest Nomsa will have earned after four years.
   e. Explain which investment is better, and give a reason for your answer.

Question taken from NC(V) Mathematics L2 Paper 1 March 2012

4. Mr Masondo wanted to put some money aside for the education costs for his three children. He had $\text{R}50\ 000$ to invest in different savings plans. Each policy matured when that child turned 18 years old. He invested $\text{R}10\ 000$ in a policy for Nzuzo (age 10) which paid 13% compound interest p.a., $\text{R}15\ 000$ in a policy for Tando (age 13) which paid 18% simple interest p.a. and $\text{R}25\ 000$ in a policy for Bamukele (age 15) which paid a flat rate of $\text{R}250$ per month.
   a. How many years did Nzuzo have to wait before she collected her money?
   b. How much did Nzuzo receive when her policy matured?
   c. Calculate the interest that Tando earned on her investment.
   d. How much did Bamukele receive when her policy matured?
   e. Which ONE of the three children received the largest amount of money at the end of their investments?

The full solutions are at the end of the unit.

Unit 1: Solutions

Exercise 1.1

1. This is a simple interest problem.
   $P = \text{R}7\ 850$, $i = 0.0623$, $n = 6$, $A = ?$
   $$\begin{aligned} A &= P(1 + in) \\ &= 7\ 850(1 + 0.0623 \times 6) \\ &= 10\ 784.33 \end{aligned}$$
   The balance after six years will be $\text{R}10\ 413.40$.
2. This is a simple interest problem.

$P = \text{R}5\ 724$, $A = \text{R}17\ 892.31$, $n = 13$, $i = ?$

$$\begin{aligned} A &= P(1+in) \\ \therefore \frac{A}{P} &= 1 + in \\ \therefore in &= \frac{A}{P} - 1 \\ \therefore i &= \frac{\frac{A}{P} - 1}{n} \\ &= \frac{\frac{17\ 892.31}{5\ 728} - 1}{13} \\ &= 0.1634 \\ &= 16.34\% \end{aligned}$$

The interest paid was 16.34% simple interest per annum.

3. This is a simple interest problem.

$P = \text{R}10\ 800$, $A = \text{R}33\ 000$, $i = 0.0825$, $n = ?$

$$\begin{aligned} A &= P(1+in) \\ \therefore \frac{A}{P} &= 1 + in \\ \therefore in &= \frac{A}{P} - 1 \\ \therefore n &= \frac{\frac{A}{P} - 1}{i} \\ &= \frac{\frac{33\ 000}{10\ 800} - 1}{0.0825} \\ &= 24.92 \end{aligned}$$

Salim will need to invest his money for 25 years.

Back to Exercise 1.1

Exercise 1.2

1. This is a compound interest problem.

$P = \text{R}4\ 378$, $i = 0.056$, $n = 7$, $A = ?$

$$\begin{aligned} A &= P(1+i)^n \\ &= 4\ 378(1+0.056)^7 \\ &= 6\ 410.96 \end{aligned}$$

The balance accumulated will be $\text{R}6\ 410.96$.

2. This is a compound interest problem.

$A = \text{R}35\ 000$, $i = 0.092$, $n = 10$, $P = ?$

$$\begin{aligned} A &= P(1+i)^n \\ \therefore P &= \frac{A}{(1+i)^n} \\ &= \frac{35\ 000}{(1+0.092)^{10}} \\ &= 14\ 515.82 \end{aligned}$$

Thobeka will need to invest $\text{R}14\ 515.82$ now.

3. This is a compound interest problem.

$P = \text{R}7\ 650$, $n = 8$, $A = \text{R}13\ 427.92$, $i = ?$

$$\begin{aligned} A &= P(1+i)^n \\ \therefore \frac{A}{P} &= (1+i)^n \\ \therefore \sqrt[n]{\frac{A}{P}} &= 1+i \\ \therefore i &= \sqrt[n]{\frac{A}{P}} - 1 \\ &= \sqrt[8]{\frac{13\ 427.92}{7\ 650}} - 1 \\ &= 0.0729 \\ &= 7.29\% \end{aligned}$$

The bank gave Bongani a compound interest rate of 7.3% p.a.

Back to Exercise 1.2

Unit 1: Assessment

1. This is a compound interest problem.
   $P = \text{R}5\ 000$, $n = 18$, $A = \text{R}26\ 292.89$, $i = ?$
   $$\begin{aligned} A &= P(1+i)^n \\ \therefore \frac{A}{P} &= (1+i)^n \\ \therefore \sqrt[n]{\frac{A}{P}} &= 1+i \\ \therefore i &= \sqrt[n]{\frac{A}{P}} - 1 \\ &= \sqrt[18]{\frac{26\ 292.89}{5\ 000}} - 1 \\ &= 0.0966 \\ &= 9.66\% \end{aligned}$$
   The bank's interest rate was 9.66% compound interest p.a.
2. This is a simple interest problem.
   $P = \text{R}30\ 000$, $A = \text{R}45\ 000$, $i = 0.075$, $n = ?$
   $$\begin{aligned} A &= P(1+in) \\ \therefore \frac{A}{P} &= 1+in \\ \therefore in &= \frac{A}{P} - 1 \\ \therefore n &= \frac{\frac{A}{P} - 1}{i} \\ &= \frac{\frac{45\ 000}{30\ 000} - 1}{0.075} \\ &= 6.67 \end{aligned}$$
   He will need to invest his money for 6.67 years.
3.
   a. Sonto's investment (simple interest):
      $P = \text{R}10\ 000$, $i = 0.15$, $n = 4$, $A = ?$
      $$\begin{aligned} A &= P(1+in) \\ &= 10\ 000(1 + 0.15 \times 4) \\ &= 16\ 000 \end{aligned}$$
      The balance after four years will be $\text{R}16\ 000$.

b. Sonto will have earned $\mathrm{R}16\ 000 - \mathrm{R}10\ 000 = \mathrm{R}6\ 000$ after four years.

c. Nomsa's investment (compound interest):
$P = \mathrm{R}10\ 000$, $i = 0.15$, $n = 4$, $A = ?$
$$\begin{aligned} A &= P(1+i)^n \\ &= 10\ 000(1+0.15)^4 \\ &= 17\ 490.06 \end{aligned}$$
The balance after four years will be $\mathrm{R}17\ 490.06$.

d. Sonto will have earned $\mathrm{R}17\ 490.06 - \mathrm{R}10\ 000 = \mathrm{R}7\ 490.06$ after four years.

e. Nomsa's investment is better as she earns a higher return (more interest) at the same rate over the same term.

4.

a. Nzuzo will need to wait eight years.

b. This is a compound interest problem.
$P = \mathrm{R}10\ 000$, $i = 0.13$, $n = 8$, $A = ?$
$$\begin{aligned} A &= P(1+i)^n \\ &= 10\ 000(1+0.13)^8 \\ &= 26\ 584.44 \end{aligned}$$
Nzuzo received $\mathrm{R}26\ 584.44$.

c. This is a simple interest problem.
$P = \mathrm{R}15\ 000$, $i = 0.18$, $n = 5$, $A = ?$
$$\begin{aligned} A &= P(1+in) \\ &= 15\ 000(1+0.18\times 5) \\ &= 28\ 500 \end{aligned}$$
Tando earned $\mathrm{R}28\ 500 - \mathrm{R}15\ 000 = \mathrm{R}13\ 500$ interest.

d. Bamukele collected on her policy after three years, which was 36 months. Each month, her investment earned $\mathrm{R}250$ for a total of $\mathrm{R}250 \times 36 = \mathrm{R}9\ 000$. Therefore, she collected a total of $\mathrm{R}25\ 000 + \mathrm{R}9\ 000 = \mathrm{R}34\ 000$.

e. Bamukele received the largest amount.

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